TODAY: Separable Equations §2.2

Exact Equations & Integrating factors §2.6

Linear vs non-linear equations §2.3.

Second Part:

Qualitative discussion §2.5

Recap: last time: strategy / solution for 1st order scalar linear ODE. Today we discuss non-linear ODEs.

Setting: $f'(t) = G(t, f)$.

SEPARABLE EQUATION:

Let's first deal w/ it when $G(t,f) = M(t) \cdot N(f)$ when $M: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. It is called

$N: \mathbb{R} \longrightarrow \mathbb{R}$

SEPARABLE because the variables t & f are separated: $\Rightarrow$ assuming $N(f) \neq 0$ in the interval considered

$$\Rightarrow \frac{1}{N(f)} df = M(t)\, dt \Rightarrow \boxed{\int \frac{1}{N(f)} df = \int M(t)\, dt + K}$$

In Lecture 1 & PSet 1 there are already examples of separable equations but let's see another one:

Example:
$$\begin{cases} y' = \dfrac{3t^2 + 4t + 2}{2(y-1)} \\ y(0) = -1 \end{cases}$$

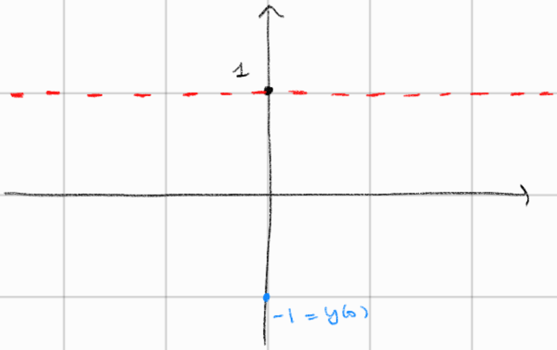
First step: clarification. 1st order - scalar - non-autonomous (there is an explicit t) - non-linear ODE.

Second step: check HPs of P-L theorem.

the function: $G(t,x) = \dfrac{3t^2 + 4t + 2}{2(x-1)}$ is continuous everywhere except at $x = 1$

& $\dfrac{\partial}{\partial x} G(t,x) = \left(\dfrac{3t^2+4t+2}{2}\right) \dfrac{\partial}{\partial x} \dfrac{1}{(x-1)} = \dfrac{3t^2+4t+2}{2} \left(-\dfrac{1}{(x-1)^2}\right)$ again

So in the ty-plane → we exclude $y=1$



Then: $2(y-1)\,dy = (3t^2+4t+2)\,dt$ ⇒ integrate both sides

$$\Rightarrow \quad y^2-2y = t^3+2t^2+2t+K \qquad y(0)=-1 \Rightarrow 1+2 = 0+K \Rightarrow K=3.$$

$$\Rightarrow \quad \underbrace{y^2-2y-(t^3+2t^2+2t+3)=0}_{\text{quadratic equation in } y \;\longrightarrow\; ax^2+bx+c=0 \;\Rightarrow\; x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}} \Rightarrow y_{1,2} = 1 \pm \sqrt{1+t^3+2t^2+2t+3} = 1\pm\sqrt{t^3+2t^2+2t+4}$$

How to decide which is the right sign? (Rmk we know that there must be one & only one solution by P-L-thm): evaluate again in $\begin{cases} t=0 \\ y(0)=-1 \end{cases}$. $-1 = 1 \pm\sqrt{4} = 1\pm 2$ ⇒ "$-$" is the right choice

More complicated ⇒ Non separable but Exact equations.

Let's consider the following ODE $2xy\cdot y' + 2x + y^2 = 0$. It is not linear & it is not separable.

However, let's notice that there exists a function $\Psi(x,y)$: $\frac{d}{dx}\Psi(x,y(x)) =$ LHS of the ODE

Namely. $\Psi(x,y(x)) = xy^2+x^2 \Rightarrow \frac{d}{dx}(xy^2+x^2) = y^2 + 2xy\cdot y' + 2x$

$\Rightarrow$ So the ODE: $2xy\cdot y' + y^2 + 2x = 0$ can be rewritten as $\frac{d}{dx}(xy^2+x^2) = 0$

In particular $\int \frac{d}{dx}\Psi(x,y(x))\,dx = \Psi(x,y(x)) = K \Rightarrow \boxed{xy^2+x^2=K}$

So in the previous example the fundamental step was to find a function $\Psi(t,y)$

($\Psi(x,y) = xy^2 + x^2$ in the example) such that $\boxed{Eq(t,y,y') = \frac{d}{dt}\Psi(t,y)}$

Definition: if this is the case the ODE is said to be EXACT.

$\Rightarrow$ Solutions to them are (usually) given implicitly by $\Psi(x,y) = k$ $\quad k \in \mathbb{R}$ a constant.

* in the example it was easy to find Ψ, but in general it may be hard to "see" whether or not

such $\Psi(t,y)$ $\exists$. Luckily we have a systematic way to check whether or not an equation is

exact!

THM: Put your (1st-order, scalar) ODE in the form $M(t,y) + N(t,y)\, y' = 0$

(therefore $y' = G(t,y) = -\frac{M(t,y)}{N(t,y)}$). Consider a rectangle of the plane ty, $I \times J$,

where M & N are continuous & $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial t}$ are continuous as well.

Then the equation is exact in $I \times J$ iff. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ in $I \times J$.

Namely $\left\{ \begin{array}{l} \exists \text{ a function } \Psi(t,x) \text{ s.t.} \\ \frac{d}{dt}\Psi(t,y(t)) = Eq(t,y,y') = M(t,y) + N(t,y)y' \end{array} \right\} \iff \{ \partial_y M = \partial_t N \}$

Proof: $\boxed{\Rightarrow}$ $\frac{d}{dt}\Psi(t,y(t)) = \left(\frac{\partial \Psi}{\partial t}\right)(t,y(t)) + \left(\frac{\partial \Psi}{\partial y}\right)(t,y(t)) \cdot y' = M(t,y) + N(t,y)\cdot y'$

RECALL: chain rule : $\frac{d}{dt} f(x(t),y(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$$\Rightarrow \frac{\partial}{\partial t}\psi(t,y) = M(t,y) \quad \Rightarrow \quad \frac{\partial}{\partial y}\frac{\partial}{\partial t}\psi(t,y) = \frac{\partial}{\partial y}M$$

since they are both continuous

$$\frac{\partial}{\partial y}\psi(t,y) = N(t,y) \qquad \frac{\partial}{\partial t}\frac{\partial}{\partial y}\psi(t,y) = \frac{\partial}{\partial t}N$$

you can swap the order of derivation

$$\partial_y\partial_t = \partial_t\partial_y \Rightarrow \frac{\partial}{\partial y}M = \frac{\partial}{\partial t}N$$

RECALL: symmetry of second derivatives

$$\left[\begin{array}{l} \psi: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R} \quad (\Omega \subseteq \mathbb{R}^2 \text{ some open rectangle}) \\ \text{if } \partial_x\psi,\ \partial_y\partial_x\psi,\ \partial_y\psi\ \exists \text{ and they are continuous} \\ \Rightarrow \partial_x\partial_y\psi\ \exists, \text{ it is continuous } \ \&\ \partial_x\partial_y\psi = \partial_y\partial_x\psi. \end{array}\right]$$

example: in the example above $\psi = xy^2 + x^2$, $M = 2x + y^2$ & $N = 2xy$.

$$\frac{\partial}{\partial y}M = 2y \quad ; \quad \frac{\partial}{\partial x}N = 2y \quad ✓.$$

[BACK TO THE PROOF]

$\boxed{\Leftarrow}$ Assume now $\partial_y M = \partial_x N$. The proof gives the construction of the ψ.

① Consider the family of functions of the form: $\int M(x,y)\,\partial x + F(y) =: \psi(x,y)$,

where by $\int M(x,y)\,\partial x$ I mean that I am thinking about y as a function which does

NOT depend on x! That is why I add $F(y)$ & not just a constant.

In particular, for the fund. theorem of calculus $\Rightarrow \frac{\partial}{\partial x}\psi(x,y) = M(x,y)$. Pose $Q(x,y) := \int M(x,y)\,\delta x$

② Compute $\frac{\partial}{\partial y}\psi(x,y)$: it is equal to $\frac{\partial}{\partial y}Q + F'(y)$ and I want it $= N(x,y)$

$$\Rightarrow F'(y) = N(x,y) - \frac{\partial}{\partial y}Q \quad (\text{It must be}).$$

Remark: $N(x,y) - \frac{\partial}{\partial y}Q$ does not have an explicit dependence on x! Indeed

$$\frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}Q = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}Q = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}M = 0$$

Therefore, after integrating w.r.t. y $\Rightarrow$ $\boxed{\varphi(x,y) = \int M(x,y)\,dx + \int \left[N - \frac{\partial}{\partial y} Q\right] dy + K}$

Examples: (1) $(2xy^2 + 2y) + (2x^2y + 3x) \cdot y' = 0$ $\quad M = 2xy^2 + 2y$, $N = 2x^2y + 3x$

(2) $(2xy^2 + 2y) + (2x^2y + 2x) \cdot y' = 0$ $\quad M = 2xy^2 + 2y$, $N = 2x^2y + 2x$

claim: one is exact & the other isn't! $\Rightarrow$ Moral: Exact & non-Exact equation may look very similar to each others $\Rightarrow$ it is very good to have a criterium to distinguish them.

Let's apply the criterium, (1): $\frac{\partial}{\partial y} M \overset{?}{=} \frac{\partial}{\partial x} N$ $\quad$ LHS $= 4xy + 2$, RHS $= 4xy + 3$ $\quad$ X

(2) LHS $= 4xy + 2$ & RHS $= 4xy + 2$ ✓ it is exact: let's find the φ then!

First step: $\int M(x,y)\,dx = \int (2xy^2 + 2y)\,dx = x^2y^2 + 2xy = Q(x,y)$

Second step: $N - \frac{\partial}{\partial y} Q = 2x^2y + 2x - 2x^2y - 2x = 0$

$\Rightarrow \boxed{\varphi(x,y) = x^2y^2 + 2xy + K}$

$\Rightarrow$ ODE: $\frac{d}{dx}\varphi = 0 \Rightarrow \varphi(x,y) = x^2y^2 + 2xy + K = \text{constant}$

$\Rightarrow \boxed{x^2y^2 + 2xy = C}$

Further Example: $\begin{cases} \overbrace{(y\cos(t) + 2te^y)}^{M\,=\,M} + \overbrace{(\sin(t) + t^2e^y - 1)}^{N\,=\,N} \cdot y' = 0 \\ y(0) = 5 \end{cases}$

Question: it is exact? Let's check: $\frac{\partial}{\partial y} M = \cos(t) + 2te^y$; $\frac{\partial}{\partial t} N = \cos(t) + 2te^y$ ✓

Then find Ψ: step 1) $\int M\,dt = \int y\cos(t) + 2te^y\,dt = y\sin(t) + t^2e^y = Q$

step 2) $N - \frac{\partial}{\partial y} Q = \sin(t) + t^2e^y - 1 - [\sin(t) + t^2e^y] = -1 \Rightarrow \int(-1)dy = y$

$\Rightarrow \Psi = y\sin(t) + t^2e^y - y + K \Rightarrow$ solution of the ode: $y\sin(t) + t^2e^y - y = C$

$t=0 \Rightarrow 0 + 0 - y(0) = C \Rightarrow C = -5 \Rightarrow$ $\boxed{y\sin(t) + t^2e^y - y = -5 \text{ is the solution to the IVP}}$

Example

Rmk: Differences between linear and non-linear:

1st linear $\Rightarrow$ they always have an explicit solution

non-linear $\Rightarrow$ they may have an implicit solution

2nd if $y' = a(t)y + b(t)$ has a & b continuous on I $\Rightarrow$ the solution is well defined over the whole I!

for non-linear a priori the interval of definition may be smaller (see example $y' = y^2$)

3rd The non-linear examples that we are seeing today are very special

$\Rightarrow$ most 1st-order, scalar ODE cannot be solved in this way & in general it is not possible to find even implicit solutions (even if we know that they exist by P-L-thm)

However: for many purposes it is enough to provide a QUALITATIVE DISCUSSION

Before moving to that, let's see one more example where we can find a (maybe implicit) solution.

Indeed, sometimes eq.s are not exact but can be made exact!

Examples: $\overbrace{\left(4\frac{x^3}{y^2}+\frac{3}{y}\right)}^{M} + \overbrace{\left(\frac{3x}{y^2}+4y\right)}^{N} y' = 0$; $\frac{\partial}{\partial y}M = -8\frac{x^3}{y^3} - \frac{3}{y^2}$ & $\frac{\partial}{\partial x}N = \frac{3}{y^2}$

so it is not exact. However if we multiply by y^2

$$\Rightarrow \quad y^2M = \tilde{M} = 4x^3 + 3y \quad \Rightarrow \quad \frac{\partial}{\partial y}\tilde{M} = 3$$

$$y^2N = \tilde{N} = 3x + 4y^3 \quad \text{ & } \quad \frac{\partial}{\partial x}\tilde{N} = 3 \quad ✓$$

Method of Integrating factor for non-exact eq.s.

Exactly as we have done for the linear eq. $f' + af + b = 0$, given a non-linear, non exact ODE $\quad M(x,y) + N(x,y)y' = 0 \quad$ we want to find a $\mu(x,y)$ such that

$$\underline{\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0} \quad \text{is EXACT}$$

From the thm we know that is possible iff. $\frac{\partial}{\partial y}\mu M = \frac{\partial}{\partial x}\mu N$, namely:

$$\mu_y M + \mu M_y = \mu_x \cdot N + \mu N_x . \qquad (*)$$

REMARK: finding $\mu(x,y)$ is HARD! In this class μ will be either a function in x only

or in y only. $\boxed{\text{So } \mu = \mu(x) \text{ or } \mu(y)}$.

Example: $(3xy + y^2) + (x^2 + xy)\cdot y' = 0$

$M = 3xy + y^2 \Rightarrow \frac{\partial}{\partial y} M = 3x + 2y$; $N = x^2 + xy \Rightarrow \frac{\partial}{\partial x} N = 2x + y$ ✗

Let's try to find a $\mu(y)$ that makes it Exact: by $(*)$ the sufficient & necessary condition

for $\mu(y)$ to exist is: $\boxed{\mu_y M + \mu M_y = \mu N_x}$

$\Rightarrow \mu_y (3xy + y^2) = \mu(2x + y - 3x - 2y) = \mu(-x - y)$ $\underset{\uparrow}{\text{"}\Rightarrow\text{"}}$ $\frac{\mu'}{\mu} = -\frac{x+y}{3xy + y^2}$

mod some non-vanishing

assumptions

$\Rightarrow \frac{1}{\mu} d\mu = -\underbrace{\frac{x+y}{3xy + y^2}}_{\text{non only in } y \text{ but both } x \,\&\, y} dy$ ↯

↯

Let's try $\mu(x)$: then $(*)$ becomes $\mu M_y = \mu_x \cdot N + \mu N_x$.

So it becomes $\mu(3x + 2y - 2x - y) = \mu'(x)(x^2 + xy) \Rightarrow \mu(\cancel{x+y}) = \mu' \cdot x \,(\cancel{x+y})$

$\Rightarrow \mu = \mu' \cdot x$ if $x \neq 0$ & $\mu \neq 0$ $\Rightarrow \frac{1}{\mu} d\mu = \frac{1}{x} dx \Rightarrow \ln|\mu| = \ln|x|$

$\Rightarrow |\mu| = |x|$ works fine. Assume $x \geq 0 \Rightarrow |\mu| = x$ both $\mu = x$ & $\mu = -x$ work fine.

Pose $\mu = x \Rightarrow x \cdot$ ODE is exact, more precisely $\Rightarrow \underbrace{(3x^2y + y^2x)}_{\tilde{M}} + \underbrace{(x^3 + x^2y)}_{\tilde{N}} \cdot y' = 0$ is exact.

Let's solve the new ODE

$$\tilde{\varphi}(x,y) = \int \tilde{M}(x,y)\,dx + \int \left[\tilde{N} - \frac{\partial}{\partial y}\tilde{Q}\right] dy + K$$

$\int 3x^2y + y^2x \, dx = x^3y + \frac{y^2x^2}{2} = \tilde{Q}$; $x^3 + x^2y - x^3 - yx^2 = 0 \Rightarrow \tilde{\varphi} = x^3y + \frac{x^2y^2}{2} + K$

$\frac{d}{dx}\tilde{\varphi} = 0 \Rightarrow x^3y + \frac{x^2y^2}{2} = C$ ← away from $x=0$.

SECOND PART : Qualitative discussion §2.5

We have seen already the first step

1st) classification of the ODE (7/3)

2nd) Equilibrium points & approximated Integral curves

Setting: for now we consider diff. equations $y' = G(y)$ where $G : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the assumptions of P-L Thm (so we have $\exists$ & ! of solution for IVP) & $y : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is the variable function.

DEF: A point $\alpha \in \mathbb{R}$ is an equilibrium point for the ODE $y' = G(y)$ if $G(\alpha) = 0$.

↳ For each eq. point $\Rightarrow$ it corresponds a constant solution $y(t) \equiv \alpha$ & it is called an equilibrium solution.

Rmk: in particular the IVP $\begin{cases} y' = G(y) \\ y(t_0) = \alpha \end{cases}$ has $y \equiv \alpha$ as unique solution

Example: $y' = ay + b$ (from last time). $\Rightarrow$ $\alpha = -\frac{b}{a}$ is an equilibrium point.
$\boxed{\& \ a \neq 0}$

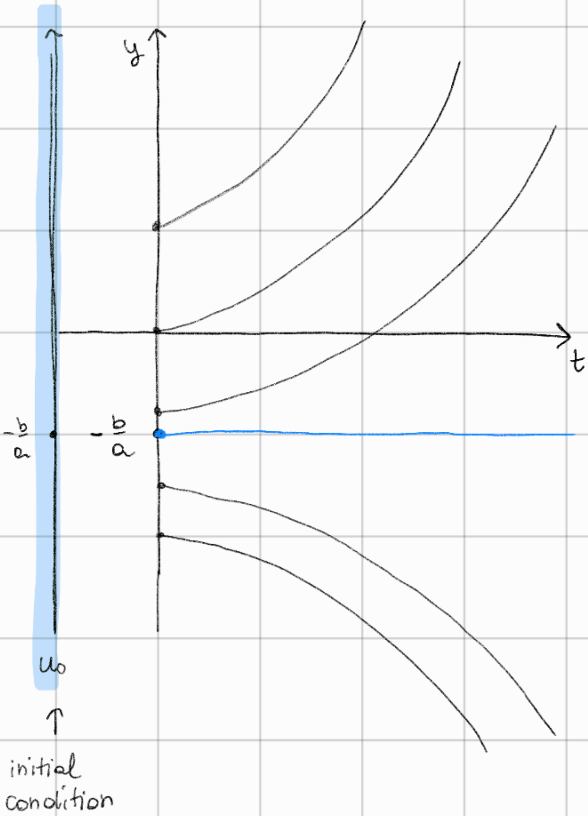
Notice that given IVP $\begin{cases} y' = ay + b \\ y(0) = u_0 \neq -\frac{b}{a} \end{cases}$ $\Rightarrow$ $y(t) = Ce^{at} - \frac{b}{a}$ is the unique solution, where

C is equal to: $y(0) = u_0 = C - \frac{b}{a}$ $\Rightarrow$ $C = u_0 + \frac{b}{a}$

$\Rightarrow y(t) = \left(u_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$

Let's display on a plane ty the different solutions that we get for different initial value.

(for simplicity, let's assume $t \geq 0$, a & $b > 0$ - the other cases are analogous



⊙ in particular $-\frac{b}{a} < 0$

⊙ if $u_0 > -\frac{b}{a} \Rightarrow \left(u_0 + \frac{b}{a}\right) > 0$

$\Rightarrow y' = a\left(u_0 + \frac{b}{a}\right)e^{at} > 0 \Rightarrow y(t)$ increasing function.

$y(t) \equiv -\frac{b}{a}$ constant / equilibrium solution.

⊙ $u_0 < -\frac{b}{a} \Rightarrow u_0 + \frac{b}{a} < 0 \Rightarrow y' = a\left(u_0 + \frac{b}{a}\right)e^{at} < 0$

$\Rightarrow y(t)$ always decreasing

Rmks: 1) By P-L Thm the graphs of the solutions for different u_0 they do not meet by uniqueness!

2) the equilibrium solution splits the halfplane in two: none of the solutions above can cross it, neither the below solutions can.

3) non-constant solutions pull away from the equilibrium one.

↑

These features are not casual & not limited to linear ODEs!

In general: given an autonomous, 1st order, scalar ODE $y' = G(y)$, $y : I \subseteq \mathbb{R} \to \mathbb{R}$ as above:

2nd step) eq. points.

3rd step) display the equilibrium points along a vertical line, which represents the possible initial conditions u_0, marking them:

For instance if $G(x) = 0$ iff. $x = \alpha$ or β $\Longrightarrow$

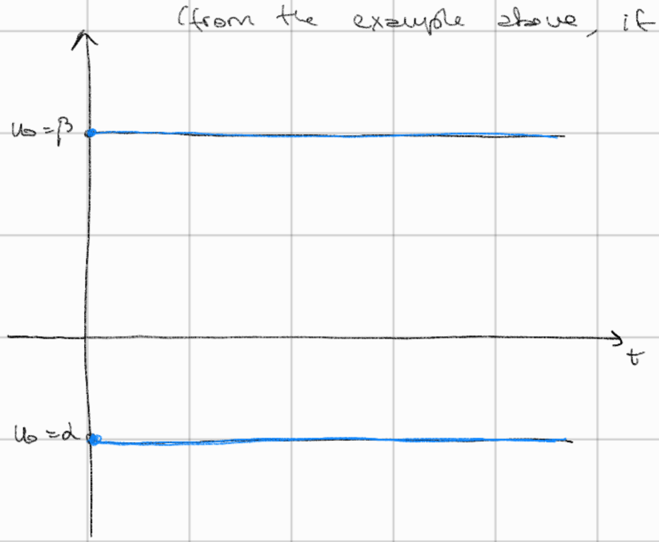
& $\beta > \alpha$

(we exclude the values $(0, u_0)$ for which G or $\frac{\partial G}{\partial x}$ are not continuous)

4th step) draw the ty-halfplane: we usually are interested in the case where $t_0 = 0$ & $t \geq 0$ (the other situations are analogous)

For each eq. point $\Longrightarrow$ you have a corresponding eq. solutions.

(from the example above, if $\alpha < 0 < \beta$ ↙)



Important Rule:

By uniqueness, the eq. solutions divide the half-plane into uncrossable regions.

For instance,

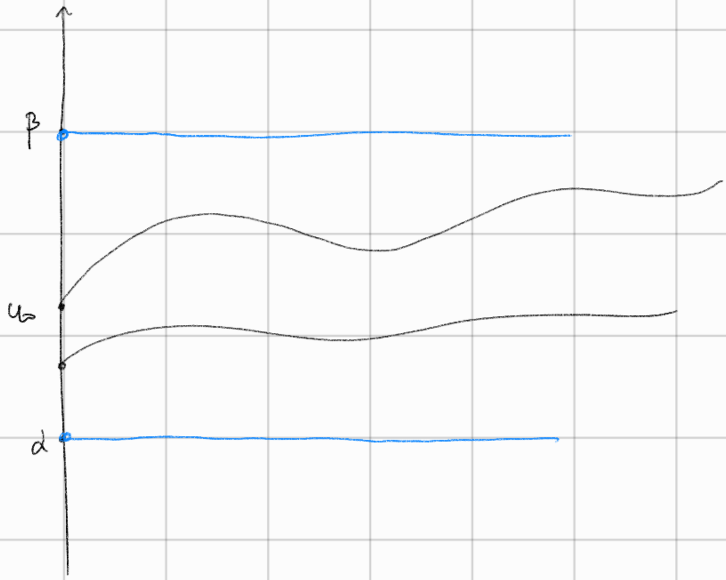
* this means that if a solution is IN

between (α, β)

$\Rightarrow$ it will stay between α & β

(it cannot meet the solution $y(t) \equiv \beta$ nor

$y(t) \equiv \alpha$)



* Analogously if it starts below α, it stays below (same for above β)

Rmk: if the hp of PL thm are satisfied for all $(0, u_0)$ with $u_0 \in (\alpha, \beta)$

then $\Rightarrow T_+ = +\infty$! They exist $\forall\, t \geq 0$!

Indeed PL Thm 2 tells you that T_+ $\exists$ and it is $T_+ = +\infty$ or $T_+ < +\infty$ & $|y(t)| \xrightarrow{t \to T_+} +\infty$

Since the second situation is not possible ($\forall t \; \alpha < y(t) < \beta$) $\Rightarrow$ the first one is the right one

$\Rightarrow$ we have just deduced a very important information (namely, $\exists \; \forall t \geq 0$)

without even knowing G, but only knowing that it has 2 eq. points (α, β)

& $u_0 \in (\alpha, \beta)$.

5th step) fill in the regions of the ty-plane with the right "growing behavior" of $y(t)$.

Namely: (*) whether or not is increasing / decreasing

(*) whether or not $\exists \lim_{t \to T_+} y(t)$ & what is it if $\exists$.

Increasing / decreasing behavior $\longleftrightarrow$ first order derivative y'

Namely: if $y' > 0 \Rightarrow y(t)$ is increasing & if $y' < 0 \Rightarrow y(t)$ is decreasing.

Notice! $y' = G(y) \Rightarrow$ enough to study the positivity of $G(y)$

In order to illustrate this, let's do an example:

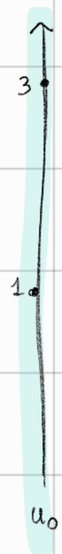
Example: consider for an unknown function $y: I \subseteq \mathbb{R} \to \mathbb{R}$ the ODE $y' = (1-y)(3-y)$

(1) Classification: autonomous, 1st order, scalar, non linear. Notice: it is separable but let's not solve it

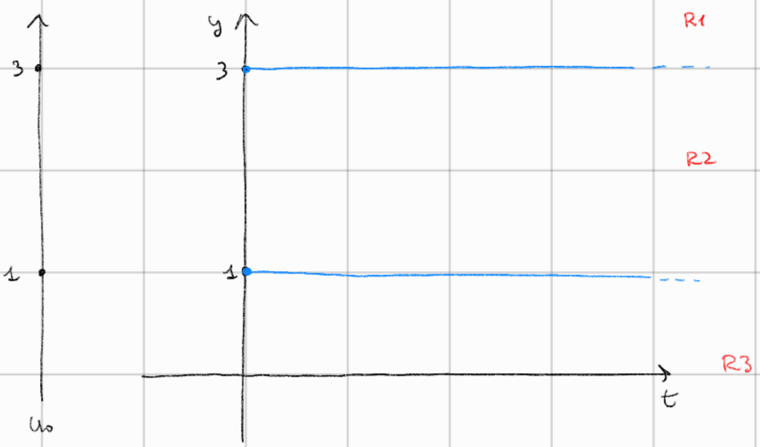
(2) Eq. points: $G(x) = (1-x)(3-x) \Rightarrow x_1 = 1$ & $x_2 = 3$.

(3) Check hp's for P-L Thm: $(1-x)(3-x)$ is C^∞

($\exists$ derivative of any order) $\Rightarrow$ any u_0 is fine.



(4) Regions in ty-plane.



(5) Behavior in different regions: $G(x) = (1-x)(3-x)$. we know already that at $x=1$ & 3 it vanishes.

x		1		3	
$1-x$	+		−		−
$3-x$	+		+		−
$G(x)$	+		−		+

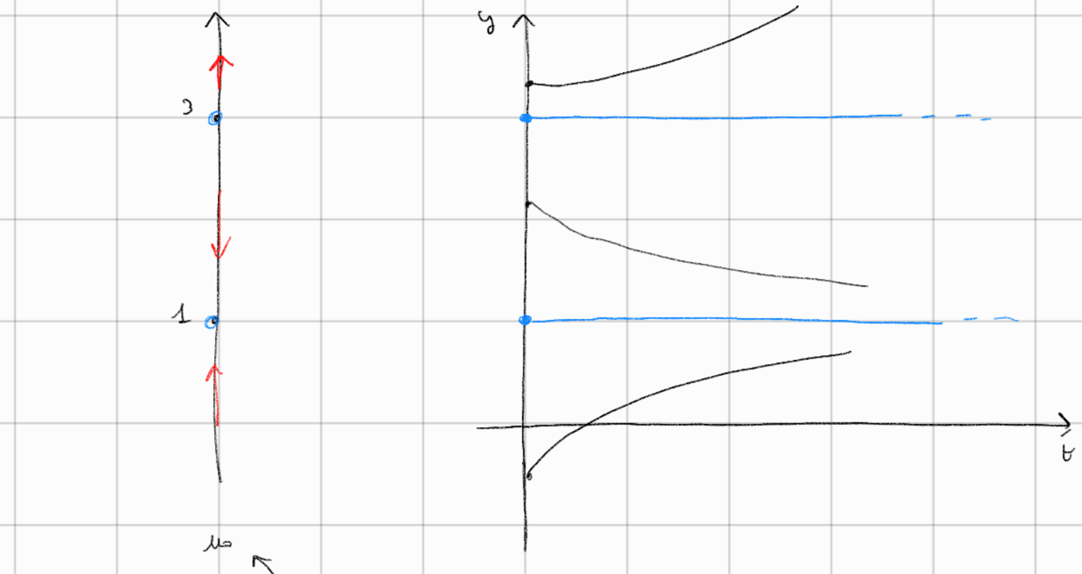
- at $y(t) = 1$ & 3 $\Rightarrow$ $G(y) = 0$
- if $y(t) < 1$ or > 3 $\Rightarrow$ $G(y) > 0$ $\Rightarrow$ y is increasing
- if $1 < y(t) < 3$ $\Rightarrow$ $G(y) < 0$ $\Rightarrow$ y is decreasing.

Notice: $u_0 > 3 \Rightarrow y(t) > 3 \quad \forall t \geq 0$ | $u_0 > 3 \Rightarrow y(t)$ is increasing

$1 < u_0 < 3 \Rightarrow 1 < y(t) < 3 \quad \forall t \geq 0$ | $u_0 \in (1,3) \Rightarrow y(t)$ is decreasing

$u_0 < 1 \Rightarrow y(t) < 1 \quad \forall t \geq 0$ | $u_0 < 1 \Rightarrow y(t)$ is increasing.

$\Rightarrow$ Moral: by uniqueness again: enough to study $G(u_0)$ in order to get the info about $G(y)$!



DEFINITION: the vertical line about u_0 with eq. points + "directed segments" (the red arrows) is called the phase line for the ODE

Notation:

upward direction $\iff G(u_0) > 0 \implies y(t)$ increasing

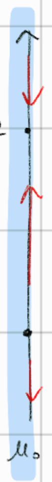
downward direction $\iff G(u_0) < 0 \implies y(t)$ decreasing.

Before moving to the question about the behavior of $y(t)$ for $t \to T_+$, let's do another example.

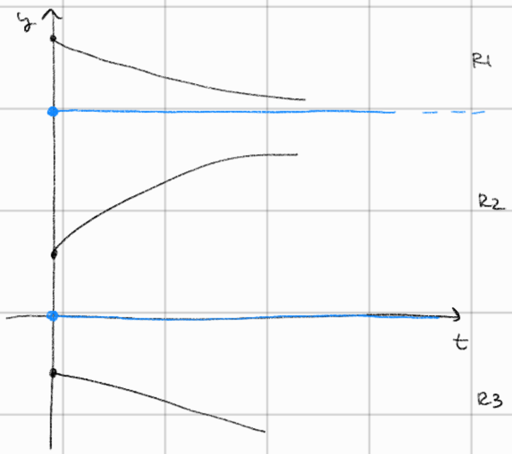
Example. $y' = \left(1 - \frac{y}{2}\right) y$ (1) classification: autonomous, 1st order, scalar, non linear

(2) eq. points: $y = 2$ & 0

(3) Lip thm ✓ (everywhere) $\implies$



$\implies$ (4)



(5) Behavior: $G(u_0) = \left(1 - \frac{u_0}{2}\right) u_0$

		0		2	
u_0	$-$		$+$		$+$
$1 - \frac{u_0}{2}$	$+$		$+$		$-$
$G(u_0)$	$-$		$+$		$-$
	$\Downarrow$ decreasing		$\Downarrow$ increasing		$\Downarrow$ decreasing

LIMIT & T_+ ⊛ We noticed already that if the region is bounded above & below

$\implies T_+ = +\infty$;

(*) if you are considering a region $R = \{u_0 < \alpha\}$ & the $G(u_0) > 0$ for $u_0 < \alpha$

$\Rightarrow$ for the same reason $\Rightarrow$ $T_+ = +\infty$. (Same if $R = \{u_0 > \beta\}$ & $G(u_0) < 0$)

Moreover since $G(u_0) \neq 0$ in that region & it is continuous, in all above cases

$$\lim_{t \to +\infty} y(t) = \text{equilibrium solution.}$$

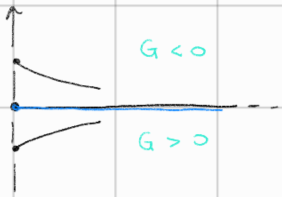
$\Rightarrow$ we are left with

$$\boxed{\begin{array}{l} \square \; R = \{u_0 < \alpha\} \quad \& \quad G(u_0) < 0 \\ \square \; R = \{u_0 > \beta\} \quad \& \quad G(u_0) > 0 \end{array}}$$

$\leftarrow$ if we have time we'll see two theorems that can help us but for now let's just say the following:

classification of stability for eq. solutions

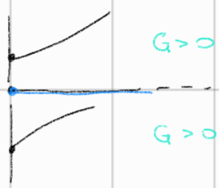
(A)



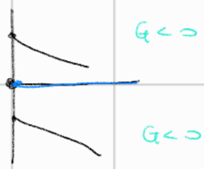
or equiv.

on the phase line $\Rightarrow$ we say that α is **asymptotically stable**

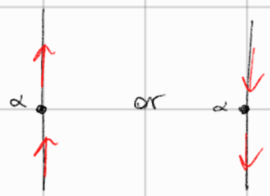
(B)



or

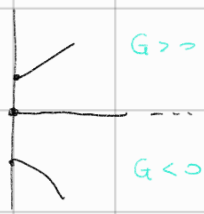


Equiv.



$\Rightarrow$ we say α is **semistable**

(C)

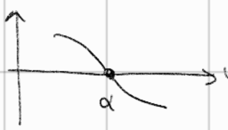


Equiv.

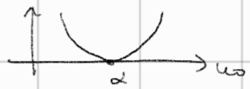
$\Rightarrow$ α is **unstable**

⇒ Rule: we can decide the stability of an equilibrium point even without the sign-discussion $G<0, =0, >0$! Indeed since $\frac{\partial G}{\partial y}$ $\exists$ & it is continuous we can use the sign of the derivative in order to conclude the stability!

Indeed: CASE (A) – Asymp. stability. G goes from >0 to $=0$ to <0

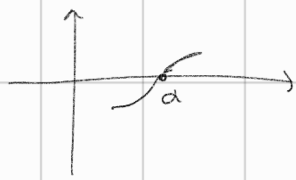
$\Rightarrow$ 

$\Rightarrow \boxed{G(\alpha) \le 0}$

CASE (B) – Semistable: $G>0 \Rightarrow G=0 \Rightarrow G>0 \Rightarrow$ 

$G<0 \Rightarrow G=0 \Rightarrow G<0 \Rightarrow$ 

$\Rightarrow \boxed{G(\alpha) = 0}$

CASE (C) – Unstable: $G<0 \Rightarrow G=0 \Rightarrow G>0 \Rightarrow$ 

$\Rightarrow \boxed{G(\alpha) \ge 0}$

Therefore: $G(\alpha) > 0 \Rightarrow$ unstable

$G(\alpha) < 0 \Rightarrow$ asymp. stable

$G(\alpha) = 0 \Rightarrow$ unconclusive

Example $G(x) = x^3$ 

$G(0) = 0$ but unstable