Today's content: §3.1 - §3.2 - §3.3 - §3.4. (we do not cover all of it, but all of it can be found in those sections)

- $\exists$/! Thm for 2nd-order linear scalar ODE
- Constant coeff.'s + homogeneous case
- Superposition / Fundamental set of solutions.
- Method of reduction of order

SECOND PART

$\Downarrow$

(*) RMKS on $\exists$ & ! Thm & Maximal I

(see pdfs uploaded)

Last week: 1st order + scalar:
- linear $\longrightarrow$ solution
- non-linear
  - $\to$ separable $\longrightarrow$ solution (maybe implicit)
  - $\to$ exact $\longrightarrow$ solution (" " " ")
  - $\to$ non-exact but $\exists$ $\mu(x)$ or $\mu(y)$ $\longrightarrow$ (implicit) solution.
  - $\to$ Qualitative discussion - we have done it for autonomous $y' = G(y)$.

THIS WEEK: 2nd order + scalar + linear.

Rmk (*) any 2nd-order - scalar - linear ODE has the following form: $c_2(t)y'' + c_1(t)y' + c_0(t)y = g(t)$

where $g : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ a given function. The $c_i(t) : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ are the coefficients.

(*) we always want to exclude those times $t \in \mathbb{R}$: $c_2(t) = 0$. $\Rightarrow$ assume $c_2(t) \neq 0$ $\forall t \in I$.

$\Rightarrow$ you can put the ODE in a standard form (coeff. of $y'' = 1$)

$$\Rightarrow \boxed{y'' + p(t)y'(t) + q(t)y(t) = g(t)}$$

(*) recall: • $p(t)$ & $q(t)$ both constant $\Rightarrow$ the ODE has constant coeff.'s : ex. $y'' + y' - 6y = \sin(t)$

• $g(t) \equiv 0$ $\Rightarrow$ the ODE is homogeneous. ex. $y'' + ty = 0$

(*) since order = 2, the IVP needs 2 values:

$$\begin{cases} \text{ODE} \\ y(t_0) = u_0 \\ y'(t_0) = u_1 \end{cases}$$

Also in this case we have a local existence & uniqueness thm.

THEOREM: assume p, q, g continuous on $I = (t_1, t_2)$. Then the I.V.P. (we don't prove it)

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = u_0 \\ y'(t_0) = u_1 \end{cases}$$

has a unique solution $\forall\ t_0 \in I$

* Existence
* Uniqueness
* maximal interval $\subseteq I$ is I.

Example: $(t^2 - 3t)y'' + t \cdot y' - (t+3)y = \sin(t)$

put it in standard form: $y'' + \frac{1}{t-3} - \frac{t+3}{t(t-3)}y = \frac{\sin(t)}{t(t-3)}$. Then $p(t) = \frac{1}{t-3}$, $q(t) = -\frac{t+3}{t(t-3)}$

and $g(t) = \frac{\sin(t)}{t(t-3)}$.

$p(t)$ is continuous everywhere except $t=3$

$q(t)$ " " " " " " " $t=0,3$

$g(t)$ " " " " " " " $t=0,3$

$\Rightarrow$ in any case we need to avoid $t=0,3$.

Therefore, if $t_0 = 1$ (for instance), for any values $u_0, u_1 \in \mathbb{R}$ we get a unique solution defined between $(0,3)$.

Now that we have this thm, let's look for the actual solution.

First case: the (linear) ODE is homogeneous & with constant coeff.

$\Rightarrow \quad y'' + py' + q = 0$ with p & $q \in \mathbb{R}$ - constant.

Ansatz method: borrowed from German word "ansetzen" = "to put to, fix, set, estimate".

It refers to the method of "assuming that the solutions have a specific form".

As done in PSET 1, let's look for solutions of the form $y(t) = e^{rt}$. $r \in \mathbb{R}$.

$\Rightarrow \quad r^2 e^{rt} + rpe^{rt} + qe^{rt} = 0 \quad \Longrightarrow$ (remove e^{rt}, since it is nowhere vanishing)

$\Rightarrow \quad r^2 + rp + q = 0 \quad \leftarrow$ necessary condition on r.

DEF: the polynomial $r^2 + r \cdot p + q = 0$ is called the characteristic polynomial of the ODE

$y'' + py' + y = 0$.

Three scenarios: 1) $r^2 + p \cdot r + q = 0$ has two distinct real solutions

2) " " " " has one real root $(r-a)^2 = r^2 + pr + q$

3) " " " " has two (distinct) complex (not real) roots.

(rmk: in case 3), the roots are conjugated!

Indeed, $r_{1,2} = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ & posing $\Delta = p^2 - 4q$ (the discriminant)

$p^2 - 4q > 0$	$p^2 - 4q = 0$	$p^2 - 4q < 0$
$\Downarrow$	$\Downarrow$	$\Downarrow$
(1)	(2)	(3)

case (1) - distinct (real) roots: call $r_1 = \frac{-p-\sqrt{p^2-4q}}{2}$ & $r_2 = \frac{-p+\sqrt{p^2-4q}}{2}$; then $e^{r_1 t}$ & $e^{r_2 t}$ are

solutions to $y''+py'+qy=0$.

Rmk: $\forall\, C_1, C_2 \in \mathbb{R}$ constant $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is a solution as well! This is a

result true for all HOMOGENEOUS - linear - ODES & it goes under the name of:

(linear)
SUPERPOSITION (Thm): Consider a linear homogeneous ode: $\sum_{j=0}^{n} c_j(t) y^{(j)} = 0$ $(g(t)=0)$.

if $\varphi(t)$ & $\psi(t)$ are solutions $\Rightarrow$ $C_1\varphi + C_2\psi$ is a solution as well $(\forall\, C_1, C_2 \in \mathbb{R})$.

proof. • derivatives are linear: $\frac{d}{dt}(af+bg) = a\frac{d}{dt}f + b\frac{d}{dt}g$

$\Rightarrow (C_1\varphi + C_2\psi)^{(j)} = C_1\varphi^{(j)} + C_2\psi^{(j)}$

• φ & ψ are solutions mean $\sum_{j=0}^{n} c_j(t)\varphi^{(j)} \equiv 0$ & $\sum_{j=0}^{n} c_j(t)\psi^{(j)} \equiv 0$

This implies: $\underbrace{C_1 \sum_{j=0}^{n} c_j(t)\varphi^{(j)} + C_2 \sum_{j=0}^{n} c_j(t)\psi^{(j)}}_{\parallel} = 0$ as well

$$\sum_{j=0}^{n} c_j(t) \underbrace{[C_1\varphi^{(j)} + C_2\psi^{(j)}]}_{= (C_1\varphi + C_2\psi)^{(j)}} = 0 \quad ✓.$$

Corollary: $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is a solution $\forall\, C_1, C_2 \in \mathbb{R}$.

Question: how can we make sure that the converse is true? Namely, can we say that any solution to the ODE $y''+py'+q=0$ has the form $C_1e^{r_1t}+C_2e^{r_2t}$ for some C_1, C_2?

YES! We use the Wronskian.

DEF: The Wronskian of two functions φ and ψ is equal to $\det\begin{pmatrix}\varphi(t) & \psi(t)\\ \varphi'(t) & \psi'(t)\end{pmatrix} = \varphi\psi' - \psi\varphi'$

It is denoted by $W(\varphi,\psi)$

THM: Consider the ODE $y''+p(t)y'+q(t)=0$. Let φ and ψ two solutions of the ODE.

If $W(\varphi,\psi)\neq 0 \Rightarrow$ any solution to the ODE has the form $C_1\varphi + C_2\psi$ for some $C_1, C_2 \in \mathbb{R}$.

DEF: if this is the case $\{\varphi,\psi\}$ is called a fundamental set of solutions.

Proof: let t_0 be a time s.t. $W(\varphi,\psi)(t_0)\neq 0$ & consider the IVP $\begin{cases} y''+p(t)y'+q(t)y=0 \\ y(t_0)=u_0, \; y'(t_0)=u_1\end{cases}$

let's consider the system: $\begin{cases} C_1\varphi(t_0)+C_2\psi(t_0)=u_0 \\ C_1\varphi'(t_0)+C_2\psi'(t_0)=u_1\end{cases}$. It can be rewritten as: $\begin{pmatrix}\varphi(t_0) & \psi(t_0)\\ \varphi'(t_0) & \psi'(t_0)\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}=\begin{pmatrix}u_0\\ u_1\end{pmatrix}$

Since $\det\begin{pmatrix}\varphi(t_0) & \psi(t_0)\\ \varphi'(t_0) & \psi'(t_0)\end{pmatrix}\neq 0$, it is invertible $\Rightarrow$ the system has unique solution

$$\begin{pmatrix}C_1\\ C_2\end{pmatrix}=\begin{pmatrix}\varphi(t_0) & \psi(t_0)\\ \varphi'(t_0) & \psi'(t_0)\end{pmatrix}^{-1}\begin{pmatrix}u_0\\ u_1\end{pmatrix}$$

In particular, $C_1\varphi + C_2\psi$ is a real-function $\ni$ $y'' + p(t)y' + q(t)y = 0$ (linear superposition)

& $y(t_0) = u_0$ & $y'(t_0) = u_1$ $\Rightarrow$ by uniqueness of solution $C_1\varphi + C_2\psi$ is the unique solution to

the above IVP.

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Rmk - linear algebra interpretation: the thm says that the set of solutions to $y'' + p(t)y' + q(t)y = 0$

is a (real) VECTOR SPACE of dim 2! And that $\{\varphi, \psi\}$ is a basis iff. $W(\varphi, \psi) \neq 0$.

rmk #2: the basis is NOT unique: example $\{1, t\}$ & $\{1+t, t\}$ span the same set of

real-functions.

Corollary: $\{e^{r_1 t}, e^{r_2 t}\}$ is a fundamental set for $y'' + py' + qy = 0$ iff $W(e^{r_1 t}, e^{r_2 t}) \neq 0$

Namely if $\det\begin{pmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t}\end{pmatrix} = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0$ iff $\boxed{r_2 \neq r_1}$.

Example: ⊛ $y'' - y = 0$ $\rightarrow$ the characteristic polynomial is $r^2 - 1 = 0$ $\rightarrow$ solutions are $r = \pm 1$

$\Rightarrow$ the general solution is $\boxed{C_1 e^{t} + C_2 e^{-t}}$

⊛ $y'' - y' = 0$ $\rightarrow$ $r^2 - r = 0$ $\rightarrow$ $r = 0, 1$ $\Rightarrow$ the general solution is $\boxed{C_1 + C_2 e^{t}}$

Case (3) - distinct complex root: $e^{r_1 t}$ & $e^{\overline{r_1} t}$ are complex-valued solutions to $y''+p(t)y'+q(t)y=0$

But we want real-valued ones:

Trick: Euler's Formula: $e^{it} = \cos(t) + i\sin(t)$. Therefore.

$$\underbrace{e^{r_1 t}} = e^{(\alpha+i\beta)t} = e^{\alpha t}\cdot e^{i\beta t} = \underbrace{e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)}$$

$$\underbrace{e^{r_2 t}} = e^{(\alpha-i\beta)t} = e^{\alpha t}\cdot e^{-i\beta t} = \underbrace{e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)}$$

. Define

$$\varphi(t) = \frac{e^{r_1 t}+e^{r_2 t}}{2} = e^{\alpha t}\cos(\beta t) \quad ; \quad \psi(t) = \frac{e^{r_1 t}-e^{r_2 t}}{2i} = e^{\alpha t}\sin(\beta t)$$

$\Leftarrow$ these two are real-valued

Moreover $W(\varphi,\psi) = \det\begin{bmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) & \alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t) \end{bmatrix} =$

$$= e^{2\alpha t}\left[\cancel{\alpha \sin\cos} + \beta\cos^2 - \cancel{\alpha\sin\cos} + \beta\sin^2\right] = \beta e^{2\alpha t} \neq 0 \, ! \Rightarrow \{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$$

is a fundamental set of solutions.

Example: $y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow \alpha = 0$ & $\beta = 3 \Rightarrow$

A fundamental set of solutions is given by $\{\cos(3t), \sin(3t)\}$

$\Rightarrow$ the general solution is given by $C_1\cos(3t) + C_2\sin(3t)$.

Ex2: $\begin{cases} y'' - 2y' + 5y = 0 \\ y(\frac{\pi}{2}) = 0, \; y'(\frac{\pi}{2}) = 2 \end{cases}$

first char. polynomial. $r^2 - 2r + 5 = 0$.

Solutions are: $r_{1,2} = 1 \pm \sqrt{1-5} = 1 \pm 2i$. $\Rightarrow \alpha = 1$ & $\beta = 2$.

$\Rightarrow$ The general solution is $C_1 e^{t}\cos(2t) + C_2 e^{t}\sin(2t)$.

Let's find the right coeff's: $y\left(\frac{\pi}{2}\right) = C_1 \cdot e^{\pi/2} \cos(\pi) + C_2 e^{\pi/2} \sin(\pi) = -C_1 e^{\pi/2} = 0 \Rightarrow C_1 = 0$

$y'(t) = C_2 e^t \sin(2t) + 2 C_2 e^t \cos(2t) \Rightarrow y'\left(\frac{\pi}{2}\right) = 2 = C_2 \cdot e^{\pi/2} \cdot \sin(\pi) + 2 C_2 e^{\pi/2} \cos(\pi) =$

$= -2 C_2 e^{\pi/2} \Rightarrow C_2 = -e^{-\pi/2}$

$\Rightarrow$ the (unique) solution is $y(t) = -e^{-\pi/2} e^t \sin(2t)$.

Case (2): the char. polynomial is $(r-r_0)^2$ $(r_1 = r_2 = r_0)$ $\Rightarrow$ so, we know $e^{r_0 t}$ is a solution but we still need another one.

Question: More generally, assume that you have found a solution $\varphi(t)$ to the ODE $y'' + p(t)y' + q(t)y = 0$

How to find another one?

Answer: look for $\psi(t) = v(t)\varphi(t)$ (method of reduction of order) or Abel's thm.

Particular case: $y'' + py' + qy = 0$ with $p^2 - 4q = 0$. $\Rightarrow r = -\frac{p}{2}$ $\Rightarrow e^{-\frac{pt}{2}} = \varphi(t)$ is one solution.

$\psi(t) = v(t) \cdot e^{-pt/2} \Rightarrow \psi'(t) = v'(t)e^{-pt/2} - \frac{p}{2} v(t) e^{-pt/2} \Rightarrow \psi''(t) = v'' e^{-\frac{pt}{2}} - \frac{p}{2} v'(t) e^{-\frac{pt}{2}} - \frac{p}{2} v'(t) e^{-pt/2} + \frac{p^2}{4} v(t) e^{-pt/2}$

$= v'' e^{-\frac{pt}{2}} - p v' e^{-\frac{pt}{2}} + \frac{p^2}{4} v \, e^{-pt/2}$

$\Rightarrow \left(v'' - pv' + \frac{p^2}{4} v\right) e^{-\frac{pt}{2}} + p\left(v' - \frac{p}{2} v\right) e^{-\frac{pt}{2}} + q v e^{-\frac{pt}{2}} = 0$

$\left(v'' \cancel{- pv'} + \frac{p^2}{4} v \cancel{+ pv'} - \frac{p^2}{2} v\right) e^{-\frac{pt}{2}} + q v e^{-\frac{pt}{2}} = 0 \Rightarrow v'' e^{-pt/2} + \left(-\frac{p^2}{4} + q\right) v e^{-pt/2} = 0$

Now, recall that $p^2 - 4q = 0$ $\Rightarrow$ the above expression becomes $v'' \cdot e^{-pt/2} = 0$ $\Rightarrow$ $v'' = 0$

$\Rightarrow v' = C$ $\Rightarrow$ $\boxed{v = Ct + d}$

$\Rightarrow \{ e^{-\frac{p}{2}t}, (Ct+d)e^{-\frac{p}{2}t} \}$ is a set of solutions. Let's check when it is a fundamental set.

$$W(e^{-\frac{p}{2}t}, (Ct+d)e^{-\frac{p}{2}t}) = \det \begin{pmatrix} e^{-\frac{pt}{2}} & (Ct+d)e^{-\frac{pt}{2}} \\ -\frac{p}{2}e^{-\frac{pt}{2}} & Ce^{-\frac{pt}{2}} - \frac{p}{2}(ct+d)e^{-\frac{pt}{2}} \end{pmatrix} =$$

$$= Ce^{-pt} - \frac{p}{2}(ct+d)e^{-pt} + \frac{p}{2}e^{-pt}(Ct+d) = Ce^{-pt}$$. it is $\neq 0$ iff. $C \neq 0$.

$\Rightarrow$ $\boxed{\{ e^{-\frac{p}{2}t}, te^{-\frac{p}{2}t} \} \text{ is a fundamental set}}$

Example $\circledast$ $y'' + 4y' + 4y = 0$: the characteristic polynomial: $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r_0 = -2$

$\Rightarrow$ a fundamental set of solution is: $\{ e^{-2t}, te^{-2t} \}$.

$\circledast$ $\begin{cases} 4y'' - 4y' + y = 0 & \Rightarrow y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2, \ y'(0) = 2 \end{cases}$ $r_{1,2} = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1}{2}$ $\Rightarrow \{ e^{t/2}, te^{t/2} \}$ is a fundamental set of solutions.

$\Rightarrow$ the general solution has the form: $C_1 e^{t/2} + C_2 te^{t/2} = y(t)$

$y(0) = C_1 + 0 = 2 \Rightarrow C_1 = 2$; $y'(t) = \frac{d}{dt}(2e^{t/2} + C_2 te^{t/2}) = e^{t/2} + C_2 e^{t/2} + \frac{C_2}{2} te^{t/2}$

$y'(0) = (1 + C_2) = 2 \Rightarrow C_2 = 1$ $\Rightarrow$ the (unique) solution is $2e^{t/2} + te^{t/2}$

Summary

$\boxed{y'' + py' + qy = 0} \longrightarrow \boxed{p^2 - 4q > 0} \Rightarrow$ 2 real roots : $r_{1,2} = \frac{-p \pm \sqrt{p^2-4q}}{2}$

general solution $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$\boxed{p^2 - 4q = 0} \Rightarrow$ 1 root $\Rightarrow r = -\frac{p}{2} \Rightarrow y(t) = C_1 e^{rt} + C_2 t \cdot e^{rt}$

$\boxed{p^2 - 4q < 0} \Rightarrow$ 2 complex -conjugate-roots $\Rightarrow r_{1,2} = \alpha \pm i\beta$

$\Rightarrow y(t) = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$

Example: $\boxed{y'' - 10y' + 21y = 0 \quad , y(0) = 5 \;\&\; y'(0) = 3}$ - Find Solution & study $\lim_{t \to +\infty} y(t)$.

1st) char. polynomial: $r^2 - 10r + 21 = 0 \Rightarrow (r-3)(r-7) = 0 \Rightarrow r_1 = 3 \;\&\; r_2 = 7$.

2nd) general solution : $C_1 e^{3t} + C_2 e^{7t}$

3rd) Use initial Conditions: $t=0 \Rightarrow C_1 + C_2 = 5 \Rightarrow C_1 = 5 - C_2 \qquad C_1 = 8$

$\Rightarrow 3C_1 + 7C_2 = 3 \qquad \underbrace{15 - 3C_2 + 7C_2}_{4C_2} = 3 \Rightarrow 4C_2 = -12 \Rightarrow C_2 = -3$

4th) the solution is: $8e^{3t} - 3e^{7t}$.

Rule) it is well defined everywhere $\Rightarrow$ interval of $\exists \Rightarrow (-\infty, +\infty)$

Moreover $y'(t) = 24e^{3t} - 21e^{7t}$ which is definetively negative for $t \to \infty$. $\Rightarrow$ the solution is always decreasing & $\lim_{t \to +\infty} y(t) = -\infty$.

REMARK on $\exists$ & ! Theorems for 1st order ODE.

From Lecture 1:

THEOREM: Picard - Lindelöf thm: "short-time existence & uniqueness"

Let $f' = G(t, f)$ be a 1st-order scalar ODE s.t.

- $\underline{G(t,x)}$ & $\underline{\frac{\partial}{\partial x} G(t,x)}$ are continuous in some rectangle $I \times J = (\alpha, \beta) \times (\gamma, \delta)$.

look: both the intervals are open!

Then $\forall (t_0, u_0) \in I \times J$ $\exists h : (t_0 - h, t_0 + h) \subseteq I$ & $\exists!$ solution

$f : (t_0 - h, t_0 + h) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ of the IVP $\begin{cases} f' = G(t, f) \\ f(t_0) = u_0 \end{cases}$

How can I use it? Or better: what kind of questions should I expect about it?

Example: you are given a specific IVP: $\boxed{\text{ODE} + \text{Initial condition}}$ For example. $\begin{cases} y' = \dfrac{1+t^2}{3y - y^2} \\ y(0) = 1 \end{cases}$

Question: clarify the ODE + find THE solution to the IVP - after having checked the hp's of P-L Thm.

Clarification: non-autón. + 1st order + scalar + non-linear.

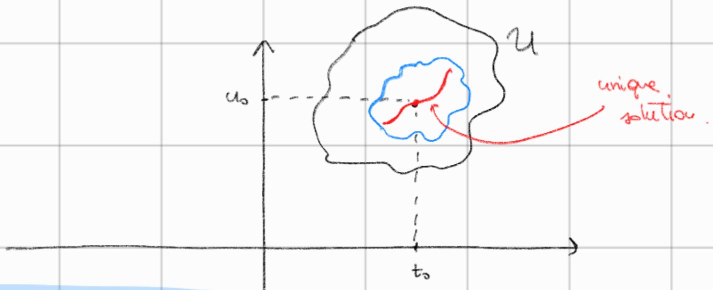
Find THE solution:

Let's understand better what the thm is telling us.

The P-L thm says that given $(*)\begin{cases} f' = G(t,f) \\ f(t_0) = u_0 \end{cases}$ as soon as you have an open $\mathcal{U} \subseteq \mathbb{R}^2$

s.t. $(t_0, u_0) \in \mathcal{U}$ & $\{G(t,x), \frac{\partial}{\partial x} G\}$ are both continuous in $\mathcal{U}$.

$\exists\ \mathcal{U}' \subseteq \mathcal{U}$ open : $\exists!$ solution to $(*)$



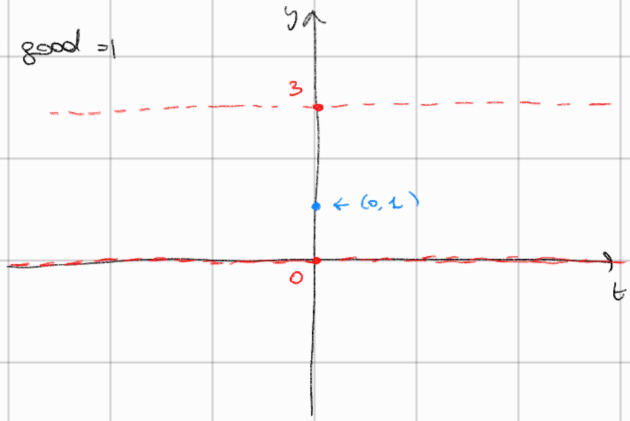
So it is enough to check whether or not your initial condition (t_0, u_0) belongs to the "discontinuity (not well-defined)" set for the function G & $\frac{\partial G}{\partial y}$.

Let's go back to the example: $G(t,x) = \frac{1+t^2}{3x-x^2}$. Here the problem is just the denominator.

When $3x-x^2 = 0$ $G(t,x)$ is not well defined. Elsewhere is continuous: $3x-x^2 = x(3-x) \Rightarrow$

$\Rightarrow x=0$ & $x=3$ are problematic $\Rightarrow$ any initial condition of the form $(t_0, 0)$ & $(t_0, 3)$

are not good $\Rightarrow$



Since the point $(0,1)$ is away from the bad region $\Rightarrow$ we have a unique solution passing by $(0,1)$

Now $\frac{\partial G}{\partial x} = \frac{\partial}{\partial x} \frac{1+t^2}{3x-x^2} = (1+t^2) \cdot \frac{1}{(3x-x^2)^2} \partial_x (3x-x^2) = \frac{-(1+t^2)}{(3x-x^2)^2}(3-2x)$ $\Rightarrow$ same issue as before at $x=0,3$.

⚠ we don't know yet the open $\mathcal{U}$ where it is defined! We only know that there exists.

Let's solve the ODE: it is separable $\Rightarrow$ $(3y-y^2)\,dy = (1+t^2)\,dt$

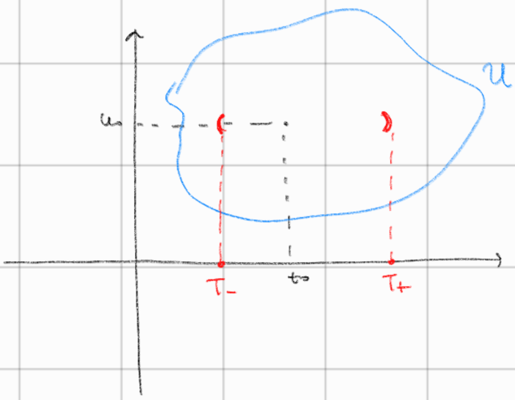
$$\Rightarrow \frac{3}{2}y^2 - \frac{1}{3}y^3 = t + \frac{t^3}{3} + K \Rightarrow \boxed{9y^2 - 2y^3 = 6t + 2t^3 + K}$$

Find K: $t=0,\ y=1 \Rightarrow 9-2=K \Rightarrow K=7$ $\Rightarrow$ $\boxed{\text{implicit solution } 9y^2 - 2y^3 = 6t + 2t^3 + 7}$

Let's now focus on the other P-L thm. from Lecture 1:

THEOREM (Picard-Lindelöf - long-time existence) - p.15 of theoryofodestimes.pdf

Same assumptions as before (namely you are given an IVP & at (t_0,u_0) $\exists$ an open $\mathcal{U}$ where G & $\frac{\partial G}{\partial x}$ are continuous). Then the IVP has a maximal interval of existence (T_-, T_+).



where you have a unique solution to the IVP.

Moreover: $T_+ = +\infty$ or $T_+ < +\infty$ & $\|\varphi(t)\| \xrightarrow{t \to T_+} \infty$ (resp. for T_-)

sloppy

Right way to say it: the graph $(t, y(t))$ leaves every closed & bounded set $\subseteq \mathcal{U}$

as $t \to T_-$ or $t \to T_+$.

What I mean by closed & bounded set K

Bounded: there is a ball B : $K \subseteq B$.

Closed: the "border of the region is included in K"

Examples in $\mathbb{R}$: * $(-1, 0)$ is open in $\mathbb{R}$

* $[2, 5]$ is closed in $\mathbb{R}$

* $[2, 5)$ is neither open or closed

Examples in $\mathbb{R}^2$: if we assume that K is a rectangle

it is open if $K = \underset{\uparrow}{(\alpha, \beta)} \times \underset{\uparrow}{(\delta, \gamma)}$

open interval x open interval

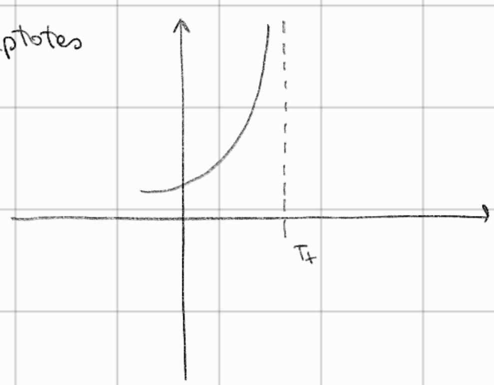
it is closed if $K = [\alpha, \beta] \times [\delta, \gamma]$

both closed interval.

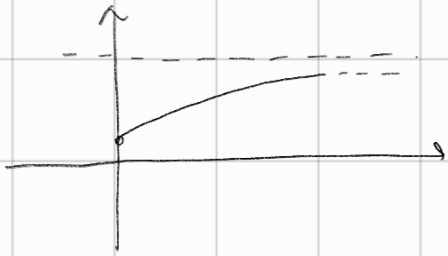
(neither open or closed otherwise)

What does it look like to "leave every compact"?

① Asymptotes

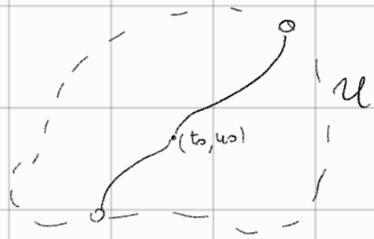


② Existing "for ever" $T_+ = +\infty$



③ But also: approaching the border of $\mathcal{U}$

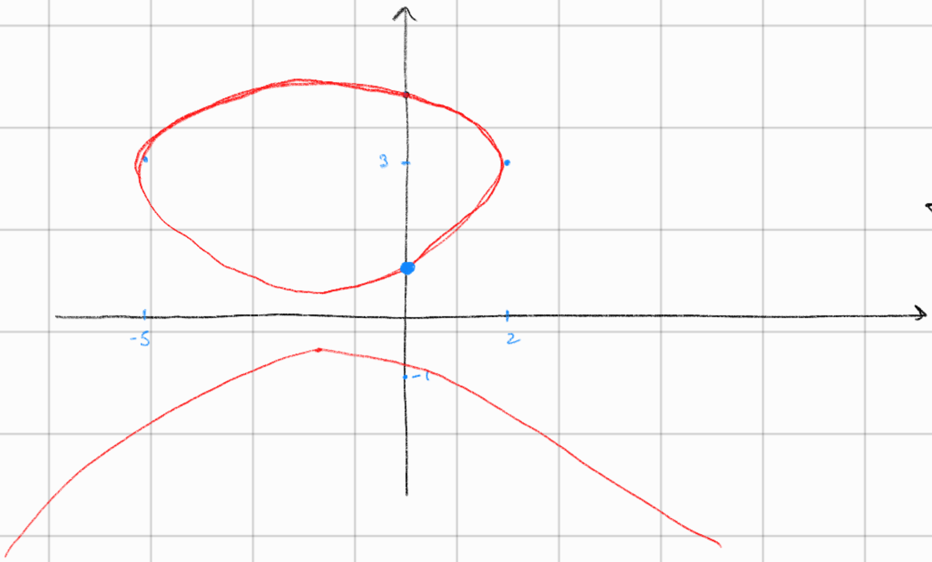
" in this case, the border is the ∞ "



Question (from PSET): you are given an IVP and you are asked to find the maximal interval of $\exists$.

This has subtleties: for instance the domain of a solution can be restricted for reasons other than the G or $\frac{\partial G}{\partial y}$ being non-continuous!

let's go back to the example: $9y^2 - 2y^3 = 6t + 2t^3 + 7$ $(\star)$: let's plot it: (I used desmos.com online calculator)



← this is the locus of the points (t, y) s.t. they satisfy $(\star)$

We know that around $(0,1)$ there is a ! solution $\Rightarrow$ there exists an explicit solution $y(t) = \varphi(t)$ such that the graph of $\varphi(t)$ overlaps with the real locus shape.

By uniqueness, $\varphi(t)$ is given by the Implicit Function theorem.

Suppose you have a function $S(t,y) : (\alpha,\beta)\times(\delta,\gamma) \longrightarrow \mathbb{R}^2$ such that

⊛ $S, \frac{\partial S}{\partial t}, \frac{\partial S}{\partial y}$ are all continuous in $(\alpha,\beta)\times(\delta,\gamma)$ (regularity assumptions)

Let (t_0,u_0) be a point on the graph $S(t,y)=0$ (i.e. $S(t_0,u_0)=0$).

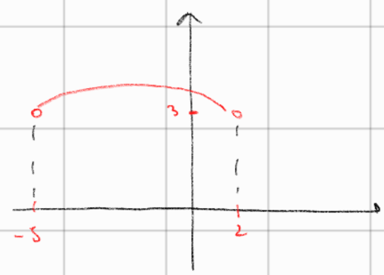
If $\left.\frac{\partial S}{\partial y}\right|_{(t_0,u_0)} \neq 0 \Rightarrow$ the curve around (t_0,u_0) can be written explicitly

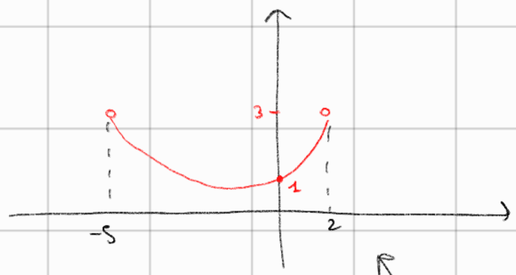
$$y = \varphi(t)$$

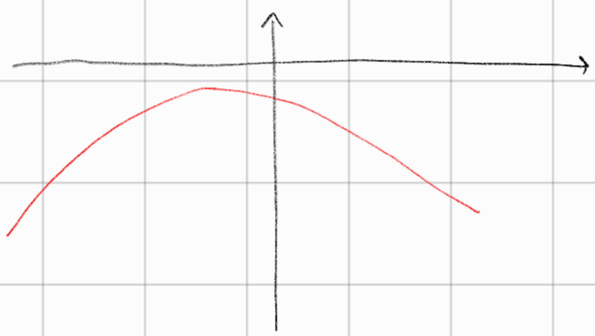
So in our case: $S(t,y) = 9y^2 - 2y^3 - (6t + 2t^3 + 7)$

$\frac{\partial}{\partial y} S(t,y) = 18y - 6y^2$: it is $=0$ iff. $\underline{y=0 \text{ or } 3}$.

$\Rightarrow$ We have the following well defined graphs:







this is the graph of the solution since we know that must pass by $(0,1)$.

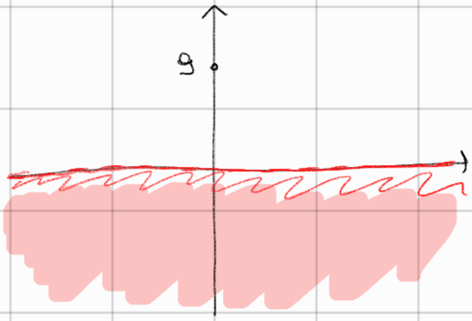
$\Rightarrow (T_-, T_+) = (-5, 2)$

Let's see other examples:

Examples: $y' = \frac{2}{\sqrt{y}}$, $y(0) = 9$: $*$ issues with continuity: only at $y \leq 0$

So we need to exclude $y \leq 0$.

$\Rightarrow$



$(t_0, u_0) = (0, 9)$ is far from $y \leq 0$. ✓

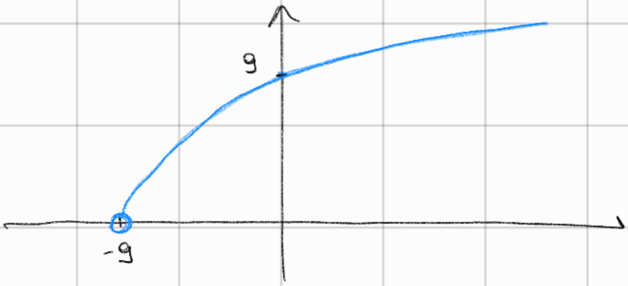
$\Rightarrow$ we have a unique solution.

you can check that you get $y^3 = 9(x+9)^2$ implicit solution.

$S(x,y) = y^3 - 9(x+9)^2 \Rightarrow \frac{\partial S}{\partial y} = 3y^2 \Rightarrow$ at $y=0$ we have issues $\Rightarrow$

$\Rightarrow$ at $9(x+9)^2 = 0$ we have issue

$\Rightarrow$ at $x = -9$ we have a problem.



$\Rightarrow T_- = -9,\ T_+ = +\infty$.

$\Rightarrow (-9, +\infty)$.

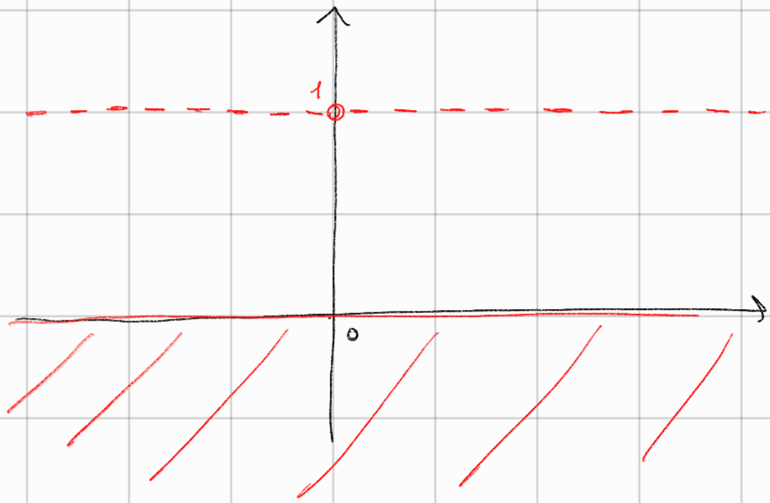
Example: $\begin{cases} y' = \frac{-xy}{\ln(y)} \\ y(0) = e^2 \end{cases}$

(*) Well-define / continuity: $\ln(y) \Rightarrow y > 0$ moreover $\ln(y)$ at denominator

$\Rightarrow \ln(y) = 0$ X $\Rightarrow y \neq 1$.

(*) You can check that $\frac{\partial}{\partial y} \frac{xy}{\ln(y)}$ has the same issues.

$\Rightarrow$



$(0, e^2)$ away from bad region.

You can check $y = e^{\sqrt{4-x^2}}$ solves the IVP. $\Rightarrow 4-x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$

However $y \neq 1$ and this happens when $x = \pm 2 \Rightarrow (-2, 2) = (T_-, T_+)$