Today: * Method of reduction of order & Abel's thm: §3.2 & §3.4

* Non-homogeneous - case §3.5 & §3.6

└ Undetermined coeff.'s & Variation of parameters.

Next time recall: ⊛ ∃/! remarks for 1ST-order ODEs.

⊛ we started discussing 2nd-order + scalar + linear ODEs. $y'' + p(t)y' + q(t)y = g(t)$

(∃/! /Maximal I Thm)

Superposition thm for homogeneous

(φ, ψ sol $\Rightarrow C_1\psi + C_2\varphi$ sol.)

$W(\varphi,\psi)$

↓

fundamental set of solution (again for homogeneous)

⊛ Constant coeff. + homogeneous: $y'' + py' + qy = 0$

⊛ Ansatz for $r_1 = r_2$ case $\boxed{\psi(t) = v(t)\varphi(t)}$ - in that case $\varphi = e^{rt}$ & $v = t$ $\Rightarrow te^{rt} = \psi$.

However we haven't seen what happens for a general ODE of the form $y'' + p(t)y' + q(t)y = 0$

when we substitute $\psi(t) = v(t)\varphi(t)$ (& $\varphi(t)$ is a solution)

Method of reduction of order

let's do it: $\psi' = v'\varphi + v\varphi'$; $\psi'' = v''\varphi + v'\varphi' + v'\varphi' + v\varphi'' = v''\varphi + 2v'\varphi' + v\varphi''$

Now let's substitute in $y'' + p(t)y' + q(t)y = 0$

$\Rightarrow$ $v''\varphi + 2v'\varphi' + \underbrace{v\varphi''}$ $+ p(t)v'\varphi + \underbrace{p(t)v\varphi'} + \underbrace{q(t)v\varphi} = v''\varphi + 2v'\varphi' + p(t)v'\varphi$

$v\varphi'' + v\,p(t)\varphi' + v\,q(t)\varphi$

$\Rightarrow$ $v[\varphi'' + p(t)\varphi' + \varphi] = 0$ (because φ is a solution)

Therefore: I want v so that $v''\cdot\varphi + v'(2\varphi' + p(t)\varphi) = 0$

Rmks: ⊛ there is dependence on $q(t)$ only on $p(t)$ & on the known solution $\varphi(t)$.

⊛ φ is indeed known $\Rightarrow$ in $v''\cdot\varphi + v'(2\varphi' + p(t)\varphi) = 0$

both the coefficients are known!

⊛ We have reduced the order! Indeed you can pose $u(t) = v'(t)$ & the above ODE (in v)

becomes: $\underbrace{u'\cdot\varphi + u(2\varphi' + p(t)\varphi) = 0}_{(*)}$ $\leftarrow$ $\boxed{\text{1st order, scalar, linear ODE}}$

& we can apply formulas from last week!

⊛ Remember that after you solve $(*)$, you still need to do two steps:

1) solve $v' = u$ $\Rightarrow$ $v(t) = \int u(t)\,dt + K$

2) pick the solution $\psi(t) = v(t)y_1(t)$ & make sure $\{y_1(t), v(t)y_1(t)\}$ is a fundamental set

Examples: 1) Consider the ODE $2t^2y'' + 3ty' - y = 0$, for $t>0$; $\varphi(t) = \frac{1}{t}$ is a solution.

Find a fundamental set of solutions.

⊛ Try $\psi(t) = \frac{v(t)}{t}$: pose $u(t) = v'(t)$: (a) solve $u'\cdot\varphi + u(2\varphi' + p(t)\varphi) = 0$

$$\rightsquigarrow \quad \frac{u'}{t} + u\left(2\cdot\left(-\frac{1}{t^2}\right) + p(t)\cdot\frac{1}{t}\right) = 0 .$$

Who is $p(t)$? It is NOT $3t$, because you need first to divide by $2t^2$!

$\Rightarrow p(t) = \frac{3t}{2t^2} = \frac{3}{2t} \Rightarrow$ the 1st order ODE for u becomes $u' + u\cdot t\left(-\frac{2}{t^2} + \frac{3}{2t^2}\right) = 0$

$\Rightarrow u' = u\left(\frac{2}{t} - \frac{3}{2t}\right) = u\left(\frac{4-3}{2t}\right) = \frac{u}{2t}$

- $u \equiv 0$ is a solution
- assume u never zero $\Rightarrow \frac{1}{u}u' = \frac{1}{2t}$

$\Rightarrow \ln|u| = \frac{1}{2}\ln|t| + K = \frac{\ln(t)}{2} + K$ because $t>0$.

$\Rightarrow |u| = \exp\left(\frac{\ln(t)}{2}\right) + K = C\cdot\sqrt{t}$, $C>0 \Rightarrow u = C\sqrt{t} \quad \forall C \in \mathbb{R}^*$.

Since also $u\equiv 0$ is a solution $\Rightarrow$ the general form is $u = C\sqrt{t} \quad \forall C\in\mathbb{R}$.

(b) Solve $v' = u \Rightarrow v = C\frac{3}{2}t^{3/2} + d \Rightarrow y(t) = C\frac{3}{2}t^{3/2}\cdot\frac{1}{t} + \frac{d}{t} = C\sqrt{t} + \frac{d}{t}$

(c) Check $\left\{\frac{1}{t}, C\sqrt{t} + \frac{d}{t}\right\}$ fundamental set: notice that the

$\text{Span}\left\{\frac{1}{t}, C\sqrt{t}+\frac{d}{t}\right\} = \text{Span}\left\{\frac{1}{t}, C\sqrt{t}\right\}$ So let's check $\left\{\frac{1}{t}, C\sqrt{t}\right\}$

$$W\left(\frac{1}{t}, C\sqrt{t}\right) = \det\begin{pmatrix} \frac{1}{t} & C\sqrt{t} \\ -\frac{1}{t^2} & \frac{C}{2\sqrt{t}} \end{pmatrix} = \frac{C}{2t\sqrt{t}} + \frac{C\sqrt{t}}{t^2} = \frac{C}{2t\sqrt{2}} + \frac{C}{t\sqrt{2}} = \left(\frac{3C}{2}\right)\frac{1}{t\sqrt{t}} \not\equiv 0 \text{ if } C\neq 0.$$

$\Rightarrow \left\{\frac{1}{t}, \sqrt{t}\right\}$ is a fundament set of solutions.

A cool trick: Suppose that you have a 2nd-order scalar linear homogeneous ODE

$y'' + p(t) y' + q(t) y = 0$ & you know a solution $\varphi(t)$ & you want to find ψ (a second solution).

You have • reduction of order (just seen)

• Abel's thm: Assume $p(t)$ & $q(t)$ are continuous over an open interval I, let φ & ψ two solutions: then $W(\varphi, \psi) = \text{Constant} \cdot e^{-\int p(t)\,dt}$ (where Constant depends on φ & ψ)

but not on t!

Moreover: $W(\varphi,\psi)$ is either ZERO $\forall t \in I$ $(c=0)$

or Never ZERO $\forall t \in I$ $(c \neq 0)$ & $\{\varphi, \psi\}$ is a fundamental set of solutions.

before the proof, let's an example in order to compare the two methods.

Example: $(t-1)y'' - ty' + y = 0$ & $\varphi(t) = e^t$ is a solution.

(*) Put it in the standard form: $y'' - \frac{t}{t-1} y' + \frac{1}{t-1} y = 0$.

$\Rightarrow p(t) = -\frac{t}{t-1}$ & $q(t) = \frac{1}{t-1}$. These functions are well-defined & continuous $\forall t \neq 1$.

So I is either $(-\infty, 1)$ or $(1, +\infty)$.

Method of reduction of order: $v'' \cdot \varphi + v'(2\varphi' + p(t)\varphi) = 0$ $\boxed{\psi = v \cdot e^t}$

$\Rightarrow v'' \cdot e^t + v'\left(2e^t - \frac{t}{t-1} \cdot e^t\right) = 0 \Rightarrow v'' + v'\left(\frac{2(t-1) - t}{t-1}\right) = 0$

$v'' + v'\left(\frac{2t - t - 2}{t-1}\right) = 0 \Rightarrow v'' = -\frac{v'(t-2)}{(t-1)}$

solve for $u = v'$

$v = \int u + k \Rightarrow \psi = \left(\int u + k\right)\varphi$

check $W(\varphi, \psi) \not\equiv 0$.

$\frac{1}{u} u' = \left[-\frac{(t-1)}{t-1} + \frac{1}{t-1}\right] \Rightarrow \ln|u| = \int -1 + \frac{1}{t-1}\, dt = -t + \ln|t-1|$

$\Rightarrow$ Assume $t > 1 \Rightarrow |u| = e^{-t}(t-1) \Rightarrow u = C e^{-t}(t-1)$.

$\Rightarrow \psi = \left(C \cdot \int e^{-t} t - e^{-t}\, dt + k\right)e^t$

$= \left(C \cdot \left[e^{-t} - e^{-t} t - \int -e^{-t} dt\right] + k\right)e^t = \left(C\left[e^{-t} - e^{-t} t - e^{-t}\right] + k\right)e^t$

$= -Ct + ke^t \Rightarrow \psi = t$ works fine.

Abel's thm: $W(\varphi, \psi) = W(e^t, \psi) = e^t\psi' - e^t\psi = C \cdot e^{-\int\left(-\frac{t}{t-1}\right)dt}$

$\Rightarrow \psi' - \psi = C \cdot e^{-t} \cdot e^{\int\left(\frac{t-1}{t-1}\right) + \left(\frac{1}{t-1}\right)dt} = C \cdot e^{-t} \cdot e^{\int 1 + \frac{1}{t-1}\, dt} =$

$= C \cdot e^{-t} \cdot e^{t} \cdot e^{\ln|t-1|} = C \cdot |t-1|$

Assume $t > 1 \Rightarrow \psi' - \psi = C(t-1) \Rightarrow \psi' = \psi + C(t-1)$

$\Rightarrow \psi = e^{\int a(t)dt}\left[\int e^{-\int a(t)dt}\, b(t)\, dt + k\right]$ $a(t) = 1$ & $b(t) = C(t-1)$

$$\Rightarrow \psi = e^{t}\left[\int e^{-t}\cdot C\,(t-1)\,dt + k\right] = Ce^{t}\left[\int e^{-t}t - e^{-t}\,dt\right] + ke^{t} =$$

$$= Ce^{t}\left[-e^{-t}t - e^{-t} + e^{-t}\right] + ke^{t} = -Ct + ke^{t} \Rightarrow \psi = t \text{ works fine.}$$

Proof: φ & ψ are solutions $\Rightarrow$ $\psi'' + p(t)\psi' + q(t)\psi \equiv 0$ ← multiply by φ

(of Abel's Thm)

$\varphi'' + p(t)\varphi' + q(t)\varphi \equiv 0$ ← multiply by ψ

& substract them: you get

$$\varphi''\psi + p(t)\varphi'\psi + \cancel{q(t)\varphi\psi} - \psi''\varphi - p(t)\psi'\varphi - \cancel{q(t)\psi\varphi} \equiv 0$$

$$\Rightarrow (\varphi''\psi - \psi''\varphi) = p(t)\left[\psi'\varphi - \varphi'\psi\right]$$

$= W(\varphi,\psi) = \det\begin{pmatrix}\varphi & \psi \\ \varphi' & \psi'\end{pmatrix}$

This is $-\frac{d}{dt}W(\varphi,\psi) = -\left(\psi''\varphi + \cancel{\psi'\varphi'} - \varphi''\psi - \cancel{\varphi'\psi'}\right) = -\psi''\varphi + \varphi''\psi$. ✓

$$\Rightarrow \frac{d}{dt}W = -p(t)\cdot W \Rightarrow \frac{1}{W}dW = -p(t)\,dt \Rightarrow \ln|W| = -\int p(t)\,dt + c$$

$$\Rightarrow W = C\cdot e^{-\int p(t)\,dt}$$

§3.5 NEXT - Step: Constant coefficients still but not homogeneous.

$$\boxed{y'' + py' + qy = g(t)}$$

Examples: (1) $y'' + 3y' + 5y = t$

(2) $y'' - y = \sin(t)$

(3) $y'' - 2y' + y = e^t$

Rmk: Assume Y_1 & Y_2 are solutions to $y'' + py' + qy = g(t)$ then $Y_1 - Y_2$ is a solution to the homogeneous equation: indeed

$$\left.\begin{aligned} Y_1'' + pY_1' + qY_1 &\equiv g(t) \\ Y_2'' + pY_2' + qY_2 &\equiv g(t) \end{aligned}\right\} \text{subtract one to the other} \qquad Y_1'' - Y_2'' + p(Y_1' - Y_2') + q(Y_1 - Y_2) \equiv 0$$

rmk #2: the same argument works fine even when p and q are NOT constant.

In particular we know that for hom. ones we have a fundamental set of solutions $\{\psi, \varphi\}$.

$\Rightarrow$ $Y_1 - Y_2$ solution means that there exist 2 constants C_1 & $C_2 \in \mathbb{R}$ s.t. $Y_1 - Y_2 = C_1\psi + C_2\varphi$.

$$\Rightarrow Y_1 = \underbrace{(C_1\psi + C_2\varphi)}_{\text{general solution to } y'' + p(t)y' + q(t)y = 0} + \overbrace{Y_2}^{\text{particular solution to } y'' + p(t)y' + q(t)y = g(t)}$$

THEOREM: The general solution to $y'' + p(t)y' + q(t)y = g(t)$ has the form

$Y(t) = C_1 \cdot \psi + C_2\varphi + Y_p$ where $\{\psi, \varphi\}$ is a fundamental set of sol.s to $y'' + p(t)y' + q(t)y = 0$

Y_p is ANY solution to $y'' + p(t)y' + q(t)y = g(t)$

MORAL: after studying the homogeneous case, enough to find one particular solution to the non-homogeneous one and you find all of them.

STRATEGY	$y''+py'+qy=g(t)$ [const. coeff.s]	$y''+p(t)y'+q(t)y=g(t)$ & $\varphi(t)$ is a solution to the homog. one
step 1	⊙ Find general solution to $y''+py'+qy=0$	⊙ Find general solution to $y''+p(t)y'+q(t)y=0$
How to do step 1	⊙ Char. polynomial $\Rightarrow$ roots $\Rightarrow$ 3 cases $\Rightarrow$ pick the right form	⊙ either Abel's theorem $W = Ce^{-\int p(t)}$ or Ansatz $\psi = v\varphi$
step 2	⊙ Find ONE solution y_p to $y''+py'+qy=g(t)$	⊙ Find ONE solution y_p to $y''+p(t)y'+q(t)y=g(t)$
How to do step 2	⊙ Method of undetermined coefficients ⊙ Variation of parameters.	⊙ Variation of parameters.

Rmk:

Undetermined coeff.'s:	PROS: Easy	CONS: doesn't always work
Variation of parameters:	PROS: works always	CONS: Maybe bad computations

Method of undermined coefficients. for $y'' + py' + cy = g(t)$

(*) it requires us to make an initial assumptions about the shape of the particular solution Y_p

The assumption made depends on $g(t)$

shape of $g(t)$	shape of the guessed Y_p
① polynomial. $g(t) = a_n t^n + \dots + a_1 t + a_0$	① polynomial of the form $t^s(\underbrace{A_n t^n + A_{n-1} t^{n-1} + \dots + A_0}_{\text{same degree as } g(t)})$ See later what s is.
② $g(t) = e^{\alpha t}$	② exponential: $t^s \cdot A_0 e^{\alpha t}$ same α
③ $g(t) = \sin(\beta t)$	③ trigonometric: $[A\cos(\beta t) + B\sin(\beta t)]t^s$
④ $g(t) = \cos(\beta t)$	④ 〃 〃 〃 〃 〃 〃 〃 〃
⑤ $P_n(t) \cdot e^{\alpha t}$ = polynomial (degree n) · exponential	⑤ $t^s(A_n t^n + \dots + A_0)e^{\alpha t}$
⑥ $P_n(t)\, e^{\alpha t} \cdot \cos(\beta t)$ (or $P_n(t) e^{\alpha t} \cdot \sin(\beta t)$) ← polynomial, exponential, trigonometric	⑥ $t^s[(A_n t^n + \dots + A_0)e^{\alpha t}\cos\beta t + (B_n t^n + \dots + B_0)e^{\alpha t}\sin(\beta t)]$

Therefore: as far as your $g(t)$ is a linear combination of the above functions

(trigonometric functions, polynomials, exponentials, & product of them) then you can

use the method and determine the RED coefficients.

let's go back to Examples: (1) $y'' + 3y' + 5y = t$.

(2) $y'' - y = \sin(t)$

(3) $y'' - 2y' + y = e^t$

(1) step 1: solve $y'' + 3y' + 5y = 0$. : char. polynomial is $r^2 + 3r + 5 = 0$ $r_{1,2} = \frac{-3 \pm \sqrt{9-20}}{2}$

$= \frac{-3 \pm i\sqrt{11}}{2}$

$\Rightarrow \alpha = -\frac{3}{2}$ & $\beta = \frac{\sqrt{11}}{2}$ $\Rightarrow$ the general solution has the form

$$e^{-3t/2}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}t\right)\right)$$

(2) step 2: find one particular solution: $g(t) = t$ $\Rightarrow$ polynomial of degree 1

$\Rightarrow \boxed{Y_p = t^s(A_1 t + A_0)}$

Try $s = 0$: $Y_p = A_1 t + A_0$: then $Y_p' = A_1$ & $Y_p'' = 0$

$\Rightarrow 0 + 3A_1 + 5(A_1 t + A_0) = t \Rightarrow 3A_1 + 5A_0 + 5A_1 t = t$ $\Rightarrow$ compare the

coefficients: $5A_1 = 1 \Rightarrow A_1 = 1/5$

$\frac{3}{5} + 5A_0 = 0 \Rightarrow A_0 = -\frac{3}{25}$ $\Rightarrow Y(t) = e^{-\frac{3t}{2}}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}t\right)\right)$

$+ \frac{t}{5} - \frac{3}{25}$

(2) $y'' - y = \sin(t)$ step 1: $y'' - y = 0$ $\Rightarrow$ char. polynomial $r^2 - 1 = 0$ $\Rightarrow$ $r = \pm 1$

$\Rightarrow$ general solution $C_1 e^{t} + C_2 e^{-t}$

step 2: find ONE solution to the not-homogeneous one.

$g(t) = \sin(t) \Rightarrow Y_p = [A\cos(t) + B\sin(t)]t^s$

Try $s = 0$ $\Rightarrow Y_p' = -A\sin(t) + B\cos(t)$, $Y_p'' = -A\cos(t) - B\sin(t)$

$-A\cos(t) - B\sin(t) - A\cos(t) - B\sin(t) = \sin(t)$ $\Rightarrow$ comparing the coefficients: $A = 0$, $B = -\frac{1}{2}$

$\Rightarrow$ the general solution: $C_1 e^{t} + C_2 e^{-t} - \frac{\sin(t)}{2}$

(3) $y'' - 2y' + y = e^{t}$: 1) solve $y'' - 2y' + y = 0$ $\Rightarrow$ $r^2 - 2r + 1 = 0$ $\Rightarrow$ $(r-1)^2 = 0$ $\Rightarrow r = 1$

$\Rightarrow$ general solution: $C_1 e^{t} + C_2 \cdot t \cdot e^{t}$.

2) find one solution to $y'' - 2y' + y = e^{t}$ $\Rightarrow$ $g(t) = e^{t} \rightsquigarrow Y_p = t^s . A_0 \cdot e^{t}$

Try $s = 0$: but now we have a problem: for $s = 0$, $Y_p = A_0 \cdot e^{t}$ & we know from step 1 that such function is a solution to the homogeneous case.

$\Rightarrow \left(y'' - 2y' + y \right)\Big|_{y = A_0 \cdot e^{t}} \equiv 0 \neq e^{t}!$ $\Rightarrow$ $t = 0$ cannot work.

For analogous reason: $s=1$, $Y_p = A_0 \cdot t \cdot e^t$ is already a solution to the homogeneous one

$\Rightarrow$ cannot work! $\Rightarrow$ try $s=2$: $Y_p = A_0 \cdot t^2 \cdot e^t$

$\Rightarrow Y_p' = A_0(2t \cdot e^t + t^2 e^t)$, $Y_p'' = A_0(2e^t + 2te^t + 2te^t + t^2 e^t)$

$\Rightarrow A_0(2e^t + 4te^t + t^2e^t) - 2A_0(2te^t + t^2e^t) + A_0(t^2e^t) = e^t$

$A_0[2e^t + 4te^t + t^2e^t - 4te^t - 2t^2e^t + t^2e^t] = e^t$

$\Rightarrow 2e^t A_0 = e^t \Rightarrow 2A_0 = 1 \Rightarrow A_0 = 1/2$ $\Rightarrow$ the general solution is

$Y = C_1 e^t + C_2 t e^t + \frac{t^2}{2} e^t$

Let's do one more example: (4) $y'' - 3y' - 4y = 2e^{-t}$

- homogeneous case: $y'' - 3y' - 4y = 0$: char polynomial $r^2 - 3r - 4 = 0$

$\Rightarrow$ roots $r_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2}$ $\begin{cases} 8/2 = 4 \\ -2/2 = -1 \end{cases}$ $\Big| \Rightarrow C_1 e^{-t} + C_2 e^{4t}$ general solution to the homog. ODE.

- non-homogeneous case: $g(t) = 2e^{-t} \Rightarrow Y_p = t^s \cdot A_0 \cdot e^{-t}$

Try $s=0$: $Y_p = A_0 \cdot e^{-t}$: but is already a solution of the homogeneous one X

$\hookrightarrow$ try $s=1$: $Y_p = A_0 \cdot t \cdot e^{-t} \Rightarrow Y_p' = A_0 e^{-t} - A_0 t \cdot e^{-t}$

$\Rightarrow Y_p'' = -A_0 e^{-t} - A_0 e^{-t} + A_0 t \cdot e^{-t}$

$$\Rightarrow \ -2A_0e^{-t} + A_0te^{-t} - 3(A_0e^{-t} - A_0te^{-t}) - 4(A_0te^{-t}) = 2e^{-t}$$

$$-2A_0e^{-t} + \underbrace{A_0t \cdot e^{-t}} - 3A_0e^{-t} + \underbrace{3A_0te^{-t} - 4A_0te^{-t}} = 2e^{-t}$$

$$\Rightarrow \ -5A_0e^{-t} = 2e^{-t} \quad \Rightarrow \quad A_0 = -\frac{2}{5} \quad \Rightarrow \text{ the general solution is}$$

$$C_1e^{-t} + C_2e^{4t} - \frac{2}{5}te^{-t} = Y(t)$$

Meaning of t^s: s = the smallest integer (here $s = 0, 1, 2$) that ensures that you don't get a solution to the homogeneous case.

Ex (3): e^t & te^t sol.s for the homog. one $\Rightarrow$ we need $t^2 \cdot e^t$

Ex (4): e^{-t} sol. for the homog. one $\Rightarrow$ we need $t \cdot e^{-t}$.

Finally: what happens if $g(t)$ = more than one function from the above list?

Example: $y'' - y = t^2 + \sin(t)$ Here $g(t) = g_1(t) + g_2(t)$ where $g_1(t) = \sin(t)$ & $g_2(t) = t^2$

As before: first the homog. case $y'' - y = 0 \Rightarrow C_1e^t + C_2e^{-t}$

then we need Y_p: $Y_p'' - Y_p = t^2 + \sin(t)$

we know that $Y_1 = -\frac{\sin(t)}{2}$ is such that $Y_1'' - Y_1 = \sin(t)$

Analogously we can solve $Y_2'' - Y_2 = t^2$: $Y_2 = A_2 t^2 + A_1 t + A_0$

$\Rightarrow -(A_2t^2 + A_1 t + A_0) + (2A_2) = t^2 \Rightarrow A_1 = 0, A_2 = -1$

$t^2 - A_0 - 2 = t^2 \Rightarrow A_0 = -1 \Rightarrow Y_2 = -t^2 - 2$

Now notice that if $Y_1'' - Y_1 = \sin(t)$ & $Y_2'' - Y_2 = t^2$

$\Rightarrow (Y_1 + Y_2)'' - (Y_1 + Y_2) = \sin(t) + t^2 \Rightarrow Y_p = Y_1 + Y_2$ works fine.

$\Rightarrow Y_p = -\frac{\sin(t)}{2} - t^2 - 2$

In general: decompose $g(t) = \Sigma g_i(t)$ such that each $g_i \in$ to the table.

$\Rightarrow$ find Y_i for any $g_i \Rightarrow Y_p = \Sigma Y_i$

$\Rightarrow \boxed{Y = \text{General sol to the hom} + Y_p}$

§3.6 Variation of Parameters

Rmks: Due to Lagrange

(a) It complements the undetermined coefficients.

(b) In principle, it can be applied to any ODE $y''+p(t)y'+q(t)y=g(t)$

⇝ But: bad integrals!

THM: given $y''+p(t)y'+q(t)y=g(t)$ & p,q,g continuous on I. Fix $\bar{t}\in I$.

Let $\{\varphi,\psi\}$ be a fundamental set of solutions for $y''+p(t)y'+q(t)y=0$ on I.

Let $W := W(\varphi,\psi)$ be the Wronskian

Then $Y_p = -\varphi \cdot \int_{\bar{t}}^{t}\left[\frac{\psi(s)g(s)}{W(s)}\right]ds + \psi\int_{\bar{t}}^{t}\left[\frac{\varphi(s)g(s)}{W(s)}\right]ds$ is a particular solution.

proof: compute Y_p' & Y_p'' from the above formula & plug everything in the ODE.

Rmks: The formula makes sense because

1) φ & ψ are fund set $\Rightarrow$ $W\neq 0$

2) ψ, g, W, φ are all continuous over I $\Rightarrow$ they are integrable.

3) t_0 must be picked $\in I$

Example: $\odot$ $y'' + 4y = \frac{3}{\cos(t)}$

$\odot$ homogeneous: $y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$\Rightarrow$ the general solution is $C_1 \cos(2t) + C_2 \sin(2t)$

$\odot$ Particular solution: since it is <u>NOT</u> of the form Polynomial / exp. / sin / cos.

$\Rightarrow$ Variation of parameters is needed!

0th-step: continuity of the coefficients: 4 ✓ $\frac{3}{\cos(t)}$: $t \neq \frac{\pi}{2} + k\pi$ $k \in \mathbb{Z}$: for simplicity

let's pick $(-\pi/2, \pi/2)$ & $\bar{t} \Rightarrow \in I = (-\pi/2, \pi/2)$

(1st) Set of fund. solutions: $\varphi = \cos(2t)$ & $\psi = \sin(2t)$

(2nd) The Wronskian $W(\cos(2t), \sin(2t)) = \det \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix} = 2\cos(2t)^2 + 2\sin^2(2t) = 2$

(3rd) Write the formula: $y_p = -\cos(2t) \int_0^t \frac{\sin(2s) \cdot 3}{\cos(s) \cdot 2} ds + \sin(2t) \int_0^t \frac{\cos(2s) \cdot 3}{2\cos(s)} ds =$

$= -\cos(2t) \int_0^t \frac{2\cos(s)\sin(s) \cdot 3}{2\cos(s)} ds + 3\sin(2t) \int_0^t \frac{2\cos^2(s) - 1}{2\cos(s)} ds =$

↑ & they are both integrable! ↑

Let's put all together, IVPs

$$\text{ODE}: \begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ \text{IV}: \quad y(t_0) = u_0, \quad y'(t_0) = u_1 \end{cases}$$

Find the solution & the maximal interval of $\exists$.

⊙ Focus only on the ODE & use the right method to find the general solution

$$Y = C_1\varphi + C_2\psi + Y_p$$

⊙ Use t_0, u_0, u_1 to find the coefficients.

Example: $\begin{cases} y'' + 2y' + 5y = 4 \cdot e^{-t}\cos(2t) \\ y(0) = 1, \; y'(0) = 0 \end{cases}$

$p = 2$, $q = 5$, $g = 4e^{-t}\cos(2t)$

they are continuous everywhere

$\Rightarrow$ maximal $I = (-\infty, +\infty)$.

⊙ homogeneous solution: $r^2 + 2r + 5 = 0 \Rightarrow r_{1,2} = -1 \pm \sqrt{1-5} = -1 \pm 2i$

$$\Rightarrow e^{-t}(C_1\cos(2t) + C_2\sin(2t))$$

⊙ particular solution to non homog. one: $e^{-t}[A\cos(2t) + B\sin(2t)] \cdot t$

($s=1$ because $s=0$ already a solution of the homogeneous one).

Plugging in you can check that $A = 0$, $B = 1$ $\Rightarrow Y_p = e^{-t} \cdot t \cdot \sin(2t)$

• Initial conditions $y(0)=1$ & $y'(0)=0$

$$Y(t) = e^{-t} \cdot t \cdot \sin(2t) + e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$$

$$Y(0) = 1 = 0 + 1(C_1) \Rightarrow C_1 = 1.$$

$$Y'(t) = -e^{-t} \cdot t \sin(2t) + e^{-t}[\sin(2t) + 2t\cos(2t)]$$

$$-e^{-t}(\cos(2t) + C_2 \sin(2t)) + e^{-t}(-2\sin(2t) + 2C_2 \cos(2t)) =$$

$$\Rightarrow Y'(0) = 0 + 0 - 1(1) + 1(2 \cdot C_2) = 0$$

$$= -1 + 2C_2 = 0 \quad \Rightarrow C_2 = \frac{1}{2}$$

$$\Rightarrow Y(t) = e^{-t} \cdot t \cdot \sin(2t) + e^{-t}(\cos(2t) + \frac{1}{2}\sin(2t))$$