So FAR:

I week: Clarification + 1st-order linear, seperable, exact, $\exists\, \mu(x,y)$

I/II week. Qualitative discussion ~ $\exists$/! Thms.

II week: 2nd-order linear

This week we take a step further in two direction: §4.1 – §4.2

Ⓘ nth-order – scalar – linear

+ constant coeff.'s + homogeneous

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

IVP: $\begin{cases} \text{ODE} + \\ y(t_0) = u_0 \\ \vdots \\ y^{(n-1)}(t_0) = u_{n-1} \end{cases}$

Ⓘⓘ 1st order – system – linear

+ constant coeff.'s + homogeneous.

$\vec{x}' = A \cdot \vec{x}$ & we assume $A$ $n \times n$-matrix

IVP: $\begin{cases} \text{ODE} \\ \vec{x}(t_0) = \begin{pmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} \end{cases}$

Today: review Linear Algebra §7.2 – §7.3

application to ODEs ⊆ §4.1, §4.2, §7.5

Next time: ⊛ Integration tricks for the midterm

⊛ IVP for today's discussion

⊛ Review for the midterm.

## REVIEW on Linear Algebra

**DEF:** A matrix with $m$ rows & $n$ columns is an $m \times n$ array of numbers $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{nm} \end{bmatrix}$

$a_{ij} =$


**Rmk:** we will mostly look at the following two cases: $2 \times 2$ $\begin{bmatrix} a_{1} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $3 \times 3$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(no $m = n$ & $\leq 3$)

**Particular cases:** $m \times 1$ - matrix $\rightarrow$ (column) vector  Example: $\begin{bmatrix} 3 \\ 5 \\ 2-i \end{bmatrix}$ a $3 \times 1$.

$1 \times n$ - matrix $\rightarrow$ (row) vector  Example: $[1 \quad 2]$ a $1 \times 2$.

### Operations:

$\boxed{\text{SUM}}$: it is component wise: $A = (a_{ij})$ & $B = (b_{ij})$ matrices of the same type ($m \times n$)

$\Rightarrow \quad A + B = (a_{ij} + b_{ij})$

Example: $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

**MULTIPLICATION by a SCALAR:** again, component wise: $\alpha A = (\alpha a_{ij})$

Example: $\alpha \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \alpha & 3\alpha \\ 0 & 2\alpha \end{bmatrix}$

**TRANSPOSE**: flip the matrix " w.r.t. the diagonal "   $A = (a_{ij}) \Rightarrow A^T = (a_{ji})$

Example: $\begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$ ; $\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$

**CONJUGATE**: again component wise: $A = (a_{ij}) \Rightarrow \overline{A} = (\overline{a_{ij}})$

Example: $A = \begin{bmatrix} 2+i & 0 \\ 1 & 5-i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} \overline{2+i} & \overline{0} \\ \overline{1} & \overline{5-i} \end{bmatrix} = \begin{bmatrix} 2-i & 0 \\ 1 & 5+i \end{bmatrix}$

Rule: if the entries of a matrix are real $\Rightarrow A = \overline{A}$

$\Rightarrow$ **CONJUGATE-TRANSPOSE**: compose transposition & conjugation: $A = (a_{ij})$

$$\begin{tikzcd} & (a_{ji}) \arrow[dr, "\overline{(\ )}"] & \\ (a_{ij}) \arrow[ur, "(\ )^T"] \arrow[dr, "\overline{(\ )}"'] & \circlearrowright & (\overline{a_{ji}}) \\ & (\overline{a_{ij}}) \arrow[ur, "(\ )^T"'] & \end{tikzcd}$$

Example: $\begin{bmatrix} 2+i & \sqrt{3}-i \\ 5 & 7 \\ i & -13i \end{bmatrix}^* = \overline{\begin{bmatrix} 2+i & 5 & i \\ \sqrt{3}-i & 7 & -13i \end{bmatrix}} = \begin{bmatrix} 2-i & 5 & -i \\ \sqrt{3}+i & 7 & 13i \end{bmatrix}$

notation: $A^*$

**MULTIPLICATION**: $P = A \cdot B$   $A = m_A \left\{ \overbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}^{n_A} \right.$ , $B = m_B \left\{ \overbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}^{n_B} \right.$

If $n_A = m_B$ we can take the product and $p_{ij} = \sum_{k=1}^{n_A} a_{ik} b_{kj}$

Examples: $\begin{bmatrix} 4 & 3 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4+9 & 0+6 \\ 0-3 & 0-2 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ -3 & -2 \end{bmatrix}$

$\neq$

$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4+0 & 3+0 \\ 12+0 & 9-2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 12 & 7 \end{bmatrix}$

$\Rightarrow$ In particular, multiplication is NOT commutative.

Particular cases: VECTOR MULTIPLICATION : ⊛ dot product: $\vec{x} \cdot \vec{y} = x^T \cdot y = \sum_{i=1}^{n} x_i y_i$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ ⊛ scalar product: $\langle \vec{x}, \vec{y} \rangle = x \cdot \bar{y} = \sum_{i=1}^{n} x_i \bar{y}_i$

rmk: $y_i$ all real $\Rightarrow$ $\langle x, y \rangle = x \cdot y$.

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More advanced: INVERSE of a MATRIX : given $A$, the inverse of $A$ (if exists) is a matrix $B$ such that $A \cdot B = B \cdot A = I$ ; Denote $B = A^{-1}$.

How to find it (if $\exists$): you can use any method.

(I) if $A$ is 2x2 : $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\Rightarrow$ it has an inverse iff $\underset{(\det A)}{ad - cb} \neq 0$ & $A^{-1} = \frac{1}{ad-cb} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

rmk: there is an analogue in higher dimension but bad from the computational point of view

(II) $n \geq 3$ : Gauss elimination ⊛ pick your $A$ & form the augmented matrix $[A \mid I]$

Example: $\begin{bmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$

⊛ apply Row operations =
1. Interchange 2 rows
2. Multiply a row by $\alpha \in \mathbb{C} \setminus \{0\}$
3. add any multiple of one row to another one

For instance, in the example above: you can substitute $R_2$ with $R_2 - 3R_1$ & you obtain:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

rmk: you change this side as well.

*) Keep going till your augment matrix looks like $\left[\begin{array}{c:c|c} Id & * & \\ \hdashline 0 & 0 & B \end{array}\right]$. If on the LHS you don't have zeroes on the diagonal $\Rightarrow$ a) $A$ is invertible

b) $B$ is the inverse $A^{-1}$.

Finish the example:

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right] \xrightarrow[-\frac{R_3}{5}]{\frac{R_2}{2}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right] \xrightarrow{R_1 + R_2 + R_3} \left[\begin{array}{c|ccc} & \frac{3}{2} - \frac{4}{5} & \frac{2}{5} - \frac{1}{2} & \frac{1}{2} - \frac{1}{5} \\ Id & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$$

$$\Rightarrow \quad B = \begin{bmatrix} \frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} = A^{-1}$$

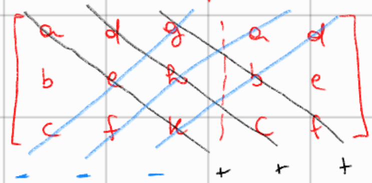
Criterion for a matrix to be invertible: $A \in M(n \times n; \mathbb{C})$ invertible iff $\det(A) \neq 0$.

## Review on determinant:

* Definition in 2x2 & 3x3 cases:

□ $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb$

□ $\det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} = aek + dhc + gbf + {}$
$- gec - dbk - hfa$



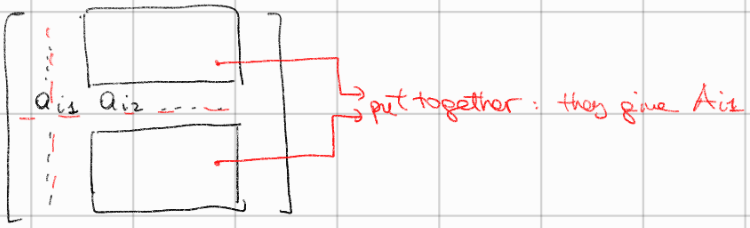
* General case: you can define det(matrix) in many way: we use a recursive definition.

  * det for 1x1, 2x2, 3x3 ✓

  * assume you have defined the determinant of a matrix up to dim $(n-1)\times(n-1)$.

then if $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ then $\det A = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + \dots$
$+ (-1)^{n+1} a_{n1} \det(A_{n1})$

$A_{i1}$ = matrix coming from $A$ after you remove the 1st column & the ith row:




Example: we compute the formula for a 3x3: $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} = A$

$\det(A) = a \cdot \det \begin{bmatrix} e & h \\ f & k \end{bmatrix} - b \det \begin{bmatrix} d & g \\ f & k \end{bmatrix} + c \det \begin{bmatrix} d & g \\ e & h \end{bmatrix}$ ✓

$= a(ek - hf) - b(dk - gf) + c(dh - ge) = aek + bgf + cdh - ahf - bdk - gec$

**Properties:**

* $\det(AB) = \det(A) \cdot \det(B)$
* $\det(\alpha A) = \alpha^{\dim(A)} \cdot \det(A)$ $\quad \dim(A) = n$ if $A$ is $n \times n$.
* $\det(A+B) \neq \det(A) + \det(B)$
* $\det(A^T) = \det(A)$
* $\det(\bar{A}) = \overline{\det(A)}$
* Block matrices. $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) . \det(C)$.

**Some results / applications of linear algebra:**

I) Matrices can be used to rewrite / solve systems of linear equations.

**Example:** Let $x_1, x_2, x_3 \in \mathbb{R}$ be real-valued variables.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 7 \\ -x_1 + x_2 - 2x_3 = -5 \\ 2x_1 - x_2 - x_3 = 4 \end{cases} \longleftrightarrow$$

linear algebra / matrix interpretation

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \quad \& \quad \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$$

**In general,**

$$\begin{matrix} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \\ a_{n1} x_1 + \dots + a_{nn} x_n = b_n \end{matrix} \longleftrightarrow \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{\vec{b}} \longleftrightarrow Ax = b$$

**Recall:** if $\det(A) \neq 0 \Rightarrow$ the system has a unique solution [indeed, $\det \neq 0 \Rightarrow \exists A^{-1}$ ] $\Rightarrow \vec{x} = A^{-1} b$ ]

**Recall:** you can use the Gauss elimination in order to solve a system in general.

Example: $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$. Represent the system as: $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ -1 & 1 & -2 & 1 \\ 2 & -1 & 3 & -5 \end{array}\right]$

Then try to obtain $\begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$ : $\overset{\text{DO}}{\circledast}$ $R_2 \to R_2 + R_1$ $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & -3 & -9 \end{array}\right]$ $\overset{\text{DO}}{*}$ $R_3 \to R_3 + 3R_1$

$\circledast$ $R_3 \to R_3 - 2R_1$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + x_3 = -4 \\ x_2 - x_3 = -3 \end{array}$

$x_3 = \alpha \Rightarrow x_1 = -\alpha - 4$, $x_2 = \alpha - 3 \Rightarrow \begin{pmatrix} -\alpha - 4 \\ \alpha - 3 \\ \alpha \end{pmatrix}$ is a solution $\forall \alpha \in \mathbb{R}$

II) Check if a set a vector is linearly dependent: recall: $v_1 \dots v_n$ are linearly independent vectors

iff. $\sum_{i=1}^{n} c_i v_i = 0$ $c_i$ constant $\Rightarrow c_i = 0\ \forall i$. Otherwise $\exists\, c_i$ constant not all zeroes: $\sum c_i v_i = 0$.

You can check $\{v_1 \dots v_n\}$ lin. independent $\Leftrightarrow$ $\begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix}$ putting the vectors as rows of a matrix

and then doing row Gauss-elimination: you get $(\mathrm{Id} \mid 0) \Leftrightarrow$ they are lin. independent.

In particular if $n$ = length of $v_i$ $\Rightarrow$ $n \times n$-matrix $\underbrace{\begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix}}_{= V}$ & they are lin. independent iff. $\det(V) \neq 0$.

Rmk: equivalently, you can put the vectors as columns of a matrix & then use column-Gauss elimination

Example: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_2$ $\Rightarrow$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$

Matrices with non-constant entries. we have already seen them: $f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_m(t) \end{pmatrix}$

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \qquad \text{example: } \begin{bmatrix} \sin(t) & t \\ 1 & \cos(t) \end{bmatrix}$$

(*) Everything that we have said / done so far for constant-entries matrix is true for a matrix of functions.

Moreover: def 1: $A(t)$ continuous if all $a_{ij}(t)$'s are

def 2: $A(t)$ differentiable if all $a_{ij}(t)$'s are : $A'(t) = \frac{dA(t)}{dt} = \left( \frac{d a_{ij}(t)}{dt} \right) = (a_{ij}'(t))$

Example: $\begin{pmatrix} \sin(t) & t \\ 1 & \cos(t) \end{pmatrix}' = \begin{pmatrix} \cos(t) & 1 \\ 0 & -\sin(t) \end{pmatrix}$

Properties: $\frac{d}{dt} \alpha A(t) = \alpha \frac{d}{dt} A$ ; $\frac{d}{dt}(A+B) = \frac{d}{dt}A + \frac{d}{dt}B$ ; $\frac{d}{dt} AB = \frac{dA}{dt}B + A\frac{dB}{dt}$.

(non-commutative) $\neq B\frac{dA}{dt} + \frac{dB}{dt}A$

def: in the same way, i.e. component wise, you can define $\int A(t)\,dt$ - primitive &

$\int_a^b A(t)\,dt$ the definite integral:

Example: $$\int \begin{pmatrix} \sin(t) & t \\ 1 & \cos(t) \end{pmatrix} dt = \begin{pmatrix} -\cos(t) + k_{11} & \frac{t^2}{2} + k_{12} \\ t + k_{21} & \sin(t) + k_{22} \end{pmatrix} = \begin{pmatrix} -\cos(t) & t^2/2 \\ t & \sin(t) \end{pmatrix} + \underset{\substack{\uparrow \\ 2\times 2 \text{ - matrix}}}{K}$$

Eigenvalues / eigenvectors: $A$: $n\times n$ -matrix

def: $\lambda \in \mathbb{C}$ is an eigenvalue for $A$ if $\exists x \in \mathbb{R}^n \setminus \{0\}$ s.t. $A\vec{x} = \lambda \vec{x}$. In this case we say that $\vec{x}$ is an eigenvector of $\lambda$.

how to find the set of eigenvalues of $A$? $\exists x \in \mathbb{R}^n \setminus \vec{0}$: $Ax = \lambda x$ $\Leftrightarrow$ $\exists x \in \mathbb{R}^n \setminus \vec{0}$: $(A - \lambda Id)x = 0$

$\Leftrightarrow$ $(A - \lambda Id)$ has a non-trivial kernel. $\Leftrightarrow$ $\det(A - \lambda Id) = 0$

Def: $\det(A - t Id) =: p_A(t)$ the characteristic polynomial of $A$ & the eigenvalues are the roots of it

Example: $A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \Rightarrow A - t Id = \begin{bmatrix} 3-t & -1 \\ 4 & -2-t \end{bmatrix} \Rightarrow p_A(t) = (3-t)(-2-t) + 4$

$= -6 + 2t - 3t + t^2 + 4 = t^2 - t - 2$

$= (t-2)(t+1) \Rightarrow \lambda = 2, -1$ are the eigenvalues.

Given $\lambda$ eigenvalue, how to find all its corresponding eigenvectors? We need to solve the system $(A - \lambda Id)x = 0$!

In the example: $\underline{\lambda = 2}$ $\Rightarrow$ $A - 2\cdot Id = \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix}$ : $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 4 & -4 & 0 \end{array}\right) \xrightarrow{R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ $x_1 - x_2 = 0$ is the only condition $\Rightarrow x_2 = \alpha$, $x_1 = \alpha$ $\Rightarrow \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \text{Span}\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \setminus \{\vec{0}\}$ are all the eigenvectors.

$\lambda = -1$ $\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} \Rightarrow \left(\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array}\right) \xrightarrow{R_2 - 4R_1} 4x_1 = x_2 \Rightarrow \begin{pmatrix} \alpha \\ 4\alpha \end{pmatrix} \Rightarrow \alpha \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\Rightarrow \text{Span}\left\langle \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\rangle \setminus \{\vec{0}\}$.

Example: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find eigenvalues / eigenvectors.

$* \; A - \lambda Id = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \Rightarrow p_A(t) = det(A - \lambda Id) = (-\lambda)^3 + 1 + 1 - (-\lambda - \lambda - \lambda)$

$= -\lambda^3 + 2 + 3\lambda$

Find the roots to $\lambda^3 - 3\lambda - 2 = 0$

Trick: always try integers $\alpha$ : $\alpha \mid$ known term (in this case $-2$)

$\Rightarrow \alpha = \pm 1, \pm 2 \Rightarrow \alpha = 1 : 1 - 3 - 2$ ✗ ; $\alpha = -1 : -1 + 3 - 2 = 0$ ✓ $\Rightarrow (\lambda+1) \mid \lambda^3 - 3\lambda - 2$

$\Rightarrow$ polynomial division.

$$\begin{array}{l|l} \lambda^3 - 3\lambda - 2 & \lambda + 1 \\ \hline -\lambda^3 - \lambda^2 & \lambda^2 - \lambda - 2 \\ \hline -\lambda^2 - 3\lambda - 2 & \\ \lambda^2 + \lambda & \\ \hline -2(\lambda+1) & \end{array}$$

$\Rightarrow (\lambda+1)\underbrace{(\lambda^2 - \lambda - 2)}_{= (\lambda-2)(\lambda+1)}$

$\Rightarrow (\lambda+1)^2(\lambda-2) = 0 \Rightarrow \lambda = -1, 2.$

Rmk: we say that $-1$ is an eigenvalue with algebraic multiplicity 2

" " " " 2 " " " " " " " " 1.

def: if $(t-\lambda)^m \mid p_A(t)$ but $(t-\lambda)^{m+1} \nmid p_A(t)$ ($\& \; m \geq 1$) $\Rightarrow \lambda$ is an eigenvalue of algebraic multiplicity $m$.

Notation: $\mu_a(\lambda)$

Find eigenvectors for $\lambda = 2$ $\rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$

$\Rightarrow \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow Span \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle \setminus \{0\}.$

Find eigenvectors for $\lambda = -1$ : $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$.

$$\begin{pmatrix} -\alpha-\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \therefore \quad \text{Span} \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \setminus \{0\}.$$

we say that the geometric multiplicity of $-1$

def: For an eigenvalue $\lambda$, the number of linearly independent eigenvectors is the geometric multiplicity of $\lambda$. Notation: $\mu_g(\lambda)$

Proposition: $\mu_a(\lambda) \geq \mu_g(\lambda)$.

BACK to ODEs: I) Let's first address how to solve $x' = Ax$. (& consequently $\begin{cases} x' = Ax \\ x(t_0) = \vec{v}_0 \end{cases}$

RECALL THEOREMS: 1) Superposition: still true that $\vec{x}_1$ & $\vec{x}_2$ solutions of $\vec{x}' - A\vec{x} = \vec{0}$

$\Rightarrow C_1\vec{x}_1 + C_2\vec{x}_2$ solution as well (because the ODE is linear & homogeneous)

2) $\exists$ / ! / Maximal interval of $\exists$: $\exists!$ solution to any IVP & $I = \mathbb{R}$. (A is a constant matrix $\Rightarrow$ it is continuous everywhere)

3) The set of solutions to $x' - Ax = 0$ forms a vector space of dim $= \dim(A) = n$. This means:

(a) $\exists$ $\vec{x}_1 \dots \vec{x}_n$ solutions to $x' - Ax = 0$ s.t. any other solution can be written as

$\phi(t) = \sum_{i=1}^{n} C_i \vec{x}_i(t)$ (for some $C_i$)

(b) $\vec{x}_1 \dots \vec{x}_n$ are linearly independent, namely $\sum_{i=1}^{n} c_i \vec{x}_i(t) = 0 \Rightarrow c_i = 0 \quad \forall i = 1 \dots n$.

DEF: we call any set of lin. ind. solutions of cardinality $n$ a fundamental set of solutions.

4) Wronskian: moreover we can check $\vec{x}_1(t) \dots \vec{x}_n(t)$ to be a fundamental set of solution.

taking the $\det \begin{bmatrix} | & & | \\ \vec{x}_1(t) & \dots & \vec{x}_n(t) \\ | & & | \end{bmatrix} =: W(\vec{x}_1, \dots, \vec{x}_n)$ - called the Wronskian of $\vec{x}_1 \dots \vec{x}_n$.

It is a fundamental set iff $W(t) \neq 0$.

Rmk: the proofs of these theorems are analogous to the case $y'' + p(t)y' + q(t)y = 0$.

Conclusion: so now we know that we need to find $n$ solutions & if $W \neq 0$ $\Rightarrow$ we have done.

(1) Find candidates for the solutions: we want to use the same ansatz as for the scalar case

$\rightsquigarrow e^{rt}$ with $r \in \mathbb{C}$ but we need a vector ($\vec{x}(t)$ needs to be a vector).

$$\Rightarrow x(t) = e^{rt} \cdot \vec{v}, \quad \vec{v} \text{ constant vector}.$$

necessary condition to be a solution: $x'(t) = \begin{pmatrix} e^{rt} v_1 \\ \vdots \\ e^{rt} v_n \end{pmatrix}' = \begin{pmatrix} r e^{rt} v_1 \\ \vdots \\ r e^{rt} v_n \end{pmatrix} = r e^{rt} \vec{v}$

$\Rightarrow r e^{rt} \vec{v} = A e^{rt} \vec{v} \iff r \vec{v} = A \vec{v} \iff$ $r$ is an eigenvalue for $A$ and

$\vec{v}$ is an eigenvector w.r.t. $r$.

$\Rightarrow$ therefore in order to find solutions to $x' = Ax$ we need to find:

* the eigenvalues $\longrightarrow$ find roots to $p_A(\lambda) = \det(A - \lambda Id)$
* their eigenvectors $\longrightarrow$ solve $(A - \lambda Id)\vec{v} = 0$.

How to make sure that we have found all of them?

Results from linear algebra: eigenvectors from different eigenvalues are linearly independent.

Moral: if $A$ has $n$ lin. independent eigenvector $\Rightarrow$ Wronskian $\neq 0$.

$\Rightarrow$ we have found a fundamental set of sol.s

Flow-chart in order to find a fundamental system of solutions.

$\boxed{x' = Ax}$ ← $n \times n$ (usually $3\times3$ & $2\times2$)

find eigenvalues: $\lambda_1 \dots \lambda_k$ with alg. multiplicity $\mu_a(\lambda_1) \dots \mu_a(\lambda_k)$

rmk: $\prod_{i=1}^{k} (t - \lambda_i)^{\mu_a(\lambda_i)} = \pm p_A(t) \Rightarrow \boxed{\sum \mu_a(\lambda_k) = n}$

all $\lambda_i$ are real

some $\lambda_i$ are complex

rmk: they come in pairs $(\lambda, \bar{\lambda})$

All $\mu_a(\lambda_i) = 1$.

$\Downarrow$

$\forall \lambda_i$, $\{v : Av = \lambda_i v\} = \langle v_i \rangle$

$\Rightarrow$ the general solution is

$$\sum_{i=1}^{n} c_i e^{\lambda_i t} \vec{v}_i$$

$\mu_a(\lambda_i) \geq 1$ but $\mu_a(\lambda_i) = \mu_g(\lambda_i)$

$\Downarrow$

$\forall \lambda_i$, $\{v : Av = \lambda_i v\} = \langle v_i^{(1)}, \dots, v_i^{(\mu_g)} \rangle$

$\Rightarrow$ the general solution is

$$\sum_{i=1}^{k} e^{\lambda_i t} [c_i^{(1)} v_i^{(1)} + \dots + c_i^{(\mu_g)} v_i^{(\mu_g)}]$$

$\exists i : \mu_a(\lambda_i) > \mu_g(\lambda_i)$

we'll see how to deal with these 2 cases

Example: $x' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} x$. Find solutions of the form $e^{rt} v$.

(*) eigenvalues r: $\det \begin{bmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{bmatrix} \Rightarrow (\lambda+3)(\lambda+2) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1)$

$$\underbrace{\Rightarrow \lambda = -1, -4.}$$

(*) find eigenvectors: $\boxed{r=-1}$ : $(A - \lambda Id)v = 0$ : $(A + Id)v = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} v = 0$

$$\left[\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right] \xrightarrow[\text{Gauss elimination}]{} \left[\begin{array}{cc|c} 1 & -\frac{1}{\sqrt{2}} & 0 \\ 1 & -\frac{1}{\sqrt{2}} & 0 \end{array}\right] \Rightarrow v_1 - \frac{v_2}{\sqrt{2}} = 0 \Rightarrow v_1 = \frac{v_2}{\sqrt{2}}$$

$$\Rightarrow \begin{pmatrix} \alpha/\sqrt{2} \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1/\sqrt{2} \\ 1 \end{pmatrix}$$

$\Rightarrow v = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \Rightarrow$ a solution is $e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \vec{x}_1$

$\boxed{r=-4} \Rightarrow (A + 4Id)v = 0$ : $\left[\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{array}\right] \Rightarrow v_1 + \sqrt{2} v_2 = 0 \Rightarrow \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = v \Rightarrow e^{-4t}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = \vec{x}_2$

$\Rightarrow$ the general solution is: $c_1 e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$.

Example 2: $\vec{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}$ : (*) eigenvalues of $A$: $\det(A - \lambda Id) = 0$.

$$A - \lambda Id = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} \Rightarrow p_A(\lambda) = -\lambda(3-\lambda)^2 + 16 + 16 - \left(-16\lambda + 4(3-\lambda) + 4(3-\lambda)\right) =$$

$$= -\lambda(9 - 6\lambda + \lambda^2) + 32 + 16\lambda - 8(3-\lambda) = -9\lambda + 6\lambda^2 - \lambda^3 + 32 + 16\lambda - 24 + 8\lambda =$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0.$$

Apply trick for integer roots: $\pm 1, \pm 2, \pm 4, \pm 8$ : $\lambda = 1$ : $-1+6+15+8$ ✗

$\lambda = -1$ : $1 + 6 - 15 + 8 \Rightarrow$ ✓

$$\begin{array}{l|l} -\lambda^3+6\lambda^2+15\lambda+8 & \lambda+1 \\ \hline \lambda^3+\lambda^2 & -\lambda^2+7\lambda+8 \\ 0+7\lambda^2 & \\ -7\lambda^2-7\lambda & \\ 0 \quad 8(\lambda+1) & \end{array}$$

$\Rightarrow -p_A(\lambda) = (\lambda+1)(\lambda^2 - 7\lambda - 8)$

$\overset{!}{=} (\lambda+1)(\lambda+1)(\lambda-8) \Rightarrow$ $r=-1,\ \mu_a(-1)=2$ ; $r=8,\ \mu_a(8)=1$

⊛ Find eigenvectors for $\boxed{r=-1}$ : solve $(A+Id)v=0$ : $\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$

$R_3 - 2R_2$ & $\Rightarrow R_1 - 2R_2$ : $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow 2V_1 + V_2 + 2V_3 = 0 \Rightarrow V_2 = -2V_1 - 2V_3 \Rightarrow \begin{pmatrix} \alpha \\ -2\alpha - 2\beta \\ \beta \end{pmatrix}$

$\Rightarrow \alpha \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \Rightarrow e^{-t}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ & $e^{-t}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ are solutions.

$\Uparrow$

$\mu_g(-1)=2$

for $\boxed{r=8}$ : solve $(A - 8Id)v=0$ : $\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$

$R_1 - R_2 \Rightarrow \left[\begin{array}{ccc|c} -9 & 0 & 9 & 0 \\ 1 & -4 & 1 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \Rightarrow$ $V_1 = V_3$, $2V_2 = V_3$ : $\begin{pmatrix} 2\alpha \\ \alpha \\ 2\alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow e^{8t}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is a solution.

$\Rightarrow$ In particular $\left\{ e^{-t}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, e^{-t}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, e^{8t}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a fundamental set of solutions.

Before the two cases left open ( $\mu_a > \mu_g$ & complex root ) , let's see the setting (II)

(II) $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$ : pose $x_1 = y$ $: \mathbb{R} \to \mathbb{R}$

$x_2 = y'$

$\vdots$

$x_n = y^{(n-1)}$

$\Rightarrow$ $x_1' = x_2$

$x_2' = x_3$

$\vdots$

$x_n' = y^{(n)} = \frac{-1}{a_n} [ a_{n-1} x_n + \dots + a_1 x_2 + a_0 x_1 ]$

$$\Rightarrow \quad x' = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\frac{a_0}{a_n} & \cdots & & -\frac{a_{n-1}}{a_n} \end{bmatrix} x$$

$\Rightarrow$ all the things about $\exists$ / ! / superposition / wronskian / Ansatz $e^{rt}$ $\leftarrow$ all true in this case.

Therefore : also in this case : (*) look at the roots of the characteristic polynomial

$p(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 = 0$ [ which btw coincides with the determinant of the above matrix ]

(*) $a_n \cdot \prod_{i=1}^{n} (t - \lambda_i)^{\mu_a(\lambda_i)} = p(t)$ —— some $\lambda_i$ are complex

$\lambda_i$ all real

— all different ( $\mu_a(\lambda_i) = 1$ ) $\Rightarrow$ general solution: $C_1 e^{\lambda_1 t} + \dots + C_n e^{\lambda_n(t)}$ .

— some repetition ( $\mu_a(\lambda_i) \geq 1$ ) $\Rightarrow$ $\forall \lambda_i$ : $\mu_a(\lambda_i) = m_i > 1$

$\Rightarrow$ $e^{\lambda_i t}, t e^{\lambda_i t}, \dots, t^{m_i - 1} e^{\lambda_i t}$