Today: Finish discussion about higher-order case } § 4.1 - § 4.2)

Examples of IVPs.

Review for the midterm: ⊛ Flow chart

⊛ Tips to avoid common mistakes

⊛ Integration tricks

Resume: higher-order - scalar - linear ODE with constant coeff.s & homogeneous

(II) $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$: it can be seen as a particular case of a 1-st order - SYSTEM - linear ODE.

Indeed: pose $x_1 = y : \mathbb{R} \to \mathbb{R}$, $x_2 = y', \dots, x_i = y^{(i-1)}, \dots, x_n = y^{(n-1)}$. Then the original ODE becomes: $y^{(n)} = -\frac{a_{n-1}}{a_n} y^{(n-1)} - \frac{a_{n-2}}{a_n} y^{(n-2)} - \dots - \frac{a_1}{a_n} y' - \frac{a_0}{a_n} y$

$$\Rightarrow y^{(n)} = (y^{(n-1)})' = \begin{cases} x_n' = -\frac{a_{n-1}}{a_n} x_n - \frac{a_{n-2}}{a_n} x_{n-1} - \dots - \frac{a_1}{a_n} x_2 - \frac{a_0}{a_n} x_1 \\ x_{n-1}' = x_n \\ x_{n-2}' = x_{n-1} \\ \vdots \\ x_1' = x_2 \end{cases}$$

Then we need to remember how the x_i's are related to each other

$$\Rightarrow x' = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\frac{a_0}{a_n} & \cdots & & -\frac{a_{n-1}}{a_n} \end{bmatrix} x$$

Pose $X = \begin{pmatrix} x_n \\ \vdots \\ x_1 \end{pmatrix} \Rightarrow X' = \begin{pmatrix} -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & \cdots & -\frac{a_0}{a_n} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} X$

$\underbrace{}_{= Id_{n-1}}$

Example: $y^{(3)} - 2y'' + y' + 7y = 0$: $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$\Rightarrow$ $x_3' - 2x_3 + x_2 + 7x_1 = 0$
$x_2' = x_3$
$x_1' = x_2$

$\Rightarrow X' = \begin{pmatrix} 2x_3 - x_2 - 7x_1 \\ x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$

$\Rightarrow$ All the thms / strategies seen last time are still valid for this particular case.

Let's see how they translate

(*) Superposition: φ & ψ solutions to $a_n y^{(n)} + \ldots + a_0 y = 0$ $\Rightarrow$ any $c_1\varphi + c_2\psi$ solution as well.

Rmk: also IVPs can be translated $\begin{cases} ODE \\ y(t_0) = u_0 \\ \vdots \\ y^{(n-1)}(t_0) = u_{n-1} \end{cases} \Rightarrow \begin{cases} X' = AX \\ X(t_0) = \begin{pmatrix} x_n(t_0) \\ \vdots \\ x_1(t_0) \end{pmatrix} = \begin{pmatrix} u_{n-1} \\ \vdots \\ u_0 \end{pmatrix} \end{cases}$

(*) $\exists!$ / maximal I of $\exists$: any IVP: $\begin{cases} ODE \\ y(t_0) = u_0 \\ \vdots \\ y^{(n-1)}(t_0) = u_{n-1} \end{cases}$ has unique solution & that solution is defined over $\mathbb{R}$.

(*) Set of solutions = vector space of dim n. & you can check if $\{\varphi_1, \ldots, \varphi_n\}$ is a basis computing the Wronskian:

$$\left\{\begin{array}{c}\text{Solution to the higher order ODE}\\[1em] \varphi_1 \; \dots \; \varphi_n\end{array}\right\} \iff \begin{pmatrix}\varphi_1^{(n-1)}\\ \vdots \\ \varphi_1\end{pmatrix} = x_1 \,, \dots , \begin{pmatrix}\varphi_n^{(n-1)}\\ \vdots \\ \varphi_n\end{pmatrix} = x_n \quad \text{solutions to the associated system}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

lin. independent $\iff$ lin. independent.

$$\Updownarrow$$

$$\det\begin{pmatrix}\varphi_1 & \dots & \varphi_n\\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & & \varphi_n^{(n-1)}\end{pmatrix} \neq 0 \quad \Longleftarrow \quad W(x_1 \dots x_n) \neq 0$$

$=: W(\varphi_1 \dots \varphi_n)$ (definition of the Wronskian)

Rmk: this recovers the 2nd-order case $W(\varphi,\psi) = \det\begin{pmatrix}\varphi & \psi \\ \varphi' & \psi'\end{pmatrix}$. (the same as above, after setting $n=2$).

⊛ The Ansatz: exponential again: indeed the Ansatz for $x' = Ax$ is $e^{rt}\vec{v}$.

$$\Rightarrow \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}e^{rt}v_1\\ \vdots \\ e^{rt}v_n\end{pmatrix} \quad \& \quad y = x_1 \Rightarrow e^{rt}\cdot\text{constant} \Rightarrow e^{rt} \text{ is the right Ansatz also in this case.}$$

The right r? $\Rightarrow$ Characteristic polynomial. (enough to plug in $y = e^{rt}$)

$\Rightarrow p(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$ & the roots are the right exponents.

Rmk:
$$\det\begin{pmatrix} -\frac{a_{n-1}}{a_n} - t & -\frac{a_{n-2}}{a_n} & \dots & -\frac{a_0}{a_n} \\ 1 & -t & 0 \dots & 0 \\ 0 & 1 & -t & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & \\ 0 & \dots & 0 \;\; 1 & -t \end{pmatrix} = \left(-\frac{a_{n-1}}{a_n} - t\right)\det\begin{pmatrix} -t & & 0 \\ 1 & \ddots & \\ 0 & 1 & -t\end{pmatrix}$$

$$- \underbrace{\det\begin{pmatrix} -\frac{a_{n-2}}{a_n} & \dots & -\frac{a_0}{a_n} \\ 1 & -t & 0 \\ 0 & 1 & -t \end{pmatrix}}$$

$$= -\frac{a_{n-1}}{a_n} \cdot (-1)^{n-1} t^{n-1} - (-1)^{n-1} t^n - \left[-\frac{a_{n-2}}{a_n} (-1)^{n-2} t^{n-2} - \det \begin{pmatrix} -\frac{a_{n-3}}{a_n} & -t & \cdots & -\frac{a_0}{a_n} \\ 1 & -t & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & -t \end{pmatrix} \right]$$

same type as before but

$(n-2)\times(n-2)$ instead of $(n-1)\times(n-1)$

$$\Rightarrow (-1)^n \left[\frac{a_{n-1} t^{n-1} + t^n + a_{n-2} t^{n-2}}{a_n} \right] + \det\begin{pmatrix} \\ \end{pmatrix} \quad \text{etc} \ldots$$

one by one we recover all the terms in the polynomial.

$\Rightarrow$ (*) $a_n \prod_{i=1}^{n} (t-\lambda_i)^{\mu_a(\lambda_i)} = p(t)$ — some λ_i are complex

λ_i all real

— all different ($\mu_a(\lambda_i)=1$) $\Rightarrow$ general solution: $C_1 e^{\lambda_1 t} + \ldots + C_n e^{\lambda_n(t)}$.

— some repetition ($\mu_a(\lambda_i) \geq 1$) $\Rightarrow$ $\forall \lambda_i$: $\mu_a(\lambda_i) = m_i > 1$

$\Rightarrow e^{\lambda_i t}, t e^{\lambda_i t}, \ldots, t^{m_i-1} e^{\lambda_i t}$

Examples (1) $y^{(4)} - y^{(2)} = 0$. char. polynomial: $t^4 - t^2 = 0 \Rightarrow t^2(t^2-1) = 0 \Rightarrow t=0 \quad \mu_a(0)=2$

$t=1 \quad \mu_a(1)=1$

$t=-1 \quad \mu_a(-1)=1$

$\Rightarrow$ the general solution is $C_1 + C_2 t + C_3 e^t + C_4 e^{-t}$.

(2) $y^{(5)} - 2y^{(4)} + y^{(3)} = 0$. char. polynomial $t^5 - 2t^4 + t^3 = 0 \Rightarrow t^3(t^2-2t+1) = 0$

$\quad\quad \parallel$

$t^3 (t-1)^2 = 0$

$\Rightarrow \lambda_1 = 0$ & $\mu_a = 3$ & $\lambda_2 = 1$ & $\mu_a = 2$

$\Rightarrow$ the general solution is $\underbrace{c_1 + c_2 t + c_3 t^2}_{\text{from } \lambda_1 = 0} + \underbrace{c_4 e^t + c_5 t \cdot e^t}_{\text{from } \lambda_2 = 1}$.

Example of an IVP with matrices: $x' = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(*) Solve the system: 1st) $\det \begin{pmatrix} 1-t & -1 \\ -1 & 1-t \end{pmatrix} = (1-t)^2 - 1 = t^2 - 2t + 1 - 1 = t^2 - 2t = t(t-2)$

$\Rightarrow \boxed{\lambda = 0 \ \& \ \lambda = 2}$

$\lambda = 0$ $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} v = 0$: enough $v_1 - v_2 = 0$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works fine

$\lambda = 2$ $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v = 0$: enough $v_1 + v_2 = 0$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

$\Rightarrow$ general solution $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

constant vector.

Finally: use IVP to find c_1 & c_2: $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 1 \end{cases}$

$\Rightarrow c_1 = -c_2$ $\quad -2c_2 = 1 \Rightarrow c_2 = -\frac{1}{2}$ & $c_1 = \frac{1}{2}$ $\Rightarrow$ the (unique) solution is $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Review for the midterm: 3 types of questions: I) qualitative question = qualitative discussion

II) $\exists / ! $ / Maximal Interval of $\exists$ Q.

III) solve "THIS" Question

III Look at the ODE & understand 3 things: $\boxed{\text{scalar / system}}$, $\boxed{\text{order}}$, $\boxed{\text{linearity}}$

- $\boxed{\text{System:}}$ so far only one type: $x' = Ax$ with A constant. $\to$ $\exists / !$ / maximal I: always & $I = \mathbb{R}$ (II)
  - $\to$ So far, no qualitative discussion (no (I))
- $\boxed{\text{Scalar:}}$ order?
  - $\boxed{\text{1st}}$ — linear: $a_1(t)\,y' + a_0(t)\,y = g(t)$ ($\exists$! solution when $a(t) = -\frac{a_0}{a_1}(t)$ & $b(t) = \frac{g(t)}{a_1(t)}$ are continuous)
    - non-linear
      - separable: $M(y)\,y' = N(t)$
      - exact: $M(t,y) + N(t,y)\,y' = 0$
      - non-exact but μ: same form above.
      - $\exists / !$ local $\Rightarrow$ enough to check G & $\frac{\partial G}{\partial y}$ BUT for the maximal interval of $\exists$, need either implicit $\leadsto$ explicit solution or implicit function theorem + study of the "implicit locus"
  - $\boxed{\text{2nd}}$ — non-linear $\Rightarrow$ not treated so far
    - linear
      - $g \equiv 0$
        - const. coeff.s $\to$ char. polynomial
        - non const. coeff.s: method of reduction of order (or) Abel's thm.
      - $g \neq 0$
        - const. coeff. + nice g: Undermined coeff.
        - const. coeff. + bad g
        - non-constant coeff.
        - Variation of parameters.

higher order - $\boxed{\text{cont. coeff. + hom. + linear}}$ - char polynomial.

Tips for 1st order scalar linear ODE: *) Remember to put it in the standard form

$$a_1(t)y' + a_0(t)y = g(t) \quad \rightsquigarrow \quad y' = -\frac{a_0(t)}{a_1(t)}y + \frac{g(t)}{a_1(t)}$$

*) Make sure at the previous step that you avoided the locus where $a_1(t)=0$!

*) Make sure to use the formula with the right signs & remember the constant.

$$y(t) = e^{\int a(t)dt}\left[\int e^{-\int a(t)dt} b(t)dt + K\right]$$

Tips for 2nd-order scalar-linear: *) make sure to put everything in the standard form.

$$\left[y'' + p(t)y' + q(t)y = g(t)\right]$$

*) If $p(t)$ or $q(t)$ are non constant, even if $g(t) \in$ "Table" you need to use Variation of parameters.

Example: $t^2y'' - t(t+2)y' + (t+2)y = t^3$, $t>1$.

a) show $y_1(t) = t$ is a solution for the homogeneous one

b) find fundamental set of solutions for the homogeneous

c) find a particular solution.

a) $y_1 = t$, $y_1' = 1$, $y_1'' = 0$ $\Rightarrow$ $0 - t(t+2) + (t+2)t = 0$ ✓

b) Make it into the standard form: 1st of all, I can divide by t^2 since $t > 1$:

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$$

Abel's thm: $W(y_1, \psi) = C \cdot e^{-\int p(t)dt}$: $t\cdot\psi' - \psi = C\cdot e^{\int \frac{t+2}{t}dt} = C\cdot e^{\int 1 + \frac{2}{t}dt}$

$\Rightarrow$ $t\psi' - \psi = C\cdot e^{t + 2\ln|t|}$ since $t>1$, I can remove $|.|$ & divide by t

$$\psi' = \frac{1}{t}\psi + C\cdot e^t \cdot \frac{t^2}{t} = \frac{\psi}{t} + C\cdot t\cdot e^t$$

$$\Rightarrow \psi(t) = e^{\int \frac{1}{t}dt}\left[\int e^{-\int \frac{1}{t}dt}\cdot C\cdot t\cdot e^t\, dt + K\right]$$

$$= e^{\ln(t)}\left[\int e^{-\ln(t)}\cdot C\cdot t\cdot e^t\, dt + K\right]$$

$$= t\left[C\int e^t dt + K\right] = Cte^t + Kt.$$ $\Rightarrow$ pose $K=0$ & $C=1$

$\Rightarrow$ $\{t, te^t\}$ is a fund. set.

c) standard form: $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = t$ ← nice & but non-constant coeff's.

$\Rightarrow$ Variation of parameters: $Y_p = -t\int_{\bar{t}}^{t} \frac{s\cdot e^s\cdot s}{e^s\cdot s^2}ds + t\cdot e^t\int_{\bar{t}}^{t}\frac{s\cdot s}{e^s s^2}ds$

$W(t, te^t) = e^t t^2$

$$= -t\int_{\bar{t}}^{t} 1\, ds + te^t\int_{\bar{t}}^{t} e^{-s}ds = -t[t-2] + te^t\left[-e^{-t} + e^{-2}\right]$$

Pose $t = 2$ $\qquad = -t^2 + 2t + t + e^{-2}t\cdot e^t$

Rmk: You can use indefinite integrals instead of definite ones [the part which comes from the constant E is a solution to the homogeneous eq.].

(I)

Integration tricks: $\frac{1}{\text{polynomial}}$: 1) $\frac{1}{ax+b} \to \frac{1}{a(x+\frac{b}{a})} \to \frac{1}{a} \cdot \frac{1}{x+\frac{b}{a}}$

the primitive is $\ln|x+\frac{b}{a}|$ ← absolute value

2) $\frac{1}{x^2+ax+b}$: 3 cases: $\Delta>0$ poly $=(x-\alpha_1)(x-\alpha_2)$ & $\alpha_1 \neq \alpha_2$

$\Delta=0$ poly $=(x-\alpha)^2$ $\Delta<0$ complex roots.

$\Delta>0$: $\Rightarrow$ $\frac{(x-\alpha_1)-(x-\alpha_2)}{x^2+ax+b} \frac{1}{(\alpha_2-\alpha_1)}$ is the same expression but now

$\Rightarrow \frac{1}{(\alpha_2-\alpha_1)} \cdot \left[\frac{1}{x-\alpha_2} - \frac{1}{x-\alpha_1}\right]$ ← reduced to the previous case.

Example: $\frac{1}{(x-1)(x-3)} = \frac{(x-3)-(x-1)}{2(x-1)(x-3)} = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x-3}\right] \Rightarrow \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x-3|$

$\Delta=0$ $\Rightarrow$ $\frac{1}{(x-\alpha)^2} \Rightarrow -\frac{1}{(x-\alpha)}$

$\Delta<0$, rewrite $x^2+ax+b = x^2+ax+\frac{a^2}{4}+b-\frac{a^2}{4} = (x+\frac{a}{2})^2 + b-\frac{a^2}{4}$

rmk: $\Delta < 0$: it means $a^2 - 4b < 0 \Rightarrow 4b - a^2 > 0 \Rightarrow b - \frac{a^2}{4} > 0$ Call it $\frac{1}{k^2}$ (it is > 0)

$$\Rightarrow \frac{1}{\frac{1}{k^2} + \left(x + \frac{a}{2}\right)^2} = k^2 \cdot \left[\frac{1}{1 + k^2\left(x + \frac{a}{2}\right)^2} \right].$$

Recall: primitive of $\frac{1}{1 + c^2x^2}$ is $\frac{1}{c}\arctan(cx)$

$c > 0$

$\Rightarrow k^2 \cdot \left[\frac{1}{k} \arctan\left(k \cdot \left(x + \frac{a}{2}\right)\right)\right]$ is the primitive.

II Useful trigonometric identities: $\cos^2 t + \sin^2 t = 1$

$$\cos(2t) = 2\cos^2 t - 1 = 1 - 2\sin^2(t) = \cos^2 - \sin^2$$

$$\sin(2t) = 2\sin(t)\cos(t)$$

For instance: $\int \sin^2(t) \cdot e^t \, dt = \int \frac{e^t}{2} - \frac{\cos(2t)e^t}{2} \, dt.$

use $\cos(2t) = 1 - 2\sin^2(t)$

$\Rightarrow \sin^2(t) = \frac{1 - \cos(2t)}{2}$

III Integration by parts: $\int fg \, dt = fG - \int f'G \, dt$

remember the sign

$= Fg - \int Fg' \, dt$

This case: $\int \cos(2t) e^t \, dt = \frac{1}{2}\sin(2t) \cdot e^t - \int \frac{1}{2}\sin(2t)e^t \, dt = \frac{1}{2}\sin(2t)e^t - \left[-\frac{1}{4}\cos(2t)e^t + \right.$

$- \int -\frac{1}{4}\cos(2t)\cdot e^t dt \Big] = \frac{1}{2}\sin(2t)e^t + \frac{1}{4}\cos(2t)e^t - \frac{1}{4}\int \cos(2t)\, e^t dt$

$\Rightarrow \frac{5}{4}\int \cos(2t)\, e^t\, dt = e^t\left[\frac{\sin(2t)}{2} + \frac{\cos(2t)}{4}\right]$ ← in the end remember always to put the constant.

(IV) Substitution: it is a situation where $\int f(g(t))\, g'(t)\, dt$ ⇝ first $F(x) = \int f(x)\, dx$ & then $\boxed{F(g(t))}$

Rule: if the $\int$ is definite you need to change the extreme of integration!

$$\int_a^b f(g(t))\, g'\, dt = \int_{g(a)}^{g(b)} f(x)\, dx$$

$x = g(t)$: $t = b \Rightarrow g(b) = x$

$t = a \Rightarrow g(a) = x$

Exemple: (a modification of an exercise in Practice Midterm 2).

$$\begin{cases} y'(t) = e^y + 3e^{-y} - 4 \\ y(0) = y_0 \end{cases}$$

(*) Find the solution.

(1st) $e^y + 3e^{-y} - 4$ has continuous derivate w.r.t. y ⇒ P-L theorem applies everywhere.

(2nd) Start solving it: it is 1st order - scalar - non linear but separable.

two situations: ▫ y_0 is such that $e^{y_0} + 3e^{-y_0} + 4 = 0$ ⇒ $y \equiv y_0$ is THE solution.

▫ y_0 is NOT ~ ~ ~ ~ ~ ~ ~ ~ ~ ⇝ around y_0 we can divide by it:

$$\int \frac{1}{e^y + 3e^{-y} - 4}\, dy = t + K$$

$\uparrow$

$x := e^y \Rightarrow \displaystyle\int \frac{1}{x + \frac{3}{x} - 4} \cdot \frac{dx}{x} = \int \frac{1}{x^2 - 4x + 3}\, dx = \int \frac{1}{(x-3)(x-1)}\, dx$

$\Downarrow$

$\ln(x) = y$

$\Rightarrow y' = \frac{1}{x} \Rightarrow dy = \frac{dx}{x}$ $\qquad \Rightarrow \frac{1}{2} \displaystyle\int \frac{(x-1)-(x-3)}{(x-3)(x-1)}\, dx = \frac{1}{2} \int \frac{1}{(x-3)} - \frac{1}{(x-1)}\, dx$

$\Rightarrow \frac{1}{2} \ln|e^y - 3| + \frac{1}{2} \ln|e^y - 1| = t + K.$ How to remove $|.|$? $\Leftarrow$ Qualitative discussion.

BACK to ODEs: $\exists/!$ / Maximal interval of existence.

$(ty^2 + 3t^2y) + (t^3 + t^2y)y' = 0, \quad y(1) = 1$

- ⊛ Check hp of P-L Thm.
- ⊛ Find the implicit solution
- ⊛ Find maximal interval of $\exists$.

* Check hp: $y' = G(t,y) = -\dfrac{y^2 + 3ty}{t^2 + ty} = -\dfrac{y(y+3t)}{t(t+y)}$

don't forget to check the derivative

$$\frac{\partial G}{\partial y} = -\frac{[y(y+3t)]'}{t(t+y)} + \frac{y(y+3t)}{t^2(t+y)^2}[t(t+y)]'$$ $\leftarrow$ sing. at worst at $t(t+y) = 0$.

$\Rightarrow \boxed{\text{Avoid } t = 0 \ \& \ t + y = 0}$. $1 \neq 0$ & $1 + 1 \neq 0$ ✓.

⊛ Solution: exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial t} = 2ty + 3t^2$ YES.

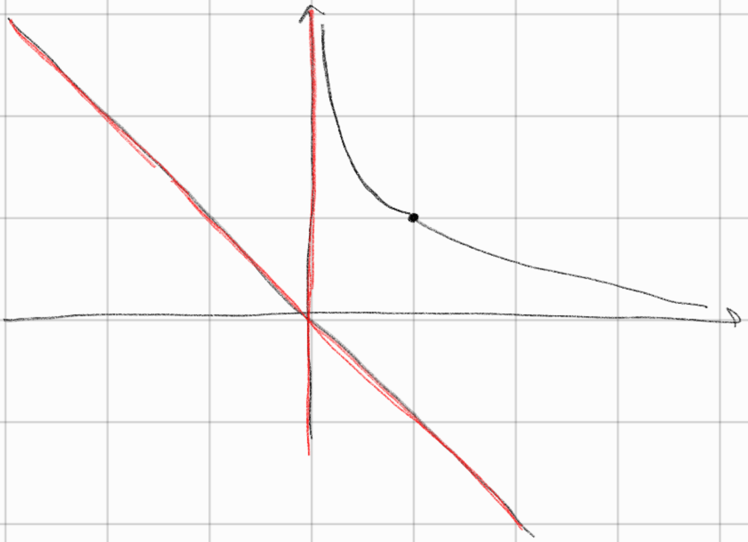
Find Ψ: $\Psi = \int M(t,y)\,\delta t + \int N - \frac{\partial Q}{\partial y}\,dy + K$

$= \frac{t^2y^2}{2} + y\cdot t^3 + \underbrace{\int t^3 + t^2y - t^2y - t^3}_{0} + K \Rightarrow \Psi = \text{Constant}.$

$\frac{t^2y^2}{2} + yt^3 = C$: $t=y=1 \Rightarrow C = \frac{3}{2}$.

Maximal interval of existence: find the right branch of the implicit locus.

$\Rightarrow y(t) = \frac{-t^3 \pm \sqrt{t^6 + 3t^2}}{t^2}$: pick + in order to match the initial conditions.



red = bad locus.

$y(t) = -t + \sqrt{t^2 + \frac{3}{t^2}}$

$\Rightarrow I = (0, +\infty)$

$\exists !$: enough to check G & $\frac{\partial G}{\partial y}$

maximal interval of $\exists$: if ODE linear: $y' = a(t)y + b(t)$ enough to avoid bad locus.

if ODE not-linear → you need to use Implicit function theorem / discuss implicit vs explicit solutions.

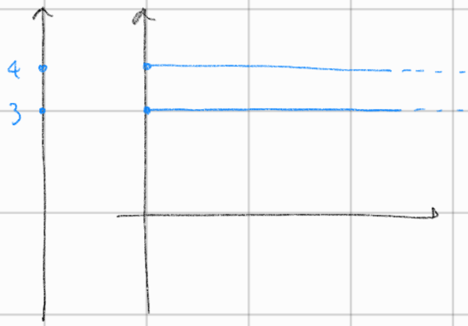
Qualitative discussion only for $\boxed{y' = F(y)}$

Example: Do qualitative discussion for $y' = (y-3)(y-4)^2$, $t \geq 0$.

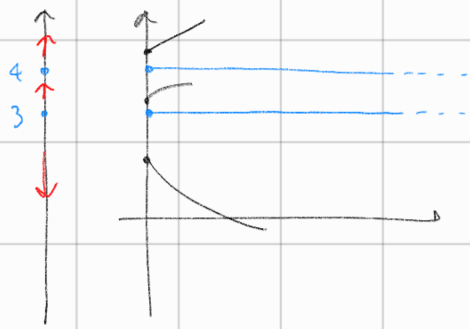
① Classification: 1st-order - autonom. - scalar - not-linear ODE.

② Eq. points: $y_0 = 3, 4$. $G(x) = (x-3)(x-4)^2$ always continuous & $G'(x)$ same as well.

⇒ ③ Phase line



④ Fill-in the gaps: $G(y_0) = (y_0-3)\underbrace{(y_0-4)^2}_{\geq 0}$ ⇒ $y_0 < 3 \Rightarrow -$; $y_0 > 3 \Rightarrow +$ ⇒



⇒ 4 semistable, 3 unstable. & $\lim_{t \to +\infty} y(t) = 3$ if $y_0 \in (3, 4)$.

RECALL: We have a trick for the stability: $G'(y_0)$ $<0 \Rightarrow$ stable ; $>0 \Rightarrow$ unstable ; BUT $=0 \Rightarrow$ unconclusive

In particular in the case above, the criterion for $y_0 = 3$ works fine, but for $y_0 = 4$ fails (you get $G'(4) = 0$!)

So you need to study the sign.