Today: linear system: 7.6 - 7.8

higher-order: 4.2 | homogeneous

higher-order: 4.3 | not homogeneous

linear system: 7.9

+ Extra (it won't be tested)

⊛ Exponential Matrix

⊛ Higher-order linear systems.

What have we left behind? (I) $x' = Ax$ & some eigenvalues of A $\begin{cases} \text{real but } \mu_a > \mu_g \\ \text{complex} \end{cases}$

(II) higher order $a_n y^{(n)} + \dots + a_0 y = 0$ & some roots are complex

$\boxed{\text{(I) Real but } \mu_a > \mu_g:}$ Example: $x' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$. $\det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 \Rightarrow \lambda = 2 \quad \mu_a = 2$.

$A - 2Id \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v = 0 \Rightarrow v_2 = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mu_g = 1$. We have found a solution tho: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$

But we need to look for another solution: scalar case: $e^{rt} \to te^{rt} \to t^2 e^{rt} \to \dots$

here: given λ eigenvalue & v eigenvector: look for a solution of the form $e^{\lambda t}(\underset{\substack{\uparrow \\ \text{unknown}}}{w} + tv)$

So, let's plug the ansatz in: $\frac{d}{dt}\left[e^{\lambda t}(w + tv) \right] = \lambda e^{\lambda t}(w + tv) + e^{\lambda t} \cdot v$ and we want this to

be equal to $A \cdot e^{\lambda t}(w + tv) = e^{\lambda t}(Aw + tAv) = e^{\lambda t}(Aw + \underbrace{t \cdot \lambda v}_{\substack{\uparrow \\ v \text{ is an eigenvector for } \lambda}})$

$$\Leftrightarrow \lambda \cancel{e^{\lambda t}}(w+tv) + \cancel{e^{\lambda t}} \cdot v = \cancel{e^{\lambda t}}(Aw + t \cdot \lambda v) \Leftrightarrow \lambda w + \cancel{t\lambda v} + v = Aw + \cancel{t\lambda v}$$

$\Leftrightarrow (A - \lambda Id)w = v$. Moreover : $c_1 e^{\lambda t} v + c_2 e^{\lambda t}(w + tv) = 0$

$$\Rightarrow c_1 v + c_2 w + \underbrace{t c_2 v}_{=0} = 0 \quad \Rightarrow c_2 = 0 \Rightarrow c_1 = 0 . \Rightarrow \text{lin. independent.}$$

$\Rightarrow e^{\lambda t}(w+tv)$ is indeed a different solution (provided that $(A - \lambda Id)w = v$).

Let's finish the previous ex. : $\lambda = 2$, $v = \begin{pmatrix}1\\0\end{pmatrix}$: $e^{2t}\left(w + t\begin{pmatrix}1\\0\end{pmatrix}\right)$ such that $\begin{pmatrix}0&1\\0&0\end{pmatrix} w = \begin{pmatrix}1\\0\end{pmatrix}$

$\Rightarrow w_2 = 1 \Rightarrow \begin{pmatrix}0\\1\end{pmatrix}$

What happens in general? : for a λ : $\mu_a(\lambda) > \mu_g(\lambda)$ you do the following :

(1) Find all the eigenvectors : $\{v_1, \dots, v_{\mu_g}\}$ $\Rightarrow$ they give you $\mu_g(\lambda)$ -solutions.

(2) Find all the solutions of the form $e^{\lambda t}(w + tv)$ with $v \in$ Eigenvectors(λ) &

$(A - \lambda Id)w = v$.

(*) Then count : do we have $\mu_a(\lambda)$ -solutions ? $\xrightarrow{\text{Yes}}$ $\boxed{\text{STOP}}$

↓ no

(3) Try $\underbrace{e^{\lambda t}\left(u + tw + \frac{t^2}{2}v\right)}$ with $\begin{cases} v : (A - \lambda Id)v = 0 \quad (\text{but } \neq 0) \\ w : (A - \lambda Id)w = v \end{cases}$

$$\Rightarrow \frac{d}{dt}[\dots] = \cancel{e^{\lambda t}}\left(\lambda u + t\lambda w + \cancel{\frac{\lambda t^2}{2}v} + w + tv\right) = \cancel{e^{\lambda t}}\left(Au + tAw + \underbrace{\frac{t^2}{2}Av}_{= \cancel{\frac{\lambda t^2}{2}v}}\right)$$

$\Rightarrow \quad (A - \lambda Id)u = t\lambda w + w + tv - tAw = t(\lambda Id - A)w + w + tv = -tv + w + tv = w$

(*) then count: do we have $\mu_a(\lambda)$ - solutions? $\xrightarrow{\text{yes}}$ $\boxed{\text{STOP}}$

$\downarrow$ no

Try: $e^{\lambda t}\left(z + tu + \frac{t^2}{2}w + \frac{t^3}{3!}v\right)$: $\frac{d}{dt}(\ ) = e^{\lambda t}\left(\lambda z + \lambda t u + \frac{\lambda t^2}{2}w + \frac{\lambda t^3}{6}v + u + tw + \frac{t^2 v}{2}\right)$

$= e^{\lambda t}\left(Az + tAu + \frac{t^2}{2}Aw + \frac{t^3}{6}Av\right)$

$\Rightarrow (A - \lambda Id)z = \lambda t u + \underbrace{\frac{\lambda t^2}{2} w} + \frac{\lambda t^3}{6}v + u + tw + \cancel{\frac{t^2 v}{2}} - tAu \underbrace{- \frac{t^2}{2}Aw} - \frac{t^3}{6}\lambda v$

$\frac{t^2}{2}(\lambda Id - A)w = -\frac{t^2}{2}v \Rightarrow$ & $t(\lambda Id - A)u = -tw$

$\Rightarrow (A - \lambda Id)z = u$ & <u>repeat the process</u> $\left[v_0 + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^k}{k!}v_k\right]$

<u>Example</u>: $x' = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} x$: $\det = (\lambda - 3)^4 = 0 \Rightarrow \lambda = 3 \quad \mu_a = 4$.

$(A - 3Id) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & \ddots & 1 & 0 \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ works fine.

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} w = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$: $w_2 = 1$, $w_3 = 0$, $w_4 = 0$: $w = \begin{pmatrix} w_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ works fine for any w_1 : pick $w_1 = 1$ (any would be good!)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} u = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} u_2 = 1 \\ u_3 = 1 \\ u_4 = 0 \end{matrix} : \begin{pmatrix} u_1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ is fine: pick } u_1 = 0.$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} z = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} z_2 = 0 \\ z_3 = 1 \\ z_4 = 1 \end{matrix} : \begin{pmatrix} z_1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad z_1 = 7 \text{ is fine:}$$

$$\Rightarrow \text{general solution: } e^{3t}\left[c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) + c_3 \left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \right.$$

$$\left. + c_4 \left(\begin{pmatrix} 7 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^3}{6} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \right]$$

I) $\boxed{\text{Complex}}$

$\det(A - \lambda Id) = p_A(\lambda) = 0$. Since $p_A(t)$ has real coeff.'s, complex roots come in conjugated pairs:

Indeed, you can always factorize a real-coeff.'s polynomial as $c \cdot \underbrace{\prod_i (t - \lambda_i)}_{\text{linear factors}} \prod_j (t^2 + a_j t + b_j)$.

where $\Delta_j = a_j^2 - 4b_j < 0$ (rmk: the only $\underbrace{\text{irreducible}}_{= \text{you cannot factorize further on } \mathbb{R} \text{ (but you can on } \mathbb{C})}$ polynomials / $\mathbb{R}$ have deg 1 or 2).

$$\Rightarrow \underbrace{\prod_i (t - \lambda_i)^{\mu_a(\lambda_i)}}_{\text{(real)}} \prod_j [(t - \lambda_j)(t - \bar{\lambda}_j)]^{\mu_a(\lambda_j)}$$

Example: $x' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} x$: $\det \begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = (1+\lambda)^2 + 4 = \lambda^2 + 2\lambda + 5$

$\Rightarrow \lambda_{1,2} = -1 \pm \sqrt{1-5} = -1 \pm 2i \leftarrow \lambda = -1 + 2i$ & $\bar{\lambda} = -1 - 2i$.

1st step: proceed as λ_j were real roots, namely

* find eigenvectors
* find extra solutions if need.

be aware: at this step you'll find vectors with complex-entries.

in the example: $(A - \lambda Id)v = 0$: $\begin{pmatrix} -1+1-2i & 2 \\ -2 & -1+1-2i \end{pmatrix} v = 0 \Rightarrow \left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & -i & 0 \end{array}\right)$

$\Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}$ works fine. - since $\mu_a(-1+2i) = 1$ we don't need to find further solutions $\Rightarrow c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t+2it}$

2nd step: take the complex conj. of the solutions you found:

$\mu_a(\lambda) = m$: $\{s_1(t), \dots, s_m(t)\}$ m-(lin. independent) solutions - they may have complex-entries.

$\Downarrow$

CLAIM: $\{\overline{s_1(t)}, \dots, \overline{s_m(t)}\}$ m-(lin-independent) solutions for $\bar{\lambda}$.

proof: linear independence does not change when you take the conjugate. &

$$s_i(t) = \begin{pmatrix} s_i^{(1)}(t) \\ \vdots \\ s_i^{(m)}(t) \end{pmatrix} e^{\lambda t} \Rightarrow \overline{s_i(t)} = \begin{pmatrix} \overline{s_i^{(1)}(t)} \\ \vdots \\ \overline{s_i^{(m)}(t)} \end{pmatrix} e^{\bar{\lambda} t}$$ (so it is w.r.t. $\bar{\lambda}$) &

$s_i' = A s_i \Rightarrow \bar{s}_i' = \bar{A} \bar{s}_i = A \bar{s}_i$ (A is real-valued) $\Rightarrow \overline{s_i(t)}$ still a solution.

3rd step: substitute complex solutions with real ones!

CLAIM #2: $\forall \lambda$, $\forall i \in \{1 \dots m\}$, there are two lin. independent real solutions $\varphi_i(\lambda), \psi_i(\lambda)$

$$\in \mathrm{Span}_{\mathbb{C}} < s_i(t), \overline{s_i(t)} > .$$

proof #2: recall from above: $s_i(t) = e^{\lambda t} \left[v_0 + t v_1 + \dots + \frac{t^k}{k!} v_k \right]$ with $k \in \{0, \dots, \mu_a(\lambda) - 1\}$

e.g. $\mu_a(\lambda) = 1 \Rightarrow s_i(t) = e^{\lambda t} v$ with v eigenvector.

So when we take the complex conjugate: $\overline{s_i(t)} = e^{\bar{\lambda} t}\left[\bar{v_0} + t\bar{v_1} + \dots + \frac{t^k}{k!}\bar{v_k}\right]$

Enough to focus on $e^{\lambda t} v$ & $e^{\bar{\lambda} t}\bar{v}$.

$\lambda = \alpha + i\beta \quad v = \vec{x} + i\vec{y} \Rightarrow e^{\alpha t} \cdot e^{i\beta t}[\vec{x} + i\vec{y}] = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\vec{x} + i\vec{y})$

$= e^{\alpha t}(\cos(\beta t)\vec{x} - \sin(\beta t)\vec{y}) + i e^{\alpha t}(\cos(\beta t)\vec{y} + \sin(\beta t)\vec{x})$ for $s_i(t)$

& the complex conj:

$= e^{\alpha t}(\cos(\beta t)\vec{x} - \sin(\beta t)\vec{y}) - i e^{\alpha t}(\cos(\beta t)\vec{y} + \sin(\beta t)\vec{x})$ for $\overline{s_i(t)}$

$\Rightarrow \frac{s_i(t) + \overline{s_i(t)}}{2}$ & $\frac{s_i(t) - \overline{s_i(t)}}{2i}$ are good replacements. $\Leftarrow \left\{ \left\{ \frac{s_i(t) + \overline{s_i(t)}}{2}, \frac{s_i(t) - \overline{s_i(t)}}{2i} \right\}_{i=1,\dots,m} \right\}$

$e^{\alpha t}(\cos(\beta t)\vec{x} - \sin(\beta t)\vec{y})$ $\quad$ $e^{\alpha t}(\cos(\beta t)\vec{y} + \sin(\beta t)\vec{x})$

[fundamental system of solutions relative to $(\lambda, \bar{\lambda}, \mu_a(\lambda) = m)$]

(in the) Example: $\begin{pmatrix}1\\i\end{pmatrix} e^{-t+2it} = s_i(t) \quad \left(\Rightarrow \overline{s_i(t)} = \begin{pmatrix}1\\-i\end{pmatrix} e^{-t-2it}\right)$

$\Rightarrow \alpha = -1, \ \beta = 2, \ \vec{x} = \begin{pmatrix}1\\0\end{pmatrix}$ & $\vec{y} = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow$ a fundamental system is:

$\left\{ e^{-t}\left(\cos(2t)\begin{pmatrix}1\\0\end{pmatrix} - \sin(2t)\begin{pmatrix}0\\1\end{pmatrix}\right), e^{-t}\left(\cos(2t)\begin{pmatrix}0\\1\end{pmatrix} + \sin(2t)\begin{pmatrix}1\\0\end{pmatrix}\right) \right\}$

Example 2: $x' = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} x$: $\det(A - \lambda I_0) = [(1-\lambda)^2 + 2](3-\lambda) = 0$.

$\Rightarrow (1 + \lambda^2 - 2\lambda + 2)(\lambda - 3) = 0 \quad \Rightarrow \quad (\lambda^2 - 2\lambda + 3)(\lambda - 3) = 0$

$\lambda_{1,2} = 1 \pm \sqrt{1-3} = 1 \pm \sqrt{2} i \quad , \lambda = 3.$

$\lambda = 3$ $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} v_2 = 0 \\ -2v_1 + v_3 = 0 \end{array} \Big| \Rightarrow \begin{pmatrix} v_1 \\ 0 \\ 2v_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ works fine

$\lambda = 1 + \sqrt{2} i$ $\left(\begin{array}{ccc|c} -\sqrt{2} i & 1 & 1 & 0 \\ -2 & -\sqrt{2} i & 1 & 0 \\ 0 & 0 & 2 - \sqrt{2} i & 0 \end{array}\right) \Rightarrow \begin{pmatrix} 1 & i/\sqrt{2} & 0 \\ 1 & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} v = 0 \Rightarrow \begin{array}{l} v_1 = -\frac{i}{\sqrt{2}} v_2 \\ v_3 = 0 \end{array} \Rightarrow \begin{pmatrix} -i \\ \sqrt{2} \\ 0 \end{pmatrix}$ works fine.

$\Rightarrow \alpha = 1, \; \beta = \sqrt{2}, \; \vec{x} + i\vec{y} = \begin{pmatrix} -i \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \; \& \; \vec{y} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$

$\Rightarrow$ general solution is: $c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{3t} + c_2 e^{t} \left[\cos(\sqrt{2}t) \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} - \sin(\sqrt{2}t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right]$

$+ c_3 e^{t} \left[\sin(\sqrt{2}t) \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} + \cos(\sqrt{2}t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right]$

II complex - similar situation: the characteristic polynomial $p(t)$ factors as

$$\underbrace{\prod (t - \lambda_i)^{ma(\lambda_i)}}_{\text{real roots}} \; \underbrace{\prod [(t - \lambda_j)(t - \bar{\lambda}_j)]^{ma(\lambda_j)}}_{\text{complex conjugate}}$$

recall: for second order ODEs: $\{e^{\lambda t}, e^{\bar{\lambda} t}\}$ with $\lambda = \alpha + i\beta \Rightarrow \{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$

= which is the same as $\left\{ \frac{e^{\lambda t}+e^{\bar\lambda t}}{2}, \frac{e^{\lambda t}-e^{\bar\lambda t}}{2i} \right\}$

$$\frac{e^{\lambda t}+e^{\bar\lambda t}}{2} = \frac{\delta_i(t)+\overline{\delta_i(t)}}{2}, \qquad \frac{e^{\lambda t}-e^{\bar\lambda t}}{2i} = \frac{\delta_i(t)-\overline{\delta_i(t)}}{2i}$$

And now we do the same thing:

(1) pretend λ is real: $\{e^{\lambda t}, te^{\lambda t}, \dots, t^{m-1}e^{\lambda t}\}$ $\quad m=\mu_a(\lambda)$

(2) take the complex conj: $\{e^{\bar\lambda t}, te^{\bar\lambda t}, \dots, t^{m-1}e^{\bar\lambda t}\}$

(3) form $(m+1)$-lin. independent

real solutions: $\left\{ \frac{e^{\lambda t}+e^{\bar\lambda t}}{2}, \frac{e^{\lambda t}-e^{\bar\lambda t}}{2i}, t\left(\frac{e^{\lambda t}+e^{\bar\lambda t}}{2}\right), t\left(\frac{e^{\lambda t}-e^{\bar\lambda t}}{2i}\right), \dots \right\}$

$\lambda = \alpha + i\beta$

$\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t), te^{\alpha t}\cos(\beta t), te^{\alpha t}\sin(\beta t), \dots\}$

II order

$e^{\lambda t}$

$e^{\bar\lambda t}$

$$\left\{ \frac{e^{\lambda t}+e^{\bar\lambda t}}{2}, \frac{e^{\lambda t}-e^{\bar\lambda t}}{2i} \right\} = \{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$$

<u>Example</u>: $y^{(5)} - 6y^{(4)} + 16y^{(3)} - 24y'' + 20y' - 8y = 0$

$\circledast$ char. polynomial: $t^5 - 6t^4 + 16t^3 - 24t^2 + 20t - 8 = p(t)$. You can check that you can factorize it

as $(t-2)(t^2-2t+2)^2$: $t=2$ is a root, $t_{1,2} = 1 \pm \sqrt{1-2} = 1 \pm i$

$\Rightarrow (t-2)(t-1-i)^2(t-1+i)^2$:

$$\left.\begin{array}{ll} \lambda = 2 & \mu_a = 1 \\ \lambda = 1+i & \mu_a = 2 \\ \lambda = 1-i & \mu_a = 2 \end{array}\right]$$

$\Rightarrow$ the general solution has the form: $c_1 e^{2t} + e^t(c_1\cos(t) + c_2\sin(t)) +$

$+ te^t(c_3\cos(t) + c_4\sin(t))$

Non-homogeneous case:

Let's start with the higher-order scalar one: as for the second order:

* constant coeff.'s & nice $g(t)$ - same procedure! ← Undetermined coeff.'s
* constant coeff.'s & bad $g(t)$ - - - - ← Variation of
* non-const coeff.'s - - - - parameters.

Ex1: $y''' - 4y' = t + 3\cos(t)$: * first solve the homog. one: so, consider the characteristic polynomial

$p(t) = t^3 - 4t \Rightarrow$ the roots are: $t(t^2-4)=0$ $t = 0, 2, -2$

$\Rightarrow$ the general solution is: $c_1 + c_2 e^{2t} + c_3 e^{-2t}$.

* Undermined coeff.: $g(t) = t + 3\cos(t)$: $g_1(t) = t$ & $g_2(t) = 3\cos(t)$

$\Rightarrow g_1(t) \rightsquigarrow t^s(A + tB)$; $g_2(t) \rightsquigarrow t^s(A\cos(t) + B\sin(t))$

↑ one root is 0 $\Rightarrow s = 1$ ↑ no root $\pm i$ $\Rightarrow s = 0$

$g_1(t)$: $At + Bt^2 = y(t)$, $y' = A + 2Bt$, $y'' = 2B$, $y''' = 0$

$\Rightarrow 0 - 4A - 8Bt = t$ $\Rightarrow A = 0$, $B = -\frac{1}{8}$ $\Rightarrow y_{p,1} = -\frac{t^2}{8}$

$g_2(t)$: $A\cos(t) + B\sin(t) = y \Rightarrow y' = -A\sin(t) + B\cos(t)$, $y'' = -A\cos(t) - B\sin(t)$, $y''' = A\sin(t) - B\cos(t)$

$A\sin(t) - B\cos(t) - 4(-A\sin(t) + B\cos(t)) = 3\cos(t)$ $\Rightarrow A = 0$ & $B = -\frac{3}{5}$ $\Rightarrow y_{p,2} = -\frac{3}{5}\sin(t)$

$\Rightarrow$ the general solution is : $c_1 + c_2 e^{2t} + c_3 e^{-2t} - \frac{t^2}{8} - \frac{3}{5}\sin(t)$

Variation of parameters: Consider the following ODE:

$$y^{(n)} + p_1(t) y^{(n-1)} + \dots + p_{n-1}(t) y' + p_n(t) y = g(t) .$$

Let $\{y_1, \dots, y_n\}$ be a fundamental set of solutions over I. Now consider the following determinants:

$$W(t) = \det \begin{pmatrix} y_1 & \dots & y_n \\ y_1' & \ddots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}$$ ← which is the Wronskian.

$$W_j(t) = \det \begin{array}{c} \begin{pmatrix} y_1 & 0 & y_n \\ \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots \\ y_1^{(n-1)} & 1 & y_n^{(n-1)} \end{pmatrix} \\ \uparrow \\ j\text{-th} \end{array}$$ ← same columns but the j-th is changed into $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ } all zeroes, ← a 1.

Then a particular solution to the not-homogeneous ODE is

$$\boxed{Y_p(t) = \sum_{j=1}^{n} y_j(t) \int \frac{g(t) W_j(t)}{W(t)} \, dt .}$$

Example: $y''' - y'' - y' + y = \frac{1}{t}$. ⊛ first, the homog. one: $\{e^t, te^t, e^{-t}\}$ is a fundamental set of solutions.

Let's compute Y_p:

⊛ compute $W(t) = \det \begin{pmatrix} e^t & te^t & e^{-t} \\ e^t & e^t(t+1) & -e^{-t} \\ e^t & e^t(t+2) & e^{-t} \end{pmatrix} = e^t \cdot \cancel{e^t} \cdot \cancel{e^{-t}} \det \begin{pmatrix} 1 & t & 1 \\ 1 & t+1 & -1 \\ 1 & t+2 & 1 \end{pmatrix}$

$$= e^t \left[(t+1)(\cancel{-t} + t + 2) - (-t-1)(\cancel{-t} + t + 2) \right] = 4e^t$$

⊛ compute $W_j(t)$: $j=1$: $W_1(t) = \det \begin{pmatrix} 0 & te^t & e^{-t} \\ 0 & e^t(t+1) & -e^{-t} \\ 1 & e^t(t+2) & e^{-t} \end{pmatrix} =$

$$= \cancel{e^t} \cdot \cancel{e^{-t}} \cdot \det \begin{pmatrix} 0 & t & 1 \\ 0 & t+1 & -1 \\ 1 & t+2 & 1 \end{pmatrix} = -t - t - 1 = -2t-1$$

$j=2$: $W_2(t) = \det \begin{pmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{pmatrix} =$

$$= \cancel{e^t} \cdot \cancel{e^{-t}} \det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} = 1 + 1 = 2$$

$j=3$: $W_3(t) = \det \begin{pmatrix} e^t & te^t & 0 \\ e^t & e^t(t+1) & 0 \\ e^t & e^t(t+2) & 1 \end{pmatrix} =$

$$= e^t \cdot e^t \det \begin{pmatrix} 1 & t & 0 \\ 1 & t+1 & 0 \\ 1 & t+2 & 1 \end{pmatrix} = e^{2t} \left[t+1-t \right] = e^{2t}$$

$$\{e^t, te^t, e^{-t}\} = \{y_1, y_2, y_3\}$$

$$\Rightarrow Y_p = e^t \int \frac{1}{t} \cdot (-2t-1) \cdot \frac{1}{4e^t} dt + te^t \int \frac{1}{t} \cdot 2 \cdot \frac{1}{4e^t} dt + e^{-t} \int \frac{1}{t} \cdot e^{2t} \frac{1}{4e^t} dt$$

Non-homogeneous first order linear systems. $x' = A(t)x + b(t)$

recall: in the scalar case: $x' = a(t)x + b(t) = e^{\int a(t)dt} \cdot \left[\int \left(e^{\int a(t)dt}\right)^{-1} \cdot b(t)\,dt + k \right]$

rmk: $e^{\int a(t)dt}$ is the solution to the homogeneous one $x' = a(t)x$.

& $K e^{\int a(t)dt}$ is the general solution.

strategy: find a replacement for the term $e^{\int a(t)dt}$ (let's call it $\Psi(t)$)

such that the general solution is of the form:

$$x(t) = \Psi(t) \cdot \left[\int \Psi(t)^{-1} \cdot b(t)\,dt + K \right]$$

rmk: (*) $b(t)$ & $x(t)$ n-dim. vector $\Rightarrow$ $\Psi(t)$ must be a $\underline{n \times n}$-matrix, $\underline{\text{invertible}}$;

K must be a constant $\underline{n\text{-vector}}$.

(*) Moreover if $b(t) = 0$ we need to recover the general solution for the homogeneous one. $\Rightarrow$ $x(t) = \Psi(t) \cdot K$

$$= \begin{pmatrix} | & & | \\ \Psi_1(t), & \dots, & \Psi_n(t) \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

$$= k_1 \Psi_1(t) + \dots + k_n \Psi_n(t)$$

In particular $\{\varphi_i(t)\} = \{$ fundamental set of solutions $\}$ works!

DEF: $\Psi(t) = \begin{pmatrix} | & & | \\ \varphi_1(t) & \cdots & \varphi_n(t) \\ | & & | \end{pmatrix}$ with $\{\varphi_1(t) \dots \varphi_n(t)\}$ fundamental system of solutions for $x' = A(t)x$

is called a fundamental matrix.

Rmk: $\det(\Psi(t)) = W(\varphi_1 \dots \varphi_n)$ is the Wronskian. Since $\{\varphi_i\}$ are lin. ind. $(\det \Psi(t) \neq 0)$

$\Rightarrow \Psi(t)$ is invertible.

Claim: such Ψ works & $x(t) = \Psi(t)\left[\int \Psi(t)^{-1} b(t) + K\right]$ is the general solution.

proof:

LHS: $x'(t) = \Psi'(t) \cdot \left[\int \Psi(t)^{-1} b(t) + K\right] + \overbrace{\Psi(t)\left[\Psi(t)^{-1} \cdot b(t)\right]}^{= b(t)}$

RHS: $A(t)x(t) + b(t) = \underbrace{A(t)\Psi(t)}_{\Psi'(t)} \cdot \left[\int \Psi(t)^{-1} b(t) + K\right] + b(t)$

$= \Psi'(t)\left[\int \Psi^{-1}(t) b(t) + K\right] + b(t)$

$\Rightarrow$ LHS = RHS ✓.

E.g: Find the general solution for $x' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} e^{-t} \\ 2 \end{pmatrix}$

(*) find $\Psi(t)$ = find a fundamental set of solutions for the homogeneous one: $x' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x$.

$\det \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (\lambda+2)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) \Rightarrow \lambda = -1, -3.$

$\lambda = -3$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works. ; $\lambda = -1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works

fundamental set: $\left\{ e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \Rightarrow \Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$

$$\Rightarrow x(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \left[\int \frac{1}{\det(\Psi(t))} \begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ 2 \end{pmatrix} dt + K \right]$$

(*) $\det(\Psi(t)) = e^{-4t} + e^{-4t} = 2e^{-4t}$

(*) $\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-2t} - 2e^{-t} \\ e^{-4t} + 2e^{-3t} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{e^{2t}}{2} - e^{3t} \\ \frac{1}{2} + e^{t} \end{pmatrix}$

$$\int \begin{pmatrix} \frac{e^{2t}}{2} - e^{3t} \\ \frac{1}{2} + e^{t} \end{pmatrix} dt = \begin{pmatrix} \int \frac{e^{2t}}{2} - e^{3t}\, dt \\ \int \frac{1}{2} + e^{t}\, dt \end{pmatrix} = \begin{pmatrix} \frac{e^{2t}}{4} - \frac{1}{3}e^{3t} \\ \frac{t}{2} + e^{t} \end{pmatrix}$$

$$\Rightarrow x(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \left[\begin{pmatrix} \frac{e^{2t}}{4} - \frac{1}{3}e^{3t} \\ \frac{t}{2} + e^{t} \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right] =$$

$$= \underbrace{k_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{solution to the homogeneous eq.}} + \underbrace{\begin{pmatrix} \frac{e^{-t}}{4} - \frac{1}{3} + \frac{t}{2}e^{-t} + 1 \\ -\frac{e^{-t}}{4} + \frac{1}{3} + \frac{t}{2}e^{-t} + 1 \end{pmatrix}}_{\text{particular solution}}$$

$$= \frac{1}{4}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{t}{2}e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Extra: (1) Alternative method for solving $\boxed{x' = A(t)\,x + b(t)}$

Start with the basic case : $x' = Ax$: in the scalar situation $x' = a\,x$, the function $x(t) = e^{at}$

is a solution (the general solution is Ce^{at}).

Hope: we can generalize the $\exp(at)$ to $\exp(At)$ - Exponential of a matrix &

have $X(t) = \exp(At)$ a solution. **Answer:** we can!

Def: $\exp(M(t)) = \sum_{n\geq 0} \frac{M(t)^n}{n!} = Id + M(t) + \frac{M(t)^2}{2!} + \dots$ $\leftarrow$ it mimic the Taylor expansion

for $e^t = \sum_{n\geq 0} \frac{t^n}{n!}$ in $t=0$.

Example: $M(t) = M.t \Rightarrow \sum_{n\geq 0} \frac{M^n . t^n}{n!}$, $M(t) = M \Rightarrow \sum_{n\geq 0} \frac{M^n}{n!}$.

Pay attention! ① It is a series : there may be convergency-issues! **Fact:** the series in the example are

convergent and in general we have convergence on any compacts under mild assumptions.

We assume always convergent.

② if $M(t)$ is $1\times 1 = f(t)$ $\exp(f(t))$ as above $\neq$ Taylor expansion of $e^{f(t)}$!!

Proposition: $\frac{d}{dt}\exp(At) = A.\exp(At)$

proof: $\frac{d}{dt}\exp(At) = \frac{d}{dt}\sum_{n\geq 0}\frac{1}{n!}A^n t^n = \sum_{n\geq 0}\frac{A^n}{n!}.n.t^{n-1} = \sum_{n\geq 1}\frac{A^{n-1}}{(n-1)!}.t^{n-1} = A\exp(At)$.

$\uparrow$ commute thanks to convergence

Corollary: by uniqueness: $e^{A\cdot t} = \Psi(t) \Rightarrow e^{At}\left[\int e^{-At}\cdot b(s)\, dt + k\right]$ is the gen. solution

for $x' = Ax + b(t)$. Rmk: $At = \int A\, dt$.

In general for $x' = A(t)x + b(t)$: $\exp\left(\int A(t)\, dt\right) \Rightarrow \frac{d}{dt}\exp\left(\int A(t)\, dt\right) = \sum_{n\geq 0} \frac{d}{dt}\frac{\left(\int A(t)\, dt\right)^n}{n!}$

$$= \sum_{n\geq 0} \frac{n}{n!}\left(\int A(t)dt\right)^{n-1}\cdot A(t) = \sum_{n\geq 1}\frac{\left(\int A(t)\, dt\right)^{n-1}}{(n-1)!}\cdot A(t) = \exp\left(\int A(t)\, dt\right)\cdot A(t).$$

IF $A(t)$ commutes with $\int A(t)dt \Rightarrow \exp\left(\int A(t)\, dt\right)$ is a solution.

(sufficient condition is $A(t)\, A(s) = A(s)\, A(t)$)

IF NOT You need to change def. of exp $\rightsquigarrow$ Ordered exponential.

$\Rightarrow$ MORAL: We know how to solve $\boxed{x' = A(t)x + b(t)}$

(2) Higher-order linear system: $x^{(n)} + P_1(t)x^{(n-1)} + \dots + P_n(t)\, x^{(0)} = G(t)$.

strategy: form a FIRST ORDER linear system and solve that instead.

Pose: $x_1(t) = x(t)$, $x_2(t) = x'(t)$, ..., $x_n(t) = x^{(n-1)}(t)$

$$\Rightarrow \boxed{\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ \vdots \\ x_n'(t) = -P_1(t)x_n(t) - \dots - P_n(t)x_1(t) + G(t) \end{cases}}$$

← First order linear system.

Example: $x''(t) + P(t)x' + Q(t)x = G(t) \Rightarrow x_1 = x \;\&\; x_2 = x'$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -Q(t)x_1 - P(t)x_2 + G(t) \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & \mathrm{Id} \\ -Q(t) & -P(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ G(t) \end{pmatrix}$$

⇒ We know how to solve linear ODEs.