Today: we study an alternative method in order to deal with <u>2nd-order</u> <u>scalar</u> <u>linear</u> homogeneous ODE:

$$P(t)\,y'' + Q(t)\,y' + R(t)\,y = 0$$ when P, Q, R are "regular enough".

<u>Reference:</u> §5.1 - 5.2 - 5.3 - 5.4

(!) Remark on regularity: for now let's assume that $P(t), Q(t), R(t)$ are polynomials

(!) Assume moreover that they do not have common factors: if they do:

$$(t-c)\tilde{P}(t)y'' + (t-c)\tilde{Q}(t)y' + (t-c)\tilde{R}(t)y = 0 .$$

eliminate it & study $\tilde{P}y'' + \tilde{Q}y' + \tilde{R}y = 0$.

<u>Aim:</u> given $t_0 \in \mathbb{R}$, we want to solve $P(t)y'' + Q(t)y' + R(t)y = 0$ <u>at least in a neighborhood of t_0</u>.

DEFINITION: a time t_0 s.t. $P(t_0) \neq 0$ is called an ordinary point.

On the other hand if t_0 s.t. $P(t_0) = 0$ is called a singular point

if P, Q, R are polynomials.

If t_0 is ordinary, then $y'' + \frac{Q(t)}{P(t)}y' + \frac{R(t)}{P(t)}y = 0$ has the following property:

Since $P(t_0) \neq 0$ & P is continuous $\Rightarrow \exists\, \delta > 0$: P over $(t_0 - \delta, t_0 + \delta)$ is never zero.

(Theorem of sign permanence for cont. functions)

This means that $\frac{Q}{P}$ & $\frac{R}{P}$ are both continuous on $(t_0 - \delta, t_0 + \delta) = I$

Then according to $\exists/!$ Theorem for II order ODEs: given (t_0, u_0, u_1), $t_0 \in I$.

$\exists!$ solution on I.

However if $P(t_0) = 0$, since either $Q(t_0)$ or $R(t_0) \neq 0$ $\Rightarrow$ either $\frac{Q}{P}$ or $\frac{R}{P}$ has a singularity

at $t = t_0$. (t_0: $\frac{Q(t_0)}{0}$, $\frac{R(t_0)}{0}$!) $\Rightarrow$ so we cannot apply thm of $\exists/!$ in this case.

However today we'll see a method to deal with this situation as well. The ideas behind it

are the same for ordinality and for singular point.

We start with ordinary point:

Ordinary point: Idea: look for the "Taylor expansion" at t_0 of the solution y.

This reduces to looking for solutions of the form

$$y = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + \dots = \sum_{n=0}^{\infty} a_n (t-t_0)^n$$

Examples: * polynomials! = finite series. (an definitively zero)

* $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ * $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ ← well defined only if $|t|<1$!

⇒ series may not be well defined everywhere!! there are convergence issues!

Review on power series, i.e. $\sum_{n=0}^{\infty} a_n (t-t_0)^n$

Recall definitions:

(1) a power series is said to converge at a point t_0 if $\lim_{N\to+\infty} \sum_{n=\infty}^{N} a_n(t-t_0)^n$ $\exists$ & it is finite.

THEOREM

any $\sum_{n=0}^{\infty} a_n(t-t_0)^n$ satisfies exactly one of the following properties.

- converges at $t=t_0$ & diverges $\forall\, t\neq t_0$

  E.g. $\sum (-1)^n n!\, t^n$

- $\exists$ a real number $R>0$ such that the series converges $\forall\, t : |t-t_0|<R$

  &

  diverges $\forall t : |t-t_0|>R$

  [at $|t-t_0|=R$ may diverge or converge]

  e.g. $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$

- converges $\forall\, t\in\mathbb{R}$

  e.g. $\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t$

  $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sin(t)$

⇒ So any time you use a series you need to make sure you are inside the Radius of convergence $=R$

What do we use in order to overcome the convergence issue?

Ratio Test thm: $\sum_{m=0}^{\infty} f_m$ a series where all $f_n \neq 0$. Then denote by $L = \lim_{m \to +\infty} \left| \frac{f_{m+1}}{f_m} \right| \in [0, +\infty]$

$L > 1 \Rightarrow \sum_{m=0}^{\infty} f_m$ diverges

$L < 1 \Rightarrow$ " " converges

$L = 1$ or L does not $\exists$ $\Rightarrow$ inconclusive

How to apply the ratio test: $\sum_{m=0}^{\infty} f_m = \sum_{n=0}^{\infty} a_n (t - t_0)^n$

* Assume $a_n \neq 0$ (definitively) then $f_n := a_n (t-t_0)^n \Rightarrow \left| \frac{a_{n+1} (t-t_0)^{n+1}}{a_n (t-t_0)^n} \right| = |t-t_0| \left| \frac{a_{n+1}}{a_n} \right|$

$\Rightarrow$ we need to consider $\lim_{n \to +\infty} |t-t_0| \left| \frac{a_{n+1}}{a_n} \right|$

$t = t_0$: $\lim_{n \to +\infty} 0 = 0$ ✓

$t \neq t_0$: $|t-t_0| \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

By the ratio test if $L < 1 \Rightarrow \sum a_n (t-t_0)^n$ converges.

We need to impose: $|t-t_0| \cdot \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. If $\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \rightarrow$ any t works fine.

If $\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = M \neq 0 \Rightarrow t$ must satisfy $|t-t_0| < \frac{1}{M} = \left(\lim_{n} \left| \frac{a_{n+1}}{a_n} \right| \right)^{-1}$

Corollary: $\frac{1}{M}$ is the radius of convergence!

Example: $\sum_{n=0}^{\infty} (-1)^{n+1} n (t-2)^n$: find the radius of convergence.

1) $t_0 = 2$ & $a_n = (-1)^{n+1} n \neq 0 \ \forall n > 0$.

2) ratio test: $|t-2| \cdot \lim_{n \to +\infty} \left| \frac{(-1)^{n+2} \ n+1}{(-1)^{n+1} \ n} \right| = |t-2| \cdot \lim_{n \to +\infty} \left| 1 + \frac{1}{n} \right| = |t-2| \cdot 1$

$\Rightarrow$ radius is $\frac{1}{1} = 1 \Rightarrow$ the series converges for t: $|t-2| < 1 \rightsquigarrow 1 < t < 3$

" " diverges " " " $|t-2| > 1 \rightsquigarrow t < 1$ or $t > 3$

inconclusive for $|t-2| = 1 \rightsquigarrow t = 1, 3$

(*) What happens if there are some a_n which keep to be zero? $\Rightarrow$ You need to do a different labelling rather than $f_n = a_n (x-x_0)^n$!

In this class, we'll see mainly the cases when

- either all even ones $= 0$. $\Rightarrow \sum_{n=0}^{\infty} a_{2n+1} (t-t_0)^{2n+1}$
- or all odd ones $= 0$. $\Rightarrow \sum_{n=0}^{\infty} a_{2n} (t-t_0)^{2n}$

Let's see what happens in these cases: let's do the second one (they are analogous.)

Label $f_n := a_{2n} (t-t_0)^{2n}$ (not $f_n = a_n (t-t_0)^n$!)

$$\Rightarrow \left|\frac{f_{n+1}}{f_n}\right| = \left|\frac{a_{2n+2}(t-t_0)^{2n+2}}{a_{2n}(t-t_0)^{2n}}\right| = |t-t_0|^2 \cdot \left|\frac{a_{2n+2}}{a_{2n}}\right| \Rightarrow \lim_{n\to+\infty} |t-t_0|^2 \left|\frac{a_{2n+2}}{a_{2n}}\right|$$

& we want this < 1. ⊙ if $t = t_0$: $\lim = 0$ ✓

⊙ if $t \neq t_0$ ⟶ $\lim \left|\frac{a_{2n+2}}{a_{2n}}\right| = 0$ ⟶ $\lim = 0$ ✓ $\Rightarrow R = +\infty$

↳ $\lim \left|\frac{a_{2n+2}}{a_{2n}}\right| \in \mathbb{R} \cup \{+\infty\} \Rightarrow |t-t_0|^2 < \left(\lim \left|\frac{a_{2n+2}}{a_{2n}}\right|\right)^{-1}$

$$\Rightarrow |t-t_0| < \left(\lim \left|\frac{a_{2n+2}}{a_{2n}}\right|\right)^{-\frac{1}{2}} = \text{radius of convergence } R$$

E.g: $\sin(t)$ & $\cos(t)$.

Now assume $\sum a_n(t-t_0)^n$ & $\sum b_n(t-t_0)^n$ they both converge at least for $|t-t_0| < r$ $r > 0$.

then:

⊛ their sum: $\sum (a_n + b_n)(t-t_0)^n$ converges at least for $|t-t_0| < r$

⊛ their product: $\sum c_n (t-t_0)^n$ with $c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$

converges at least for $|t-t_0| < r$

⊛ $y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$ is continuous & has continuous derivatives of all orders

Moreover: $y'(t) = \sum_{n=0}^{\infty} n a_n (t-t_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (t-t_0)^{n-1}$

$y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n (t-t_0)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (t-t_0)^{n-2}$

⊛ $\sum a_n (t-t_0)^n = \sum b_n (t-t_0)^n \Rightarrow a_n = b_n \ \forall n$

DEF: if $f(t)$ which has Taylor expansion $\sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n$ $(= f(t))$ with a radius of convergence $R>0$ is said to be analytic at $t=t_0$.

Eg: $\sin(t)$, e^t, $\frac{1}{t}$ $(t\neq 0)$, $\tan(t)$ $(t \neq \frac{\pi}{2}+k\pi \quad k\in\mathbb{Z})$.

Fact: f & g analytic at $t=t_0$ $\Rightarrow$ $f\pm g$, fg analytic as well at $t=t_0$.

+ if $g(t_0)\neq 0$ $\Rightarrow$ $\frac{f}{g}$

BACK to the ode - ordinary case. $P(t)y'' + Q(t)y' + R(t)y = 0$ & $P(t_0)\neq 0$.

Ansatz: assume that $y = \sum_{n=0}^{\infty} a_n (t-t_0)^n$ is a solution over some $I \subseteq (t_0-R, t_0+R)$

R = radius of convergence.

$\Rightarrow y' = \sum_{n=0}^{\infty} n a_n (t-t_0)^{n-1}$; $y'' = \sum_{n=0}^{\infty} n(n-1) a_n (t-t_0)^{n-2}$ $\Rightarrow$ plug it in

$$\sum_{n=0}^{\infty} \left[n(n-1)a_n (t-t_0)^{n-2} \cdot P(t) + n a_n (t-t_0)^{n-1} \cdot Q(t) + a_n (t-t_0)^n \cdot R(t) \right] = 0$$

Next Goal: bring ↑ in the following form: $\sum_{n=0}^{\infty} C_n \cdot (t-t_0)^n = 0$ where C_n depends on $\{a_m\}$

This implies $C_n = 0 \ \forall n$ $\Rightarrow$ imposes conditions on the coefficients $\{a_m\}$.

How? Change $P(t)$, $Q(t)$, $R(t)$ into their Taylor series expansion at $t=t_0$ and do computations.

Example: $y'' - ty = 0$. Question is: find a fundamental set of solutions. (so we expect 2 functions)

$P(t) = 1$, $Q(t) = 0$, $R(t) = -t$: hence all the points are ordinary points.

$\boxed{\text{Pick } t_0 = 0}$ $\Rightarrow$ therefore we look for solutions of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$

$\Rightarrow y' = \sum_{n=0}^{\infty} a_n \cdot n \cdot t^{n-1}$; $y'' = \sum_{n=0}^{\infty} a_n \cdot n\,(n-1)\, t^{n-2}$

$\Rightarrow$ plug them in: $\left(\sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1)\, t^{n-2} \right) - t \sum_{n=0}^{\infty} a_n t^n = 0$

$\Rightarrow \sum_{n=0}^{\infty} \left[a_n \cdot n \cdot (n-1)\, t^{n-2} - a_n t^{n+1} \right] = 0$. I want to rewrite it $\sum_{n=0}^{\infty} c_n t^n = 0$.

Let's go back to the step $\underbrace{\sum_{n=0}^{\infty} a_n\, n\,(n-1)\, t^{n-2}}_{(*)} - \sum_{n=0}^{\infty} a_n t^{n+1} = 0$

First of all notice that $(*)$ can be rewritten as $\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, t^{n-2}$ (for $n=0,1$ you get 0)

Pose $n-2 = m+1 \Rightarrow n = m+3 \Rightarrow \sum_{m+3=2}^{\infty} a_{m+3} (m+3)(m+2)\, t^{m+1} = \sum_{m=-1}^{\infty} a_{m+3} (m+3)(m+2)\, t^{m+1}$

$= \underbrace{a_2 \cdot 2 \cdot 1}_{m=-1} + \sum_{m=0}^{\infty} a_{m+3} (m+3)(m+2)\, t^{m+1}$

$$\Rightarrow \quad 2a_2 + \sum_{n=0}^{\infty} \left[a_{n+3}(n+3)(n+2) - a_n \right] t^{n+1} = 0$$

$$\Rightarrow \quad 2a_2 = 0 \quad \& \quad \underline{a_{n+3}(n+3)(n+2) = a_n} \quad \forall n \geq 0$$

$$\Rightarrow a_{n+3} = \frac{a_n}{(n+3)(n+2)}$$ ← RE CURRENCE RELATION .

Rmk: a_0 & a_1 "are free" but then

$a_0 \rightarrow$ determines $a_3 = \dfrac{a_0}{3 \cdot 2}$

$a_1 \rightarrow$ — — — $a_4 = \dfrac{a_1}{4 \cdot 3}$

$a_2 \rightarrow$ — — — $a_5 = \dfrac{a_2}{5 \cdot 4} = 0 \quad (a_2 = 0)$

$a_3 \rightarrow$ - - - - $a_6 = \dfrac{a_3}{6 \cdot 5} = \dfrac{a_0}{(6 \cdot 3)(5 \cdot 2)}$

$a_4 \rightarrow$ - - - - $a_7 = \dfrac{a_4}{7 \cdot 6} = \dfrac{a_1}{(7 \cdot 4)(6 \cdot 3)}$

$a_5 \rightarrow$ - - - - $a_8 = 0$

$\vdots$

$\Rightarrow$ Each coeff. is a function of (a_0, a_1)

Next step: find that function: the above calculations suggest

$$\forall n \geq 1 \quad a_{3n} = \frac{a_0}{(3n \cdot \ldots \cdot 6 \cdot 3)(3n-1 \cdot \ldots \cdot 5 \cdot 2)}$$

$$\forall n \geq 1 \quad a_{3n+1} = \frac{a_1}{(3n+1 \cdot \ldots \cdot 7 \cdot 4)(3n \cdot \ldots \cdot 6 \cdot 3)}$$

$$\forall n \geq 0 \quad a_{3n+2} = 0$$

prove them by induction!

$$\Rightarrow y(t) = a_0 \Big(\overbrace{1 + \sum_{n\geq 1} \frac{1}{(3n \cdots 3)(3n-1 \cdots 2)} t^{3n}}^{y_1} \Big)$$

$$\star \quad + a_1 \Big(\underbrace{t + \sum_{n \geq 1} \frac{1}{(3n \cdots 3)(3n+1 \cdots 4)} t^{3n+1}}_{y_2} \Big) \quad \text{is a solution } \forall (a_0, a_1)!$$

Rmk: two question left: $\rightarrow$ extract from $\star$ 2 lin. independent solutions.

$\rightarrow$ find the radius of convergence.

1) Let's check if $\{y_1, y_2\}$ is a fund. system of solutions.

We need $W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \not\equiv 0$. So if $W(y_1,y_2)(0) \neq 0$, we are done.

$y_1(0) = 1$, $y_2(0) = 0$. & $y_1'(0) = 0$, $y_2'(0) = 1$. $\Rightarrow$ at 0: $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$ ✓

2) Radius of convergence. $\left| \frac{f_{n+3}}{f_n} \right| = \left| \frac{a_{3n+3} \, t^{3n+3}}{a_{3n} \, t^{3n}} \right|$ & $\left| \frac{a_{3n+4} \, t^{3n+4}}{a_{3n+1} \, t^{3n+1}} \right|$

$\uparrow$ for y_1 $\qquad$ $\uparrow$ for y_2

$\Rightarrow \lim_n |t|^3 \cdot \frac{1}{(3n+3)(3n+2)}$ for y_1 ; $\lim_n |t|^3 \frac{1}{(3n+4)(3n+3)}$ for y_2

Since $\lim \frac{1}{(3n+3)(3n+2)}$ & $\lim \frac{1}{(3n+4)(3n+3)} \rightarrow 0$ $n \rightarrow +\infty$. $\Rightarrow R = +\infty$.

$\Rightarrow \{y_1, y_2\}$ is a fund. set of solutions ✓.

Rmk: we could have done the same discussion for $t_0 = 1$ $\Rightarrow$ but then we need to have the ansatz centered in 1: $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$

and the coeff's as well! $P(t) = 1 = 1 \cdot (t-1)^0$, $Q(t) = 0 \cdot (t-1)^0$

$$R(t) = -t = -1 - (t-1) = -1 \cdot (t-1)^0 - (t-1)^1$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - (1 + (t-1)) \cdot \sum_{n=0}^{\infty} a_n (t-1)^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - \sum_{n=0}^{\infty} a_n (t-1)^n - \sum_{n=0}^{\infty} a_n (t-1)^{n+1}$$

$n-2 = m+1$: $\sum_{m=-1} (m+3)(m+2) \cdot a_{m+3} t^{m+1} - a_0 - \sum_{n=1} a_n (t-1)^n - \sum_{n=0} a_n (t-1)^{n+1} = 0$

$\Downarrow$

$n = m+3$

$$2a_2 + \sum_{m=0} (m+3)(m+2) a_{m+3} t^{m+1} - a_0 - \sum_{m=0}^{\infty} a_{m+1} (t-1)^{m+1} - \sum_{m=0} a_m (t-1)^{m+1} = 0$$

$$\Rightarrow 2a_2 - a_0 = 0 \quad \& \quad \sum_{m=0} \underbrace{[(m+3)(m+2) a_{m+3} - a_{m+1} - a_m]}_{=0} (t-1)^{m+1} = 0$$

In general given $P(t)y'' + Q(t)y' + R(t)y = 0$ we say:

DEF: t_0 is an ordinary point if $p(t) = \frac{Q(t)}{P(t)}$ & $q(t) = \frac{R(t)}{P(t)}$ are analytic in t_0

Otherwise, t_0 is called singular point.

Rmk: if P, Q, R are polynomials, this definition coincides with the previous one.

THEOREM: if t_0 is an ordinary point. Then $\exists$ 2 power series $y_1 = \sum a_n (t-t_0)^n$ and

$y_2 = \sum b_n (t-t_0)^n$ s.t.

(1) the general solution is $C_1 y_1 + C_2 y_2$.

(2) R_1, R_2 = radii of convergence of y_1, y_2 are $\geq$ minimum of the radii of convergence

of $p = \frac{Q}{P}$, $q = \frac{R}{P}$

THEOREM #2: (from Complex Analysis)

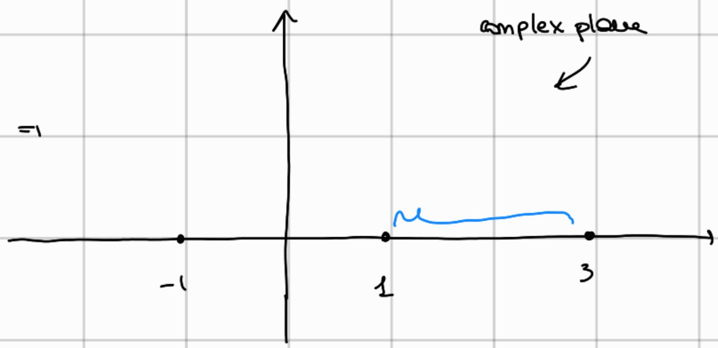
$P_1(z)$, $P_2(z)$ two polynomials. $P_1(t_0) \neq 0$ & they are coprime.

Then $\frac{P_2(z)}{P_1(z)}$ is analytic near t_0 and its radius of convergence is $=$ distance between

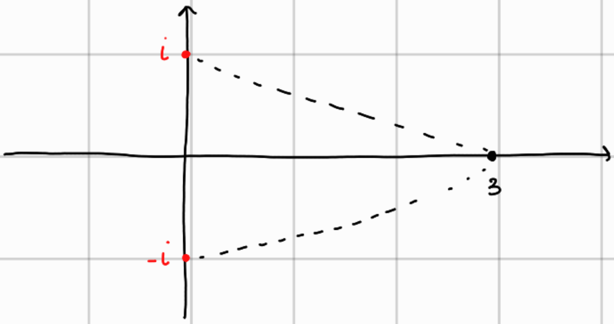
t_0 & the nearest (complex/real) root of $P_1(z)$

Examples: $\frac{1}{1-t^2}$ $t=3$: roots of $1-t^2$ are ± 1 =1

=1 $R=2$



$\frac{1}{1+t^2}$: $t=3$: roots of $1+t^2$ are $\pm i$:



distance $= \sqrt{1^2+3^2} = \sqrt{10}$

Example: $(t^2-2t+2)y'' + 2ty' + y = 0$: $P(t) = t^2-2t+2$ pick $t_0 = 0$

$Q(t) = 2t$ it is ordinary ✓

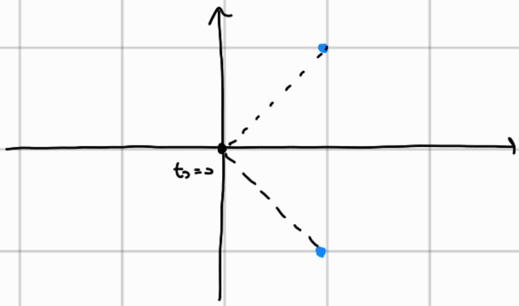
$R(t) = 1$

Then an under bound for the radii of convergence of solutions y_1, y_2 is given by:

radii of convergence of : $\frac{2t}{t^2-2t+2}$ & $\frac{1}{t^2-2t+2}$.

Roots of $t^2-2t+2 \Rightarrow$ $t_{1,2} = 1 \pm \sqrt{1-2} = 1 \pm i$.

$\Rightarrow$ $R = \sqrt{2}$



$\Rightarrow$ $R_1, R_2 \geq \sqrt{2}$.

Singular point case: pick t_0 for which either $p(t) = \frac{Q(t)}{P(t)}$ or $p(t) = \frac{R(t)}{P(t)}$ is not analytic .

(1) We need to change the previous method a little bit because we don't have analytic functions anymore .

(2) We will discuss only "mild singularities" which are the "regular singularity".

DEF: Assume t_0 is singular : it is regular singular point iff:

$$\lim_{t \to t_0} (t-t_0) \frac{Q(t)}{P(t)} \quad \exists \text{ \& it is finite}$$

&

$$\lim_{t \to t_0} (t-t_0)^2 \frac{R(t)}{P(t)} \quad \exists \text{ \& it is finite}$$

Otherwise t_0 is called irregular singular (we will not deal with this case)

Example: determine the singular points of $(1-t^2)y'' - 2ty' + y = 0$.

Since they are polynomials (P, Q, R): singularities when $P(t) = 0$. $\Rightarrow t = \pm 1$.

Let's check regularity of those points:

$t_0 = 1$

$$\lim_{t \to 1} \frac{-2t}{1-t^2} \cdot (t-1) = \lim_{t \to 1} \frac{+2t}{t+1} = 1 \checkmark$$

$$\lim_{t \to 1} \frac{1}{(1-t^2)} (t-1)^2 = \lim_{t \to 1} \frac{-(t-1)^{\cancel{2}}}{\cancel{(1-t)}(1+t)} = 0 \checkmark$$

$t_0 = -1$

$$\lim_{t \to -1} \frac{-2t \ \cancel{(t+1)}}{(1-t)\cancel{(1+t)}} = 1$$

$$\lim_{t \to -1} \frac{1}{(1-t)\cancel{(1+t)}} \cdot (t+1)^{\cancel{2}} = 0 \checkmark$$

$\Rightarrow$ both regular!

Example #2: do the same for $2t(t-2)^2 y'' + 3ty' + (t-2)y = 0$

Since P, Q, R polynomial & coprime $\Rightarrow$ singular points at $P(t_0) = 0$. $\Rightarrow t_0 = 0$ & 2.

$$p(t) = \frac{3t}{2t(t-2)^2} = \frac{3}{2(t-2)^2} \quad \& \quad p(t) = \frac{(t-2)}{2t(t-2)^2} = \frac{1}{2t(t-2)}$$

$t_0 = 0$ $\quad \lim_{t \to 0} \frac{3}{2(t-2)^2} \cdot t = 0$; $\lim_{t \to 0} \frac{t^2}{2t(t-2)} = 0$ $\Rightarrow 0$ regular singular

$t_0 = 2$ $\quad \lim_{t \to 2} \frac{3}{2(t-2)^{\cancel{2}}} \cdot \cancel{(t-2)}$ does not exist $\Rightarrow$ 2 irregular singular

Key idea for the right change: instead of $\sum_{n\geq 0} a_n(t-t_0)^n \rightsquigarrow |t-t_0|^r \cdot \sum_{n=0}^{\infty} a_n(t-t_0)^n$

with $r\in\mathbb{R}$.

For simplicity: $t>t_0 \implies y(t)=\sum_{n=0}^{\infty} a_n(t-t_0)^{n+r}$ is the new ansatz.

We need to determine: ⊛ a_n ⊛ R - radius of convergence ⊛ r as well.

Proposition: From plugging in the function $y(t)=\sum a_n(t-t_0)^{n+r}$, we get that r needs to satisfy

the following equation: $r(r-1)+\left[\lim_{t\to t_0}(t-t_0)\frac{Q(t)}{P(t)}\right]r+\left[\lim_{t\to t_0}(t-t_0)^2\frac{R(t)}{P(t)}\right]=0$

DEF: it is called the indicial equation.

Rmk: In this class we deal with the case when the roots are real. (Complex case is trickier).

After finding suitable r's, the rest is the same.

Example: Find a solution to $2t^2y''-ty'+(1+t)y=0$ for $t>0$ & near 0.

$t_0=0$ is singular : $p(t)=-\frac{t}{2t^2}=-\frac{1}{2t}$ & $q(t)=\frac{t+1}{2t^2}$

$\Rightarrow \lim_{t\to 0} -\frac{1}{2\cancel{t}}\cdot\cancel{t}=-\frac{1}{2}$ & $\lim_{t\to 0}\frac{t+1}{2\cancel{t^2}}\cdot\cancel{t^2}=\frac{1}{2} \Rightarrow t_0$ is regular singular

Indicial eq: $r(r-1) - \frac{r}{2} + \frac{1}{2} = 0$: $r(r-1) - \frac{1}{2}(r-1) = 0$: $\left(r-\frac{1}{2}\right)(r-1) \Rightarrow r = \frac{1}{2}, 1$ work.

$r=1$: $y = \sum_{n=0}^{\infty} a_n t^{n+1} \Rightarrow y' = \sum_{n=0}^{\infty} a_n (n+1) t^n$ & $y'' = \sum_{n=0}^{\infty} a_n (n+1).n.t^{n-1}$

$$\Rightarrow 2t^2 . \sum_{n=0}^{\infty} a_n (n+1) n . t^{n-1} - t \sum_{n=0}^{\infty} a_n (n+1) t^n + (1+t) \sum_{n=0}^{\infty} a_n t^{n+1} = 0$$

$$\sum_{n=2}^{\infty} 2a_n (n+1).n\, t^{n+1} - \sum_{n=0}^{\infty} a_n (n+1) t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} = 0$$

$$\sum_{n=2}^{\infty} 2a_n (n+1).n\, t^{n+1} - \sum_{n=1}^{\infty} a_n\, n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} = 0$$

t^2 : $-a_1 t^2 + a_0 t^2$

$t^{\geq 3}$: $\sum_{n=2}^{\infty} (2a_n (n+1) n - n a_n) t^{n+1} + \sum_{n=1}^{\infty} a_n t^{n+2} = 0$.

$$\sum_{m=1}^{\infty} (2a_{m+1} (m+2)(m+1) - (m+1) a_{m+1}) + a_m) t^{m+2} = 0$$

$$a_{m+1} [2(m+2)(m+1) - (m+1)] = a_{m+1} (m+1)(2m+3)$$

$$\Rightarrow \boxed{a_{m+1} = -\frac{a_m}{(m+1)(2m+3)}}$$ etc..