Qualitative discussion PART 2   §9.1-9.2   +   Laplace Transform   §6.1-6.2

Rmk: many ODE cannot be solved by analytic methods ⇒ it is important to consider qualitative info.

GOAL: know the behavior of solutions without solving the ODE.

We have discussed: autonomous + $\underbrace{\text{scalar}}$ + 1st-order ODE: $y' = G(y)$

NOW: 2x2 system! ⇒ $\vec{x}' = G(\vec{x})$

① First case: $G(\vec{x}) = A \cdot \vec{x}$, A constant matrix. Namely ODE: linear + homogeneous + const. coeff.'s. ① clarification

(Rmk: in this case, we do know the solutions)

② Eq. points: $G(\vec{x}) = 0 \Rightarrow A \cdot \vec{x} = 0 \Rightarrow$ Eq points $=$ Ker(A).

For simplicity we will consider only the case where $\ker(A) = \{0\}$, namely A is invertible

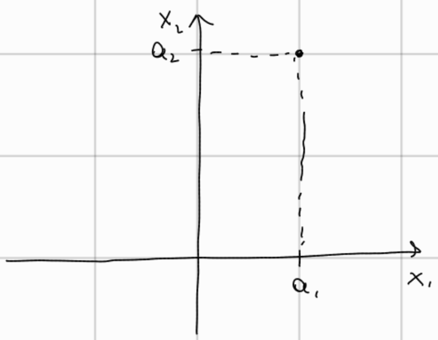
(i.e. $\det(A) \neq 0$), so we have just 1 eq. point $\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Remark: solutions of $x' = Ax$ are of the form $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$: so they can be represented in x_1x_2-plane

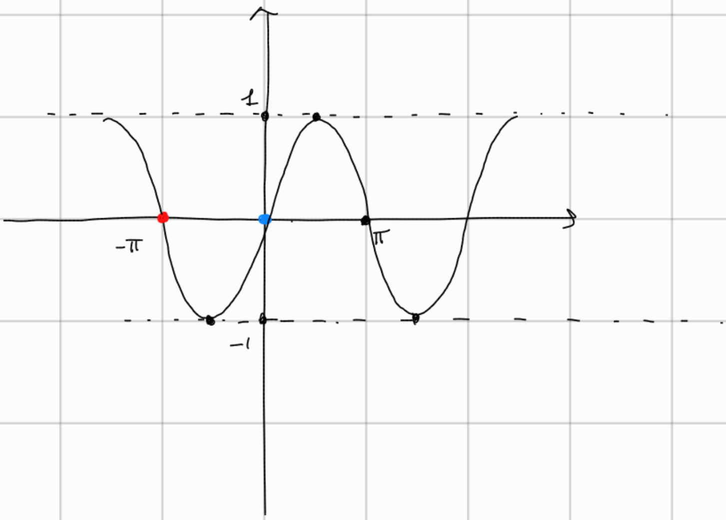
as follows:

for any time $\bar{t}$ ⟶ you get 2-coordinates $\begin{pmatrix} x_1(\bar{t}) \\ x_2(\bar{t}) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

which is a point in the x_1x_2-plane



Examples: ① $x(t) = \begin{pmatrix} t \\ \sin(t) \end{pmatrix}$

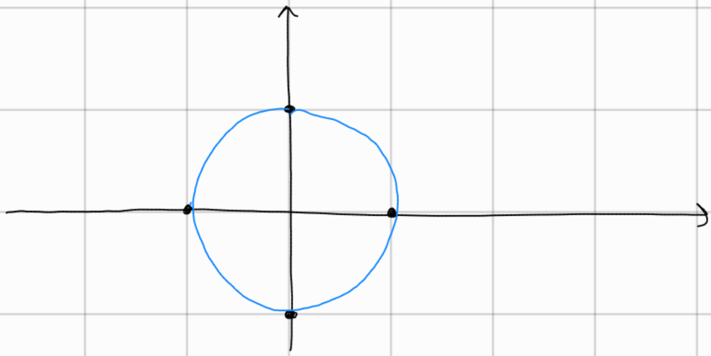


$t=0$: $\begin{pmatrix} 0 \\ \sin(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$t=-\pi$: $\begin{pmatrix} -\pi \\ \sin(\pi) \end{pmatrix} = \begin{pmatrix} -\pi \\ 0 \end{pmatrix} \Rightarrow$

$t=-\frac{\pi}{2}$: $\begin{pmatrix} -\pi/2 \\ -1 \end{pmatrix} \Rightarrow$

② $x(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$



$t=0$: $\begin{pmatrix} \sin(0) \\ \cos(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$t=\frac{\pi}{2}$: $\begin{pmatrix} \sin(\pi/2) \\ \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$t=\pi$: $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\qquad t=\frac{3}{2}\pi = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

DEFINITION: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a parametric representation (t is the parameter) of a curve in x_1x_2-plane

This curve is called the trajectory of the solution.

The plane x_1x_2 is called the phase plane

A "representative set" of trajectories is called phase portrait.

Let's see what a "representative set" is. Recall that we have 3 cases when we deal with $x' = Ax$

Ⓐ 2 different real eigenvalues λ_1, λ_2

Ⓑ 1 real eigenvalue with $\mu_a = 2$

Ⓒ complex conjugate eigenvalues.

Rmk: since $\det(A) \neq 0$, $\lambda = 0$ is never an eigenvalue!

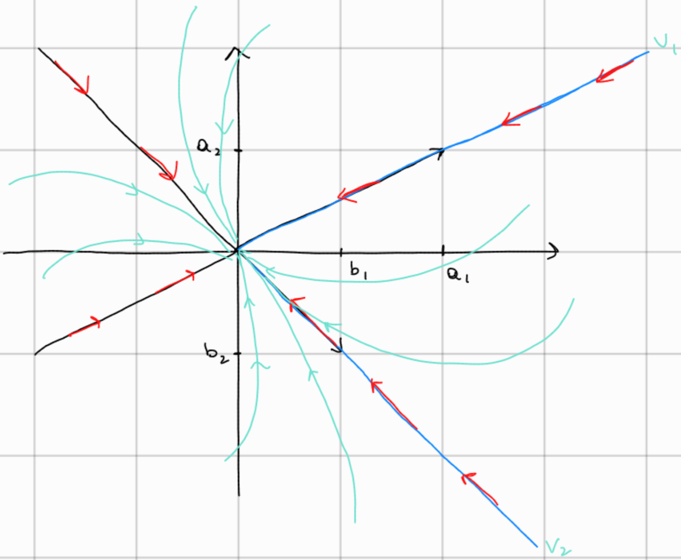
(A) Since $\lambda_1 \neq \lambda_2$: the solution is $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

A1 $\boxed{\lambda_1 < \lambda_2 < 0}$ (so $|\lambda_1| > |\lambda_2|$)

A2 $\boxed{\lambda_2 < 0 < \lambda_1}$ ($\lambda_1 > \lambda_2$)

A3 $\boxed{0 < \lambda_2 < \lambda_1}$ ($\lambda_1 > \lambda_2$)

A1 : step 1: draw in the phase plane the eigenvectors $v_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $v_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$



step 2: draw the solution $e^{\lambda_1 t} v_1 = \underbrace{\begin{pmatrix} e^{\lambda_1 t} a_1 \\ e^{\lambda_1 t} a_2 \end{pmatrix}}$ & $e^{\lambda_2 t} v_2 = \underbrace{\begin{pmatrix} e^{\lambda_2 t} b_1 \\ e^{\lambda_2 t} b_2 \end{pmatrix}}$

Notice: always a positive multiple of v_1 — always a positive multiple of v_2

step 3: draw $-e^{\lambda_1 t} v_1$, $-e^{\lambda_2 t} v_2$

step 4: put the "right arrows / directions" : since both λ_1 & λ_2 are negative

then $t \to +\infty$ $e^{\lambda_1 t} \to 0$ & $e^{\lambda_2 t} \to 0$

$\Rightarrow$ the solution $e^{\lambda_1 t} v_1 \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ & $e^{\lambda_2 t} v_2 \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ arrows pointing $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

step 5: fill in: regardless the values of c_1 & c_2 $\Rightarrow$ $c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to +\infty$

$\Rightarrow$ all the trajectories are approaching $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Moreover since $\lambda_1 - \lambda_2 < 0$: $\quad e^{\lambda_2 t} \left[\underline{c_1 v_1 e^{(\lambda_1 - \lambda_2)t}} + c_2 v_2 \right]$

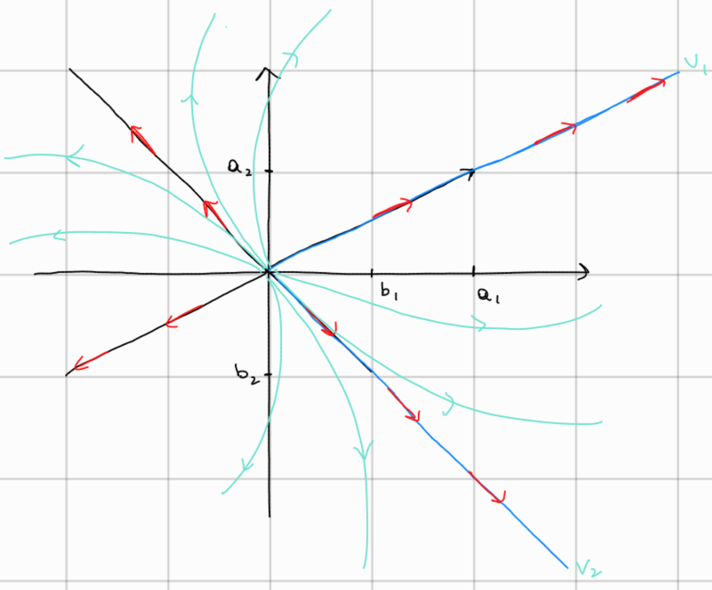
for $t \to +\infty$, this term can be forgotten w.r.t. $c_2 v_2$.

So not only the trajectories are approaching 0, they approach $e^{\lambda_2 t} c_2 v_2$ as well!

DEFINITION: in this case $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a NODE / NODAL SINK.

(A3) Same picture as before but reversed arrows.

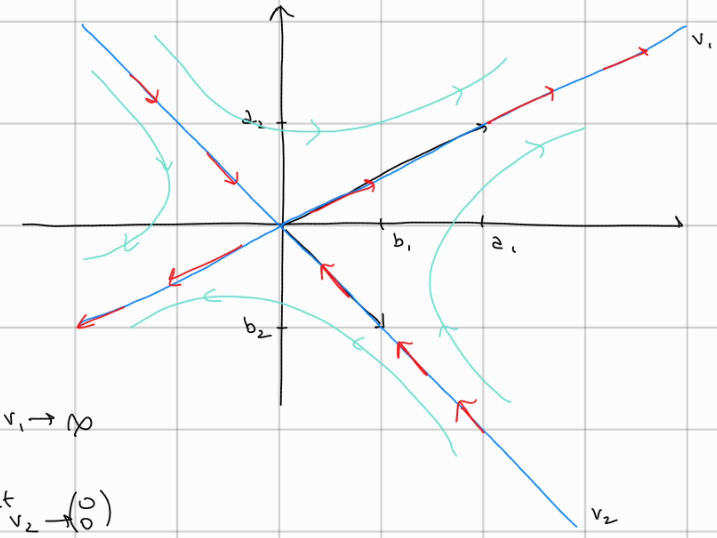
$t \to +\infty \Rightarrow e^{\lambda_1 t} v_1, \; e^{\lambda_2 t} \longrightarrow \infty$



Rule: $\boxed{\lambda_1 > \lambda_2}$!

DEF: NODE / NODAL SOURCE

(A2) Same steps: (*) eigenvectors:



(*) solutions $\pm e^{\lambda_1 t} v_1$

$\pm e^{\lambda_2 t} v_2$

(*) signs $\lambda_1 > 0$: $t \to +\infty \Rightarrow e^{\lambda_1 t} v_1 \to \infty$

$\lambda_2 < 0$: $t \to +\infty \Rightarrow e^{\lambda_2 t} v_2 \to \binom{0}{0}$

(*) fill-in the gaps

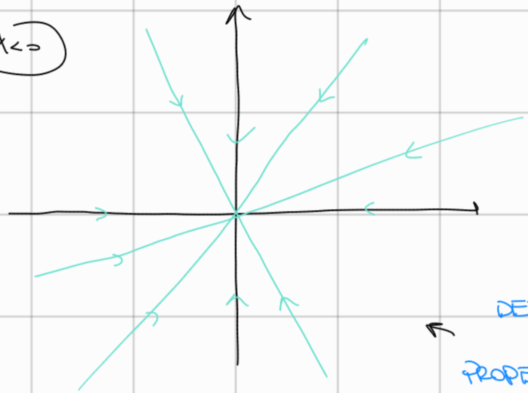
DEF: SADDLE POINT

(in particular all solution $\to \infty$ but $c_2 e^{\lambda_2 t} v_2$)

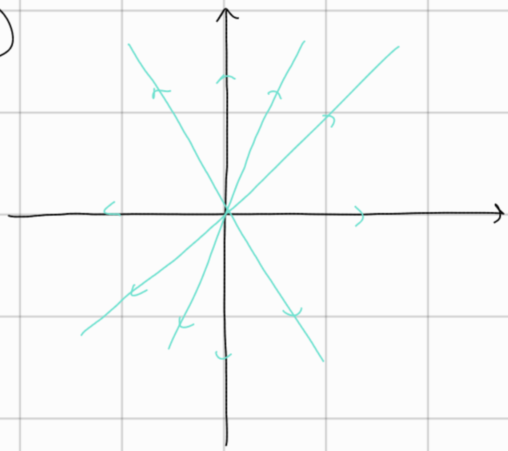
(B) B1) $\mu_a = \mu_g$, 2 lin. ind. eigenvectors. the general solution is $\underbrace{(c_1 v_1 + c_2 v_2)}_{v} e^{\lambda t}$

$\Rightarrow$ fixed c_1, c_2 : trajectory is ALWAYS a straight line.

($\lambda < 0$)



($\lambda > 0$)



DEF:

PROPER NODE / STAR POINT

(B2) $\mu_a > \mu_g$: 1 eigenvector $c_1 v e^{\lambda t} + c_2 (w + tv) e^{\lambda t}$ where $(A - \lambda Id) w = v$.

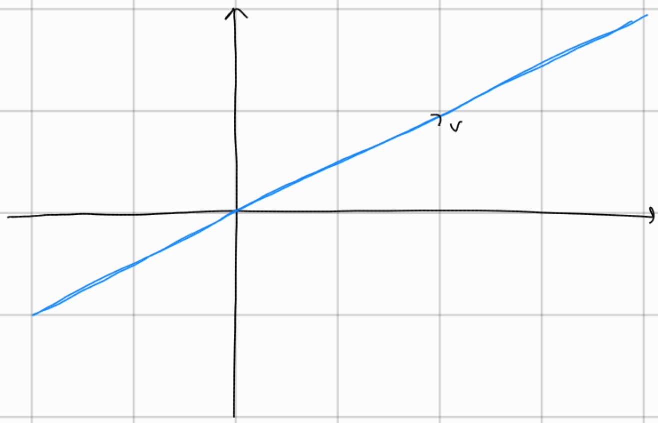
$= e^{\lambda t} [c_1 v + c_2 w + c_2 t v]$

(*) $e^{\lambda t}$ determines the limit of the solution for $t \to +\infty$

$\Rightarrow$ $\lambda > 0$: solution $\to \infty$, $\lambda < 0$: solution $\to \vec{0}$.

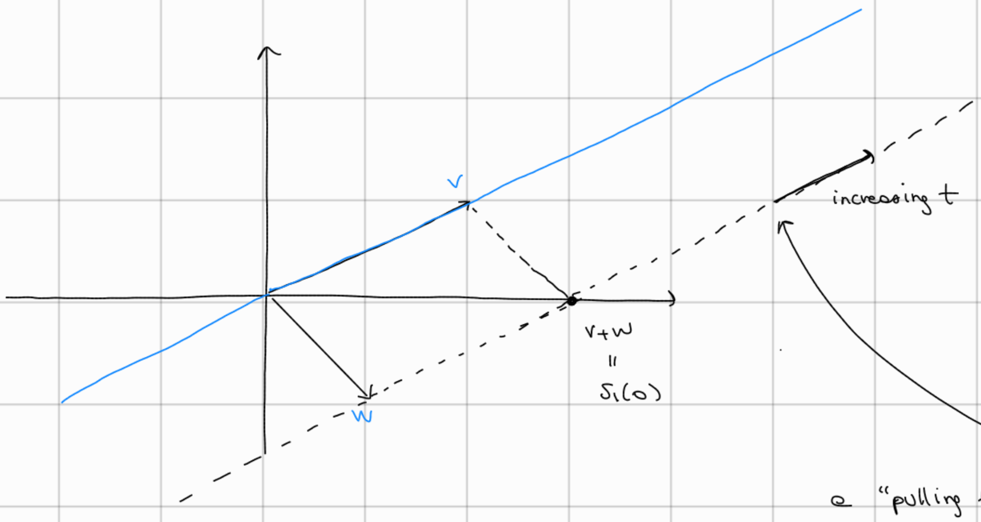
(*) in the term $c_1 v + c_2 w + c_2 t v$: $\boxed{\text{if } c_2 = 0}$: $c_1 v$ $\Rightarrow$ this corresponds to the solutions given by eigenvector $\cdot$ $e^{\lambda t}$.

$\Rightarrow$



if $c_2 \neq 0$: for $t \to \pm\infty$ the dominant term $[c_1 v + c_2 w + c_2 t v]$ is $c_2 t v$.

Draw w on the phase plane. Pose $c_1 = c_2 = 1$ & let's draw $e^{\lambda t}[v + w + tv] = s_1(t)$

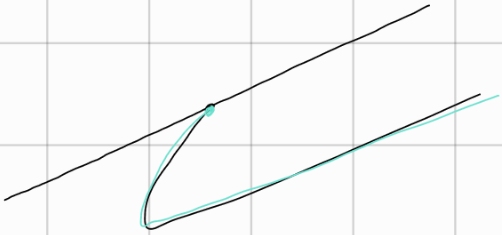


the dotted line is $(v+w) + tv$

you should imagine this line as a "pulling force" deforming $(v+w)e^{\lambda t}$ [straight line] using $vte^{\lambda t}$

$\Rightarrow$ we expect:

either



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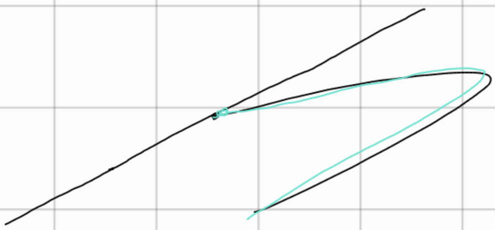
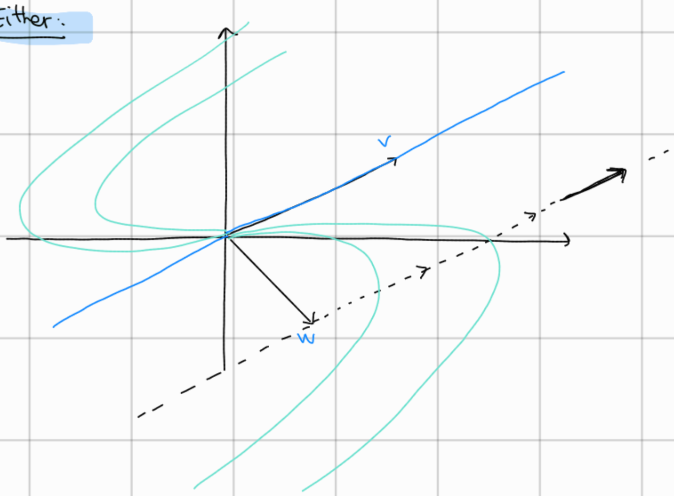


On the other side: the behavior is the same but in the opposite direction

Either:



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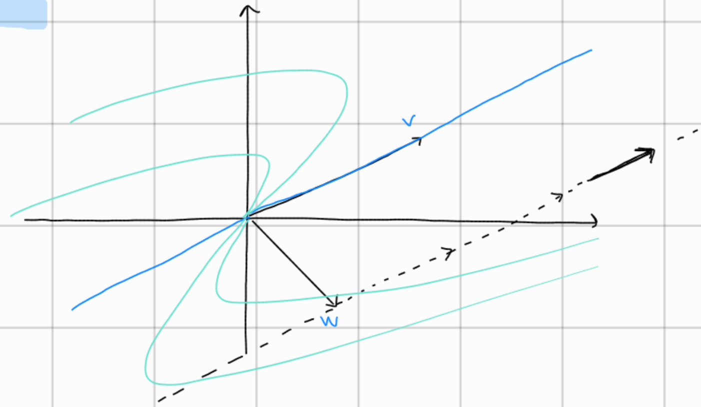


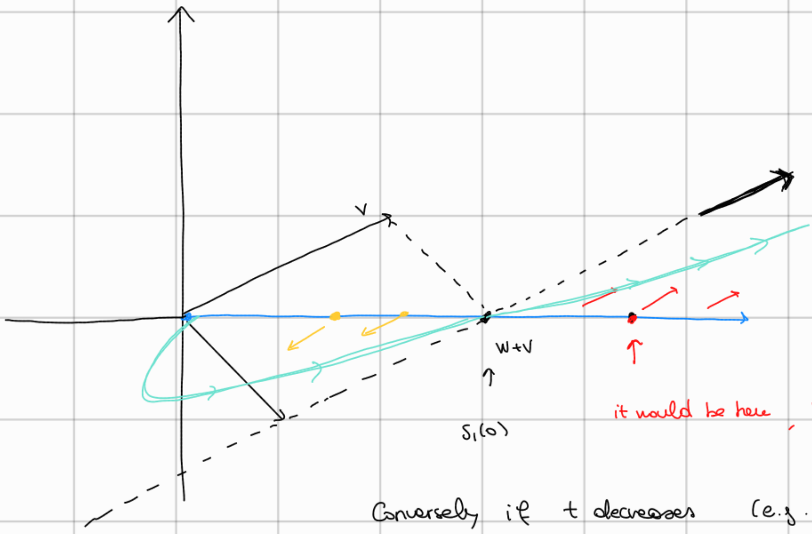
DEF: IMPROPER / DEGENERATE NODE

How to determine the right pulling?

(*) Consider the line $(v+w)+tv$ and notice the direction along which t increases (it is given by v)

$$e^{\lambda t}\left[v+w+tv\right] = \underbrace{(v+w)e^{\lambda t}}_{\text{blue line below}} + v\cdot t\cdot e^{\lambda t}$$

blue line below ↓

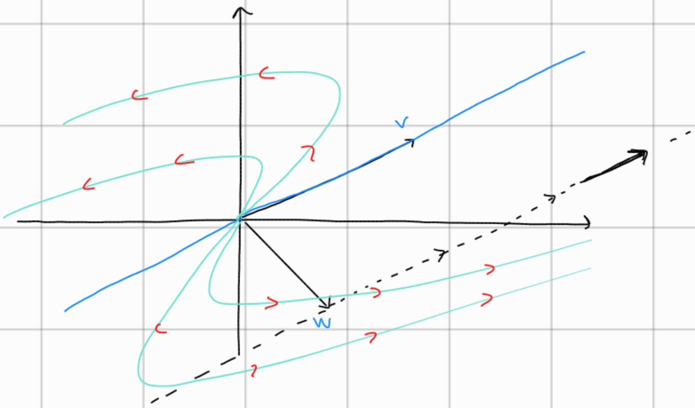


if $\lambda>0$, then t increasing (e.g. $t=1$)

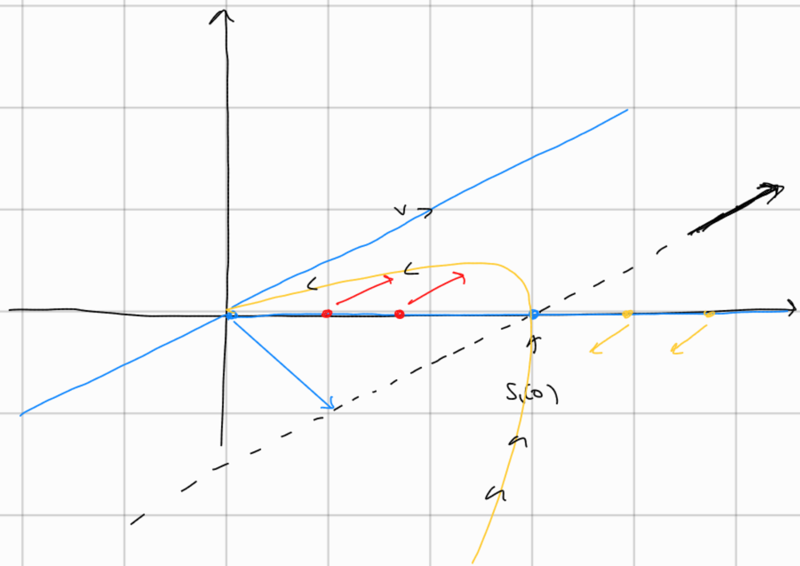
the point moves outward &

Conversely if t decreases (e.g. $t=-1$) the point moves inward (t is decreasing = pulling in the opposite direction

⇒ the right picture is then:



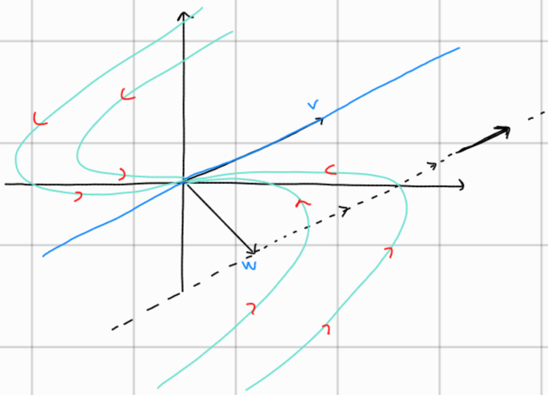
If $\lambda < 0$, the process is reversed:



t increasing ⇒ moving inward

t decreasing ⇒ moving outward

So, the right picture is:



(c) $\lambda = \alpha + i\beta$, $\bar{\lambda} = \alpha - i\beta$. & pick $\beta > 0$.

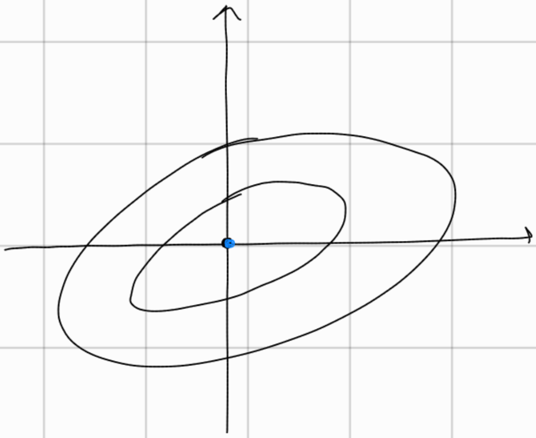
$v = \vec{x} + i\vec{y}$

(c1) $\boxed{\alpha = 0}$: general solution $s_1 = \cos(\beta t)\cdot\vec{x} - \sin(\beta t)\vec{y}$ ← periodic of period $\frac{2\pi}{\beta}$

$s_2 = \cos(\beta t)\vec{y} + \sin(\beta t)\vec{x}$

⇒ we get closed orbits! And they are around zero.

$\Rightarrow$



DEF : CENTER

clockwise / counterclockwise: you need to draw one solution.

$\boxed{C2 \quad \alpha \neq 0}$

$s_1 : \quad e^{\alpha t}\underbrace{[\cos(\beta t)\vec{x} - \sin(\beta t)\vec{y}]}$

$s_2 : \quad e^{\alpha t}\underbrace{[\cos(\beta t)\vec{y} + \sin(\beta t)\vec{x}]}$

we have a periodic component $\rightsquigarrow$ we are revolving around zero

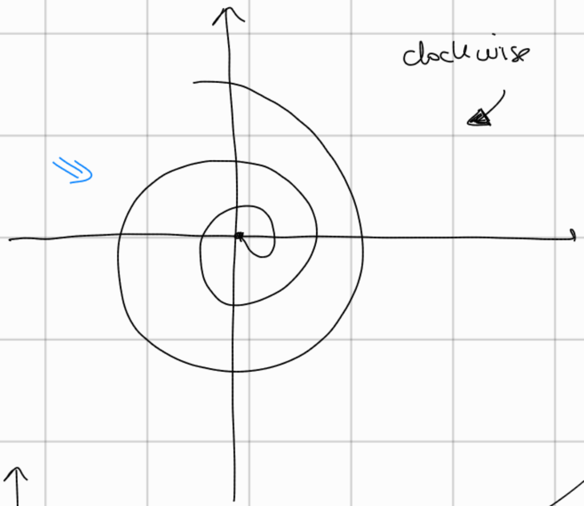
However: no more closed orbits! Indeed there is the exponential component which makes the trajectories collapse to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

SPIRAL SOURCE $\Downarrow$

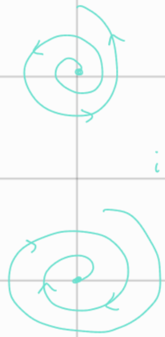
SPIRAL POINT $\Rrightarrow$

$\Rightarrow$

clockwise



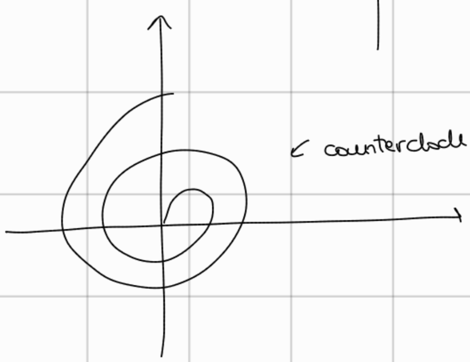
direction:



outward if $\alpha > 0$

inward if $\alpha < 0$

SPIRAL SINK $\Nearrow$



$\leftarrow$ counterclockwise. (enough to plot one to know)

- solutions escape ⇒ unstable
- solutions collapse ⇒ asymtotically stable
- solutions do not escape but do not collapse ⇒ stable

STABILITY CLASSIFICATION:

Nodal sink	asym. stable
Nodal source	unstable
Saddle point	unstable
Proper node $\lambda > 0$	unstable
Proper node $\lambda < 0$	asym. stable
Improper node $\lambda > 0$	unstable
Improper node $\lambda < 0$	asym. stable
Center	stable
Spiral sink	asym. stable
Spiral source.	unstable.

Examples: (I) $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$. $\det(A) = -6 + 4 = -2 \neq 0$.

Q1: Qualitative discussion + classify eq. points.

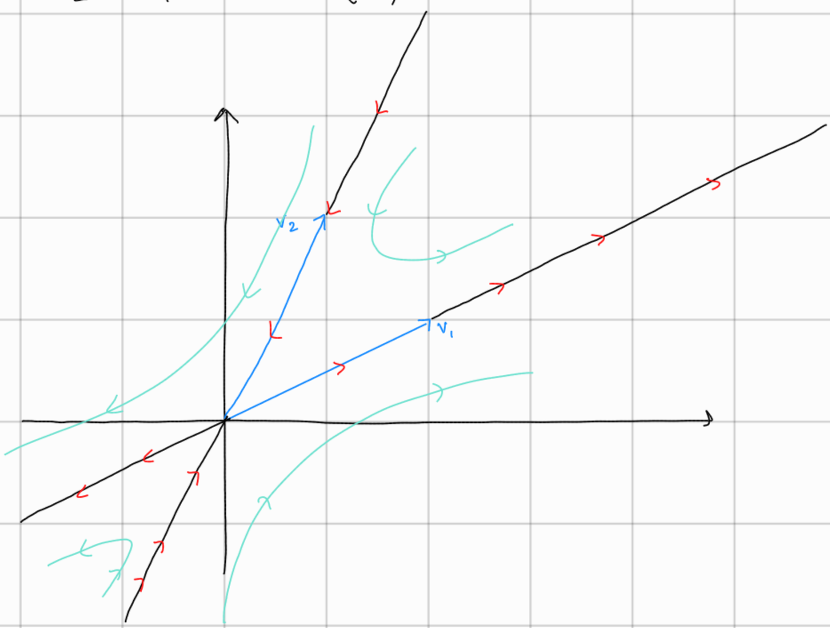
Find eigenvalues: $\det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (-2-\lambda)(3-\lambda) + 4 = -6 - 3\lambda + 2\lambda + \lambda^2 + 4 = \lambda^2 - \lambda - 2 =$

$= (\lambda - 2)(\lambda + 1) \Rightarrow \lambda = 2, -1$: $\underset{\lambda_2}{\underset{\uparrow}{-1}} < 0 < \underset{\lambda_1}{\underset{\uparrow}{2}}$

We are in the A2 - case: saddle point => unstable.

<u>Sketch</u>: we need to find the eigenvectors.

λ_1: $\lambda = 2$ $\quad \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} = v_1$ $\qquad$ λ_2: $\lambda = -1$ $\quad \left(\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



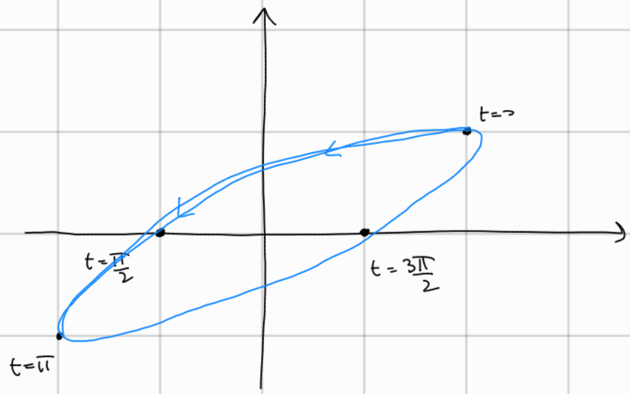
(II) $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$ $\qquad$ det $= -4+5 = 1$

Eigenvalues: $\begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} \Rightarrow \lambda^2 - 4 + 5 = \lambda^2 + 1 \Rightarrow \lambda = i, -i$ $\qquad \alpha = 0, \beta = 1.$

$\Rightarrow$ CENTER (stable)

Eigenvectors: $\left(\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right)$: $x_1 = (2+i)x_2$: $\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = v \Rightarrow x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$s_1 = \cos(t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$



$t=0 \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$t=\frac{\pi}{2} \quad \begin{pmatrix} -\sin(\pi/2) \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$t=\pi \quad \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$t=\frac{3\pi}{2} \quad \begin{pmatrix} 2\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right) \\ \cos\left(\frac{3\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

General case: no more linear: $x' = G(x) \to \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} G_1(x_1, x_2) \\ G_2(x_1, x_2) \end{pmatrix}$

Examples: $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ 2x_1x_2 \end{pmatrix}$

DEF: as before the set of points x : $G(x) = 0$ are called equilibrium / critical points

Equivalently: $G_1(x, x_2) = G_2(x_1, x_2) = 0$. $\rightsquigarrow$ they generate equilibrium solutions.

Stability: same definition as before
- stay close : stable
- escape : unstable
- collapse : asym. stable.

Aim: Use our discussion for linear ODEs for studying non linear ones in proximity of eq. points.

Key step: G_1, G_2 continuously differentiable $\Rightarrow$ the non-linear system can be approximated using Taylor expansion of G.

Recall: the first term of a Taylor expansion for a function $G: \mathbb{R}^2 \to \mathbb{R}^2$ at $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$G\begin{pmatrix}x_1\\x_2\end{pmatrix} = G\begin{pmatrix}a_1\\a_2\end{pmatrix} + \underbrace{J_G\begin{pmatrix}a_1\\a_2\end{pmatrix}}\begin{pmatrix}x_1-a_1\\x_2-a_2\end{pmatrix} + \underset{\dots}{\text{higher terms}}$$

$\Uparrow$ Jacobian of G :
$$\begin{pmatrix}\frac{\partial G_1}{\partial x_1}\Big|\begin{pmatrix}a_1\\a_2\end{pmatrix} & \frac{\partial G_1}{\partial x_2}\Big|\begin{pmatrix}a_1\\a_2\end{pmatrix}\\ \frac{\partial G_2}{\partial x_1}\Big|\begin{pmatrix}a_1\\a_2\end{pmatrix} & \frac{\partial G_2}{\partial x_2}\Big|\begin{pmatrix}a_1\\a_2\end{pmatrix}\end{pmatrix}$$

Rmk: if $\begin{pmatrix}a_1\\a_2\end{pmatrix}$ is picked to be one of the eq. points $\Rightarrow$ by definition $G\begin{pmatrix}a_1\\a_2\end{pmatrix} = \vec{0}$.

$\Rightarrow$ in order to study $\begin{pmatrix}x_1'\\x_2'\end{pmatrix} = G\begin{pmatrix}x_1\\x_2\end{pmatrix} \Rightarrow$ study $\boxed{\begin{pmatrix}(x_1-a_1)'\\(x_2-a_2)'\end{pmatrix} = J_G\begin{pmatrix}a_1\\a_2\end{pmatrix}\begin{pmatrix}x_1-a_1\\x_2-a_2\end{pmatrix}}$

Assumption: $\det\left(J_G\begin{pmatrix}a_1\\a_2\end{pmatrix}\right) \neq 0$!

This is the first case.

Example:
$$x_1' = -(x_1-x_2)(1-x_2-x_1) = G_1$$
$$x_2' = x_1(2+x_2) = G_2$$

1st) Find eq. points: $G_1 = 0 = G_2 \Rightarrow$
$G_1 \leftrightarrow x_1 = x_2$ or $x_1+x_2 = 1$
$G_2 \leftrightarrow x_1 = 0$ or $x_2 = -2$

$\Rightarrow \begin{cases}x_1 = x_2\\x_1 = 0\end{cases}$ or $\begin{cases}x_1 = x_2\\x_2 = -2\end{cases}$ or $\begin{cases}x_1+x_2 = 1\\x_1 = 0\end{cases}$ or $\begin{cases}x_1+x_2 = 1\\x_2 = -2\end{cases}$

$\Downarrow$ $p_1 = (0,0)$, $\Downarrow$ $p_2 = (-2,-2)$, $\Downarrow$ $p_3 = (0,1)$, $\Downarrow$ $p_4 = (3,-2)$

2nd) Find linear approximation: ⊛ Check continuity assumption. G_1 & G_2 are polynomials so they are cont. differentiable.

⊛ Find Jacobian:
$$\frac{\partial G_1}{\partial x_1} = \frac{\partial}{\partial x_1} -(x_1-x_2)(1-x_2-x_1) = -(1-x_2-x_1) + (x_1-x_2) = -1 + \cancel{x_2} + x_1 + x_1 \cancel{-x_2} = 2x_1 - 1$$

$$\frac{\partial G_1}{\partial x_2} = (1-x_2-x_1) + (x_1-x_2) = 1 - x_2 \cancel{-x_1+x_1} - x_2 = 1 - 2x_2$$

$$\frac{\partial G_2}{\partial x_1} = \frac{\partial}{\partial x_1} x_1(2+x_2) = 2+x_2 \quad ; \quad \frac{\partial G_2}{\partial x_2} = x_1$$

$$J_G(x_1,x_2)=\begin{pmatrix} 2x_1-1 & 1-2x_2 \\ x_2+2 & x_1 \end{pmatrix}.$$

$$J_G(p_1)=\begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix};\quad J_G(p_2)=\begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix};\quad J_G(p_3)=\begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix};\quad J_G(p_4)=\begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}$$

(1) $\begin{pmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{pmatrix} \Rightarrow -\lambda(-1-\lambda)-2=0 \quad \lambda+\lambda^2-2=0 \quad \lambda^2+\lambda-2=0 \quad (\lambda+2)(\lambda-1)=0:$

$-2<0<1 \Rightarrow$ saddle point.

(2) $\begin{pmatrix} -5-\lambda & 5 \\ 0 & -2-\lambda \end{pmatrix} \Rightarrow (-5-\lambda)(-2-\lambda)=0 \Rightarrow \lambda=-5,-2 : -5<-2<0 \Rightarrow$ nodal sink

(3) $\begin{pmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{pmatrix} \Rightarrow \lambda(\lambda+1)+3=0 \quad \lambda^2+\lambda+3=0 \Rightarrow \lambda_{1,2}=\frac{-1\pm\sqrt{1-12}}{2} : \alpha=-\frac{1}{2},\ \beta=\frac{\sqrt{11}}{2}.$

$\Rightarrow$ spiral sink.

(4) $\begin{pmatrix} 5-\lambda & 5 \\ 0 & 3-\lambda \end{pmatrix} \Rightarrow \lambda=3,5 \Rightarrow$ nodal source

Clarification for stability for non-linear

if the linearized ODE has: | we say the non linear on eq. point is:

Nodal sink	asym. stable
Nodal source	unstable
Saddle point	unstable

Proper node $\lambda > 0$	unstable	
Proper node $\lambda < 0$	asym. stable	
Improper node $\lambda > 0$	unstable	
Improper node $\lambda < 0$	asym. stable	
Spiral sink	asym. stable	
Spiral source.	unstable.	

Same as before apart from

CENTER $\longrightarrow$ indeterminate.

We say the same if $\det(J_G(p)) = 0$

Laplace Transform: Recall that

$$\underbrace{\int_0^{+\infty} f(t)\,dt}_{\text{improper integral}} = \lim_{T \to +\infty} \int_0^T f(t)\,dt \quad \text{(if this limit exists)}$$

Notation: if the limit $\exists$ & it is finite we say that the improper integral converges

" " " $\nexists$ or " " ∞ " " " " " " " " diverges.

Example: $f(t) = e^{ct}$: $c \in \mathbb{R}$

$$\int_0^T e^{ct}\,dt = \begin{cases} \left.\dfrac{e^{ct}}{c}\right|_0^T & \text{if } c \neq 0 \Rightarrow \dfrac{e^{cT} - 1}{c} \\ t\,\Big|_0^T & \text{if } c = 0 \Rightarrow T \quad \lim_{T \to +\infty} T = +\infty \quad \times \end{cases}$$

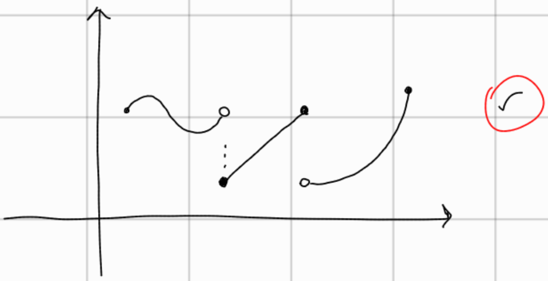
$$\lim_{T \to +\infty} \frac{e^{cT} - 1}{c} = \begin{cases} +\infty & c > 0 \quad \times \\ -\frac{1}{c} & c < 0 \quad \checkmark \end{cases} \Rightarrow \int_0^{+\infty} e^{ct}\,dt \text{ converges iff. } c < 0.$$

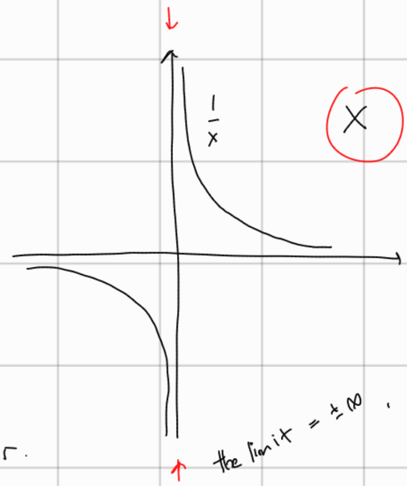
DEF: $f(t)$ is ==piecewise continuous== on $[a,b]$ if $\exists\ a < t_1 < \dots < t_{n-1} < b$ s.t.

1. $f(t)$ continuous on $[a, t_1), (t_1, t_2), (t_2, t_3), \dots, (t_{n-1}, b]$

2. $\lim\limits_{t \to t_i^+} f(t)$, $\lim\limits_{t \to t_i^-} f(t)$ both exist & finite

Example:

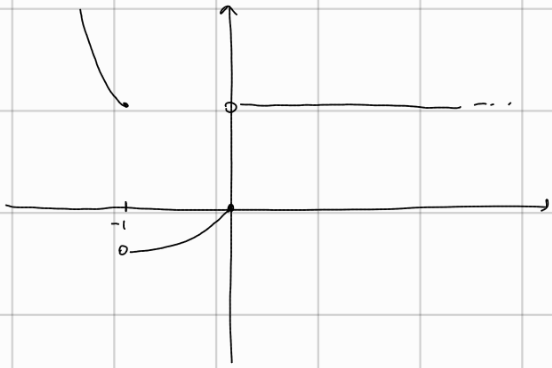




Remark: any continuous function is piecewise continuous.

In particular $f(t)$ piecewise continuous can be written as
$$f(t) = \begin{cases} f_1(t) & t \in [a, t_1) \\ \vdots & \vdots \\ f_n(t) & t \in (t_{n-1}, b] \end{cases}$$

Example:
$$f(x) = \begin{cases} x^2 & x \leq -1 \\ \sin(x) & -1 < x \leq 0 \\ 1 & x > 0 \end{cases}$$



DEF: Take $f(t)$ piecewise continuous : define the Laplace transform of f $\mathcal{L}(f)$ to be

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

s = new variable - it does <u>NOT</u> depend on t.

Rmk: $\mathcal{L}(f)$ may not be well defined for all s.

Example: $f(t) = e^{at} \Rightarrow \mathcal{L}(f)(s) = \int_0^\infty e^{-st} \cdot e^{at} = \int_0^\infty e^{(a-s)t} dt = \frac{1}{s-a}$ for $\underline{\underline{s>a}}$

(it is not well defined otherwise).

for what we have seen before

However, we have nice properties:

1. Laplace transform is linear, $\mathcal{L}(af+bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$

2. Relationship with derivatives:

a) [under some technical conditions: f continuous + f' piecewise continuous + $\exists\, k, a, M$ s.t.

$$|f(t)| \le Ke^{at} \quad \forall t \ge M \quad]$$

$\Rightarrow$ $\mathcal{L}(f'(t))\ \exists\ \forall s>a$ and $\mathcal{L}(f'(t)) = s\cdot\mathcal{L}(f) - f(0)$

b) [Assume f & f' continuous + f'' piecewise continuous + $\exists\, k, a, M$: $|f(t)|, |f'(t)| \le Ke^{at}$ $\forall t \ge M$]

$\Rightarrow$ $\mathcal{L}(f''(t))\ \exists\ \forall s>a$ &

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

c) For general n-th derivative: $\mathcal{L}(f^{(n)}(t)) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \dots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)$.

[again under technical assumptions]

How do we want to use it?

Pick an IVP: ex. $y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

Apply $\mathcal{L}$ on both sides assuming that you can. $\Rightarrow$ $\mathcal{L}(y'' - y' - 2y) = \mathcal{L}(0) = 0$.

$$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 0 . \qquad \underline{s^2 \mathcal{L}(y)} - s \underset{\substack{\uparrow \\ 1}}{y(0)} - \underset{\substack{\uparrow \\ 0}}{y'(0)} \ \underline{- s\mathcal{L}(y)} - \underset{\substack{\uparrow \\ 1}}{y(0)} \ \underline{-2\mathcal{L}(y)} = 0 .$$

$$\Rightarrow \mathcal{L}(y)(s^2 - s - 2) = s - 1 \quad \Rightarrow \mathcal{L}(y) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{1}{3}\underbrace{\frac{1}{s-2}}_{\substack{\uparrow \\ \mathcal{L}(e^{2t})}} + \frac{2}{3} \cdot \underset{\substack{\uparrow \\ \mathcal{L}(e^{-t})}}{\frac{1}{s+1}}$$

Pro: find $\mathcal{L}(y)$ of a solution is easy

$$\Downarrow$$

Cons: find $\mathcal{L}^{-1}$ - Laplace inverse of a function can be tricky. $\qquad y = \frac{e^{2t}}{3} + \frac{2}{3} e^{-t}$

[THM: if $\exists$, it is unique]

However! There is a table you can use - 317 of the book.