The Standard Conjectures

Caleb Ji

This note is meant to explain the content in the articles [2], [3], which are themselves an elaboration on the standard conjectures set out by Grothendieck in [1]. Essentially everything in this note can be found in those articles by Kleiman.

1 Algebraic correspondences

1.1 Definitions

Let X be a smooth projective variety of dimension d and let $H^*(-)$ be a Weil cohomology theory. In particular, we have Poincaré duality, a Künneth formula, a cycle map, and the Hard Lefschetz theorem.

Definition 1.1. *A correspondence $u \in H^*(X \times Y)$ is a correspondence when viewed as a map $u : H^*(X) \to H^*(Y)$ under the isomorphisms given by the Künneth formula and Poincaré duality:*

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y) \cong \mathrm{Hom}(H^*(X), K) \otimes H^*(Y) \cong \mathrm{Hom}(H^*(X), H^*(Y)).$$

We use $A^*(X)$ to denote the $\mathbb{Q}$-subspace spanned by the cycle class map from the Chow ring to $H^*(X)$. Correspondences contained in $A^*(X \times Y)$ are called **algebraic**.

The first order of business is to understand how correspondences act. If we take $u = a \otimes b \in H^{2d-i}(X) \otimes H^j(Y) \subset H^{i+j}(X \times Y)$ and $x \in H^i(X)$, we have $u(x) = a(x) \otimes b = \mathrm{Tr}(a \cup x) \otimes b = \langle x, a \rangle b$. More generally, given the projections

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \\ X & & \end{array}$$

we have the formula

$$u(x) = (p_2)_*(p_1^* x \cdot u).$$

The composition of two correspondences $u \in H^*(X \times Y)$ and $v \in H^*(Y \times Z)$ is a correspondence $vu \in H^*(X \times Z)$ given by

$$vu = p_{13*}(p_{12}^* u \cdot p_{23}^* v).$$

Note in particular that the composition of two algebraic correspondences is once again algebraic.

1.2 Examples

An important class of examples of correspondences is given by the graphs of morphisms. First, let us review the functorial properties of such a map $g : Y \to X$ (which were already used in previous formulas). First we have the pullback $g^* : H^*(X) \to H^*(Y)$. We also have a pushforward $g_* : H^*(Y) \to H^*(X)$ defined via Poincaré duality:

$$\mathrm{Tr}(x \cup g_*(y)) = \mathrm{Tr}(g^*(x) \cup y).$$

Note in particular that the degree of $g_*(y)$ is $2\dim(X) - 2\dim Y$ greater than the degree of y.

Now we claim that if we take

$$u = \mathrm{cl}(\Gamma_g) \in H^*(X \times Y),$$

then we have that $u = g^*$. To check this, we need to check that

$$g^*(a) = p_{2*}(p_1^*(a) \cdot \Gamma_g),$$

which can be done by through the definition of g_*.

Furthermore, we can consider the transpose $u^T \in H^*(Y \times X)$ which is obtained from simply flipping the Künneth components. Then $u^T = g_*$.

Let us give one important example of an algebraic correspondence before stating the standard conjectures. Let H be a hyperplane class of X. Then we define the Lefschetz operator $L : H^*(X) \to H^{*+2}(X)$ by

$$L(x) = c_1(H) \cup x.$$

Then L is indeed algebraic. Note that to show this, we have to find a class $u \in A^{d+2}(X \times X)$ that represents L; i.e.

$$p_{2*}(p_1^* x \cdot u) = x \cdot c_1(H).$$

A candidate for u is given by $p_1^*(\gamma_X(H)) = \gamma_{X\times X}(\Delta(H))$. Indeed, this is correct. To prove this, one must show that

$$p_{2*}(p_1^*(\alpha) \cup \gamma_{X\times X}(\Delta_*(H))) = \alpha \cup \Gamma_X(H).$$

One does this by using the defining property of pushforwards:

$$\langle f^*(a), b\rangle = \langle a, f_*(b)\rangle.$$

2 Statements of conjectures

2.1 Conjectures ABCDH

We recall that the Hard Lefschetz theorem gives an isomorphism

$$L^{d-i} : H^i(X) \cong H^{2d-i}(X)$$

for all $0 \le i \le d$. This motivates the first standard conjecture.

Conjecture 2.1 ($A(X)$)**.** *Hard Lefschetz on cycles. The operator L^{d-2r} on cycles:*

$$L^{d-2r} : A^r(X) \to A^{n-r}(X)$$

is an isomorphism.

Remark. Grothendieck also stated a weak form of this conjecture, which was just the Hard Lefschetz theorem. Deligne proved this for étale cohomology as a consequence of the Weil conjectures.

One can use the operator Λ to give an equivalent operation. The Λ operator

$$\Lambda : H^*(X) \to H^{*-2}(X)$$

may be defined in the following way. For $0 \le i \le n$, use the following commutative diagram.

$$\begin{tikzcd}
H^i(X) \arrow[r, "L^{d-i}\cong"] \arrow[d, "\Lambda"'] & H^{2d-i}(X) \arrow[d, "L"] \\
H^{i-2}(X) \arrow[r, "L^{d-i+2}\cong"'] & H^{2d-i+2}(X)
\end{tikzcd}$$

That is, $\Lambda(x) = (L^{d-i+2})^{-1} \circ L^{d-i+1}$ on degrees at most d, and on H^{2d-i+2} it is similarly defined by $L^{d-i+1} \circ (L^{d-i+2})^{-1}$. We see that Λ provides a on-sided inverse to L.

Conjecture 2.2 ($B(X)$)**.** *The Λ-operator is algebraic.*

A weaker form of this conjecture can be stated using the projection operators π^i. By the Künneth formula, we may write $H^*X \times X$ as $H^*(X) \otimes H^*(X)$. The projector

$$\pi^i \in \mathrm{Hom}(H^i(X), H^i(X)) \in H^{2d}(X \times X)$$

is defined by the projection of the diagonal $\gamma_{X\times X}(\Delta) \in H^{2d}(X \times X)$ onto the $H^{2d-i} \otimes H^i$ component.

Conjecture 2.3 ($C(X)$)**.** *For each i, the projector π^i is algebraic.*

The next conjecture involves two forms of equivalence of cycles: homological and numerical. An algebraic cycle $Z \subset X$ is homologically 0 if $\gamma_X(Z) = 0$, and it is numerically 0 if its intersection number is 0 with any other cycle. Because $\gamma_X(Z \cdot Z') = \gamma_X(Z) \cup \gamma_X(Z')$, we see that homological equivalence implies numerical equivalence.

Conjecture 2.4 ($D(X)$)**.** *Numerical equivalence implies homological equivalence.*

Finally we have the Hodge standard conjecture, which over $\mathbb{C}$ is known to be true by the Hodge index theorem. To state it, define the primitive subspace $P^i(X) \subset H^i(X)$ by

$$P^i(X) = (L^{d-i})^{-1}(\ker L | H^{2d-i}(X)).$$

Conjecture 2.5 ($\mathrm{Hdg}(X)$)**.** *For $i \le d$, the $\mathbb{Q}$-valued pairing on $A^i(X) \cap P^{2i}(X)$ given by*

$$(x, y) = (-1)^i \langle L^{d-2i} x \cdot y \rangle$$

is positive definite.

2.2 Primitive cohomology and additional operators

Recall that we defined the primitive subspace $P^i(X) \subset H^i(X)$ to be

$$P^i(X) = (L^{d-i})^{-1}(\ker L | H^{2d-i}(X)).$$

This gives a decomposition $H^i(X) = P^i(X) \oplus L(H^{i-2}(X))$. Continuing this process gives the Lefschetz decomposition

$$H^i(X) = \bigoplus_{j \ge i-d, 0} L^j P^{i-2j}(X).$$

We can then define Λ by
$$\Lambda(x) = \sum_{j \geq i-d,1} L^{j-1}(x_j)$$
where $x_j \in P^{i-2j}(X)$ are the primitive components of x.

In Hodge theory, the Λ-operator is defined not as an inverse operation to L, but so that $[L, \Lambda] = (k-n)\,\mathrm{id}$ on $H^k(X)$. Here we will keep our original definition of Λ but instead define
$${}^c\Lambda(x) = \sum_{j \geq i-d,1} j(n-i+j+1)L^{j-1}(x_j).$$
Up to some constant multiple, one checks that ${}^c\Lambda^{n-i}$ provides an inverse on the primitive component of $L^{n-i} : P^i(X) \to L^{n-i}P^i(X)$. It does indeed satisfy $[L, {}^c\Lambda] = (k-n)\,\mathrm{id}$. Moreover, it is algebraic if and only if Λ is.

We may also define a version of the Hodge star operator
$$* : H^i(X) \to H^{2d-i}(X)$$
by the formula
$$*x = \sum_{j \geq i-d,0} (-1)^{(i-2j)(i-2j+1)/2} L^{d-i+j} x_j.$$
This satisfies $*^2 = 1$ and $\Lambda = *L*$.

We now call attention to the fact that the definition of the Lefschetz operator L – and thus our definition of Λ – depends on the choice of a hyperplane class. However, the standard conjecture B does not. In fact, Λ begin algebraic for a single choice of H is equivalent to there existing an algebraic correspondence $v_i : H^{2r-i}(X) \cong H^i(X)$. One direction is clear. In the other direction, first we show that the existence of an algebraic inverse θ^i to L^{r-i} for every $i \leq d$ implies B. This is done through writing Λ as a sum and composition of θ^* and L. Next, we want to show that the existence of some algebraic isomorphism $v^i : H^{2d-i}(X) \cong H^i(X)$ implies the existence of such a θ^i. One sets $u = v^i L^{d-i}$ and $\theta^i = u^{-1}v^i$; this gives the desired inverse to L^{r-i} and it is algebraic through some linear algebra. (See pg. 15 of [3].)

3 Implications between the conjectures

The implications of these conjectures are given by
$$A \Leftrightarrow B \Rightarrow C, \quad \text{given } \mathrm{Hdg}: \ A \Leftrightarrow D$$

Proof of $A \Leftrightarrow B$. The fact that B implies A is simple: the operator Λ^{r-i} provides an algebraic inverse to L^{r-2i}, and it brings algebraic correspondences to algebraic correspondences, so L^{r-2i} must be an isomorphism restricted to $A^i(X)$. In the other direction, we claim:
$$A(X \times X, L \otimes 1 + 1 \otimes L) \Rightarrow B(X).$$
Indeed, ${}^c\Lambda \otimes 1 + 1 \otimes {}^c\Lambda : A^*(X \times X) \to A^{*-2}(X \times X)$ sends the class $\gamma_{X\times X}(\Delta)$ to $2({}^c\Lambda)$. Thus ${}^c\Lambda$ is algebraic, which as we have noted before, implies that Λ is algebraic. ☐

Proof of $B \Rightarrow C$. It suffices to write the projector π_i in terms of Λ and other algebraic correspondences. We have
$$\pi^i = \Lambda^{d-i}\left(1 - \sum_{j>2d-i} \pi^j\right) L^{d-i}\left(1 - \sum_{j<i} \pi^j\right).$$

Thus by induction, π^i is algebraic. $\square$

Proof that given $\mathrm{Hdg}(X)$*, we have* $A(X,L) \Leftrightarrow D(X)$. Assume A. By $\mathrm{Hdg}(X)$ we know that

$$(x,y) = (-1)^i \langle L^{d-2i} x \cdot y \rangle$$

is positive-definite, and we claim that

$$(x,y)' := \langle x \cdot *y \rangle = \sum_{j \geq i-d,0} (-1)^{(i-2j)(i-2j+1)/2} \langle x, L^{d-i+j} y_j \rangle$$

is also positive-definite. Here $y_j \in P^{i-2j}(X)$. By writing down the primitive decomposition of x and checking picking out the nonzero components of this sum, we see that each summand is of the form (x,y) as desired. Then by $A(X)$, the canonical pairing $A^i(X) \times A^{d-i}(X) \to \mathbb{Q}$. This means that if x is numerically 0, then it is homologically 0.

In the opposite direction, the canonical pairing above is nonsingular. We claim that these groups have finite dimension, which means that they have the same dimension and thus $A(X,L)$ holds (because L^{d-2i} is injective by Hard Lefschetz. But $C^i_{neq}(X)$ has finite dimension because we can embed it in $\mathbb{Z}^m$ by taking the intersection products with $y_1, \dots, y_m$ a basis of $A^{r-i}(X)$. $\square$

4 The Weil conjectures

4.1 Integrality and independence of l of the characteristic polynomials

Let X be a smooth projective variety over $\mathbb{F}_q$. The zeta function of X is defined as

$$Z(X,T) = \exp\left(\sum_{n>0} \frac{|X(\mathbb{F}_{q^n})| T^n}{n}\right) = \prod_{x \in |X|} \frac{1}{1 - T^{\deg x}}.$$

The first Weil conjecture expresses the zeta function as a rational function in terms of the following characteristic polynomials:

$$P_i(T) \det(1 - tF | H^i(X)),$$

where F is the geometric Frobenius. While the zeta function itself an integer coefficients, it is not clear that the individual $P_i(T)$ do, nor that they are independent of Weil cohomology theory. However conjecture C gives a very clean proof of these facts. To explain this, we will ened the following form of the Lefschetz trace formula.

Proposition 4.1 (Lefschetz trace formula). *Given a correspondence* $u \in H^{2d}(X \times X)$*, restricting to its action on* $H^i(X)$ *we have*

$$\mathrm{Tr}(u | H^i(X)) = (-1)^i \langle u \cdot \pi^{2r-i} \rangle.$$

One proves this in the exact same way as for the usual statement; take a basis and just compute. The fact that π^{2r-i} is algebraic given conjecture C gives our desired consequences.

Theorem 4.2. *Given that* π^{2d-i} *is algebraic, if* u *is an algebraic correspondence in* $H^*(X \times X)$*, then*

$$P_i(T) = \det(1 - uT | H^i(X))$$

is a polynomial with integer coefficients. Furthermore, these coefficients are given by polynomials in the rational numbers

$$s_n := \langle u^n \cdot \pi^{2d-i} \rangle$$

for $n = 1, 2, \dots, b_i$ *which are independent of the choice of* H^*.

Proof. By the algebraicity assumptions, there is a fixed nonzero integer m such that ms_n is an integer. By the trace formula, s_n is the sum of the nth powers of the eigenvalues of u acting on $H^i(X)$. Because this is true for every n, an elementary argument shows that the eigenvalues are algebraic integers. Thus the coefficients of $P_i(t)$ are algebraic integers too. But these can also be universally solved for using the Newton identities with rational coefficients in the s_n. Thus they are independent of H^* and also rational, and thus also integral. □

4.2 The Riemann hypothesis for function fields

Assume $B(X)$ and $\mathrm{Hdg}(X \times X)$. These will allow us to construct a form $\langle u, v\rangle = \mathrm{Tr}(u'v)$ with very nice properties. Specifically, given a correspondence u, set $u' = *u^T*$. Since $B(X) \Rightarrow *$ is algebraic, we know that u' is algebraic if and only if u is. By $B(X) \Rightarrow C(X)$, the Lefschetz trace formula implies that $\mathrm{Tr}(u'u) \in \mathbb{Q}$. Finally, one computes that $\mathrm{Hdg}(X \times X)$ implies that $\mathrm{Tr}(u'u) > 0$. This is a reasonably long but straightforward calculation done on pg. 381 of [3].

Under these assumptions, the endomorphism ring of homological/numerical motives $A^*(X \times X)$ is semisimple. For numerical motives, this has been unconditionally proven by Janssen. But through the standard conjectures, this fact also allows us to show that the eigenvalues of the Frobenius on $H^i(X)$ have absolute value $q^{i/2}$.

Proposition 4.3. *Assuming $B(X)$ and $\mathrm{Hdg}(X \times X)$, we have that $A^*(X \times X)$ is a semisimple $\mathbb{Q}$-algebra.*

Proof. Suppose to the contrary that $u \neq 0$ is nilpotent. Then $\mathrm{Tr}((u'u)^{2^n})) = \mathrm{Tr}(u^{2^n}u'^{2^n}) = 0$ for some n. By the positive-definiteness of the form $\langle u, v\rangle = \mathrm{Tr}(u'v)$, we have $u^{2^{n-1}}u'^{2^{n-1}} = 0$. We may assume that $u^{2^{n-2}}u'^{2^{n-2}} \neq 0$, but then again by the positive-definiteness of the form, we get that $u^{2^{n-1}}u'^{2^{n-1}} \neq 0$, contradiction. (If $n = 1$, a simple analogous argument works.) □

Proposition 4.4. *Assuming $B(X)$ and $\mathrm{Hdg}(X \times X)$, the eigenvalues of the Frobenius acting on $H^i(X)$ have absolute value $q^{i/2}$.*

Proof. The Frobenius Fr is an algebraic correspondence. Let $\mathrm{Fr}_i = F|H^i(X)$ and set $g = \sum_i \mathrm{Fr}_i / q^{i/2}$. Then g is an automorphism of $H^*(X)$ with $g_{2d} = \mathrm{id}$. For $a \in H^i(X)$ and $b \in H^{2d-i}(X)$, we have

$$\langle g_i^{-1}(a) \cdot b\rangle = \langle g(g_i^{-1}(a) \cdot b)\rangle = \langle a \cdot g_{2d-i}b\rangle,$$

so we have $g^{-1} = g^T$.

Next, we claim that $g(H) = H$ for a hyperplane class $H \in H^{d-2}(X)$. Indeed, since $g(H) = p_{2*}(p_1^*(H) \cup g)$ and $g = g_*(1)$, we have

$$\langle g(H) \cup b\rangle = \langle p_2^*(b) \cup p_1^*(H) \cup g_*(1) = \langle g^*p_2^*(b) \cup g^*p_1^*(H) \cup 1 = \frac{qb \cup H}{(q^{1/2})^2} = b \cup H,$$

as desired. Since $P^i(X) = \ker L^{2d-i+1}$, we see that g acts as an automorphism on each primitive subspace $P^i(X)$. Then looking at the Lefschetz decomposition

$$*x = \sum_{j \geq i-d,0} (-1)^{(i-2j)(i-2j+1)/2} L^{d-i+j} x_j,$$

we conclude that $*$ and g^T commute. Thus $g'g = **g^Tg = \mathrm{id}$. Thus returning to our form $(u, v) = \mathrm{Tr}(u'v)$, we see that it induces an inner product on the $\mathbb{Q}(q^{1/2})$-algebra $A^*(X \times X)g$. Since $g'g = \mathrm{id}$, we have that g preserves this inner product, and is thus semisimple with eigenvalues of absolute value 1. This implies that the Frobenius has eigenvalues of absolute value $q^{i/2}$ on $H^i(X)$. □

Remark. This argument works for curves (Weil) and certain Kählerian varieties (Serre).

Remark. For a long time, the standard conjectures have been known for curves, some surfaces, flag varieties, B for abelian varieties, but not much more. However, there seems to have been recent progress, e.g. the work on $\mathrm{Hdg}(X)$ for abelian fourfolds by Ancona.

References

[1] A. Grothendieck, *Standard Conjectures on Algebraic Cycles*.
http://www.numdam.org/item/10.24033/asens.1257.pdf

[2] S. Kleiman, *Algebraic Cycles and the Weil Conjectures*.
https://www.jmilne.org/math/Books/ADTnot.pdf.

[3] S. Kleiman, *The Standard Conjectures*.
http://www.numdam.org/item/10.24033/asens.1257.pdf