Semi-orthogonal Decompositions Seminar Notes

Notes taken by Amal Mattoo, who apologizes for any mistakes.

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0.1 Backgrounds from birational geometry

We work over $k = \mathbf{C}$.

First, let's consider curves. If C_1 and C_2 are smooth projective curves and $C_1 \sim_{\text{bir}} C_2$, then $C_1 \cong C_2$ (by valuative criterion of properness).

For surfaces, birational geometry is already non-trivial. Let S be a smooth projective surface, $S \dashrightarrow S'$ birational. Then does that mean $S \cong S'$? No! If $x \in S$, then $\text{Bl}_x S =: \widetilde{S} \xrightarrow{p} S$ is a birational morphism, but they are not isomorphic.

Fact: $p^{-1}(x) = E \cong \mathbf{P}^1$ and $E^2 = -1$. Then E is called a (-1)-*curve*.

Theorem 0.1 (Castelnuovo). *If S is a smooth projective surface and $E \cong \mathbf{P}^1 \subset S$ with $E^2 = -1$, then there exists $S \to \overline{S}$ blowing down E.*

So using this theorem, you can keep blowing down $S \to \overline{S}_1 \to \overline{S}_2 \to ... \to \overline{S}$ in a finite sequence of blowdowns of (-1)-curves until there are no more (-1)-curves; it terminates because each step decreases the Picard rank by 1. Call the smooth projective surface $\overline{S}$ a *minimal model*.

Theorem 0.2. *If $\overline{S}$ is a minimal surface, then $\omega_{\overline{S}} = \mathcal{O}_{\overline{S}}(K_{\overline{S}})$ is a nef line bundle, or $\overline{S}$ is $\mathbf{P}^2$ or $\overline{S}$ is a $\mathbf{P}^1$ bundle over a smooth curve.*

The first case means Kodaira dimension $K(\overline{S}) \geq 0$, and the latter two cases mean $K(\overline{S}) = -\infty$. This concludes the story of birational geometry for surfaces.

Now we turn to three-folds.

Definition 0.3. A smooth projective 3-fold X is a minimal model if K_X is nef.

Given an arbitrary X with $K(X) \geq 0$, can you develop a minimal model theory as with surfaces?

It turns out there exists a birational morphism $X \to \overline{X}_1$ that contracts a 2-dimensional subvariety that is covered by $\mathbf{P}^1$'s. Note $\overline{X}_1$ might not be smooth, but is a terminal $\mathbf{Q}$-factorial variety. Sometimes you can do so again with $\overline{X}_1 \to \overline{X}_2$. But you may not be able to do it again, and instead take $\overline{X}_2 \to *$ contracting a rational curve. But this may not even be $\mathbf{Q}$-Gorenstein, so you can't run the minimal model program further. But there is a birational map $\overline{X}_2 \dashrightarrow \overline{X}_2^+$ called a *flip* that commutes with a birational morphism $\overline{X}_2^+$ which regenerates a curve. Then you can flip again, or maybe now you can contract a divisor, and so on.

Note that flips do not change the Picard rank since contracting a curve does not change it. But it is not known in general that flips have to terminate.

For 3-folds, it has been proven that flips terminate: if X is a smooth projective 3-fold, then $\overline{X}$ is a minimal 3-fold birational to X. But $\overline{X}$ is usually not unique (and is usually quite singular).

If $\overline{X}_1 \dashrightarrow_{\text{bir}} \overline{X}_r$ are minimal models, then there exists a finite sequence of *flops* between them.

Now let's define flips and flops precisely.

Definition 0.4. Let $X^{\text{sm}} \xrightarrow{\mathcal{E}} \overline{X}$ be a contraction of a rational curve $C \subset X$ to a point $x \in \overline{X}$. This is called a *flipping contraction* if $K_X.C < 0$. It is called a *flopping contraction* if $K_X \cdot C = 0$.

Theorem 0.5. *If $X \xrightarrow{\mathcal{E}} \overline{X}$ is a flipping (resp. flopping) contraction, exists a unique $X^+ \xrightarrow{\mathcal{E}^+} X$ such that $\mathcal{E}^+$ is a contraction of a curve C^+ to $x \in \overline{X}$ and $K_{X^+} \cdot C^+ > 0$ (resp. $= 0$).*

The birational map $X \dashrightarrow X^+$ is a flip (resp. flop).

Example. Let X be a smooth projective 3-fold and

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X^+ \\ {\scriptstyle \mathcal{E}}\downarrow & \swarrow {\scriptstyle \mathcal{E}^+} & \\ \overline{X} & & \end{array}$$

is a flopping diagram contracting a rational $C \subset X$. It is a theorem that actually C is smooth, so $C \cong \mathbf{P}^1$.

By the adjunction formula,

$$\mathcal{O}(-2) = \omega_C = \omega_X|_C \otimes \det N_{C/X}$$

where $\omega_X|_C$ is a line bundle with degree $K_X \cdot C = 0$, so $\det N_{C/X} = \mathcal{O}(-2)$. Thus, $N_{C/X} = \mathcal{O}(-a) \oplus \mathcal{O}(a-2)$. Fact: either $a = 1, 0, -1$. If $a = 1$ so $N_{C/X} = \mathcal{O}(-1)^2$, then it is called a *standard flop.*

Example. Let X^{2n} be a smooth symplectic variety and a flopping diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X^+ \\ {\scriptstyle \mathcal{E}}\downarrow & \swarrow {\scriptstyle \mathcal{E}^+} & \\ \overline{X} & & \end{array}$$

contracting $Z \subset X$ of codimension ≥ 2 to $\overline{Z} \subset \overline{X}$. The fibers of $Z \to \overline{Z}$ are covered by $\mathbf{P}^1$. Fact: $\dim Z \geq n$. Moreover, if $\dim Z = n$, then $Z = \mathbf{P}^n$ and $\overline{Z} = \text{pt}$. The case of $Z = \mathbf{P}^n$ is called a Mukai flop. Fact: $N_{Z/X} \cong \Omega_{\mathbf{P}^n}$.

0.2 Standard flops

Setup: $\mathbf{P}^k \subset X^{k+\ell+1}$ with X a smooth variety and $k \geq \ell$ and $N_{\mathbf{P}^k/X} = \mathcal{O}_{\mathbf{P}^k}(-1)^{\oplus \ell+1}$ (threefold minimal model case: $k = \ell = 1$).

Construction:

$$\begin{array}{ccc} E & \overset{i}{\hookrightarrow} & \widetilde{X} = \mathrm{Bl}_{\mathbf{P}^k} X \\ \downarrow & & \downarrow p \\ \mathbf{P}^k & \underset{\text{codim } \ell+1}{\hookrightarrow} & X \end{array}$$

where $E = \mathbf{P}^k \times \mathbf{P}^\ell$. Note $\mathrm{Pic}(E) = \mathbf{Z}^2$.

Lemma 0.6. $\mathcal{O}_E(E) = \mathcal{O}_{\widetilde{X}}(E)|_E \cong \mathcal{O}(-1,-1)$.

Proof. Adunction: $\omega_E = \omega_{\widetilde{X}}|_E \otimes \mathcal{O}_E(E)$. Adjunction: $\omega_{\mathbf{P}^k} = \omega_{X|_{\mathbf{P}^k}} \otimes \mathcal{O}(-\ell-1)$.

Because $E = \mathbf{P}^k \times \mathbf{P}^\ell$, have $\omega_E = \mathcal{O}(-k-1, -\ell-1)$. Fact: $\omega_{\widetilde{X}} = p^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(\ell E)$ for blowups. Then $\mathcal{O}_E((\ell+1)E) = \mathcal{O}(-\ell-1, -\ell-1)$ so $\mathcal{O}_E(E) = \mathcal{O}(-1,-1)$. □

Theorem 0.7 (Fujiki-Nakano, Artin in the algebraic setting)**.** *Let Y be a smooth variety $E \subset Y$ a smooth divisor such that $E \to Z$ with $\mathbf{P}^k$-bundle with fiber $F = \mathbf{P}^k$. If $\mathcal{O}_F(E) \cong \mathcal{O}(-1)$, then there exists a smooth algebraic space $\overline{Y}$ such that*

$$\begin{array}{ccc} Y & \longrightarrow & \overline{Y} \\ \subset\uparrow & & \subset\uparrow \\ E & \xrightarrow{\mathbf{P}^k} & Z \end{array}$$

We showed $\mathcal{O}_E(E) = \mathcal{O}(-1,-1)$, and by the theorem we can contract E

$$\begin{array}{ccc} E & \xrightarrow{\subset} & \widetilde{X} \\ \mathbf{P}^k \downarrow & & \downarrow \text{Blowup} \\ \mathbf{P}^\ell & \xrightarrow{\subset} & X^+ \end{array}$$

Conclusion:

$$\begin{array}{ccccc} & & \widetilde{X} & & \\ & \overset{p}{\swarrow} & & \overset{p^+=q}{\searrow} & \\ X & & \overset{f}{\dashrightarrow} & & X \end{array} \qquad \begin{array}{ccccc} & & E = \mathbf{P}^k \times \mathbf{P}^\ell & & \\ & \overset{\pi}{\swarrow} & & \overset{\pi^+}{\searrow} & \\ \mathbf{P}^k & & \dashrightarrow & & \mathbf{P}^\ell \end{array}$$

Remark. Suppose

$$\begin{array}{ccccc} X & & \overset{f}{\dashrightarrow} & & X^+ \\ & \searrow & & \swarrow & \\ & & \overline{X} & & \end{array} \qquad \begin{array}{ccccc} \mathbf{P}^k & & & & \mathbf{P}^\ell \\ & \searrow & & \swarrow & \\ & & \mathrm{pt} & & \end{array}$$

Then $\deg(K_X|_{\mathbf{P}^k}) = \ell - k$. So if $k > \ell$ it is a flip, and if $k = \ell$ it is a flop. These are the standard flip and flop.

Theorem 0.8 (Bondal-Orlov 1995)**.** *$\Phi := p_* q^* : D^b(X^+) \to D^b(X)$ is fully faithful (for $k \geq \ell$). Moreover, if $k = \ell$, then Φ is an equivalence.*

Proof.

- Step 0. Recall

$$\begin{array}{ccc} E & \xhookrightarrow{i} & \widetilde{X} = \mathrm{Bl}_Y X \\ \big\downarrow{\scriptstyle \pi:\mathbf{P}^\ell\text{-bundle}} & & \big\downarrow{\scriptstyle p} \\ Y & \xhookrightarrow{\text{codim } \ell+1} & X \end{array}$$

Then $\Phi_t := \Phi_{\mathcal{O}_E(tE)} : D(Y) \to D(\widetilde{X})$ is fully faithful for all $t \in \mathbf{Z}$. Also, $p^* : D(X) \to D(\widetilde{X})$ is same.

Setting $D(Y)_t = \mathrm{im}\Phi_t \subset D(\widetilde{X})$, exists a semi-orthogonal decomposition

$$D(\widetilde{X}) = \langle D(X)_{-\ell}, D(Y)_{-\ell+1}, ..., D(Y)_{-1}, p^* D(X) \rangle$$

Applying

$$\begin{array}{ccc} \mathbf{P}^k \times \mathbf{P}^\ell & \xhookrightarrow{i} & \widetilde{X} = \mathrm{Bl}_Y X \\ \big\downarrow{\scriptstyle \pi:\mathbf{P}^\ell\text{-bundle}} & & \big\downarrow{\scriptstyle p} \\ \mathbf{P}^k & \xhookrightarrow{\text{codim } \ell+1} & X \end{array}$$

we have a semi-orthogonal $D(\widetilde{X}) = \langle \mathcal{O}(a,b), p^* D(X) \rangle$. We can write

$$\begin{aligned} D(\widetilde{X}) = \langle & \mathcal{O}(-k,-\ell), \mathcal{O}(-k+1,-\ell), ..., \mathcal{O}(0,-\ell), \\ & \mathcal{O}(-k+1), \mathcal{O}(-k+2,-\ell+1), \mathcal{O}(1,-\ell+1) \\ & \vdots \\ & \mathcal{O}(-k+\ell-1,-1), \mathcal{O}(-k+\ell-1), ..., \mathcal{O}(-\ell-1,-1), p^* D(X) \rangle \end{aligned}$$

Let $D^1 := \langle \mathcal{O}(a,b) | a < 0 \rangle$ and $D^2 := \langle \mathcal{O}(a,b) | a \geq 0 \rangle$. We want $D(\widetilde{X}) = \langle D^1, D^2, p^* D(X) \rangle$.

- Step 1. Show $\Phi := p_* q^* : D(X^+) \to D(X)$ is fully faithful. For all $E, F \in D(X^+)$,

$$\mathrm{Hom}(E,F) = \mathrm{Hom}(p_* q^* E, p_* q^* E)$$

which are equal to both of

$$\mathrm{Hom}(q^* E, q^* F) \to \mathrm{Hom}(p^* p_* q^* E, q^* F)$$

Consider the exact triangle

$$p^* p_* q^* E \to q^* E \to H$$

It is enough to show $\mathrm{Hom}(H, q^* F) = 0$ for all $F \in D(X^+)$.

Claim: $H \in D^2$.

  - $\mathrm{Hom}(p^* G, H) = \mathrm{Hom}(G, p_* H) = 0$ for all $G \in D(X)$, since since p^* is fully faithful

$$p_* p^* p_* q^* E \xrightarrow{\cong} p_* q^* E \to p_* H$$

so $p_* H = 0$.

- $\mathrm{Hom}(H, \mathcal{O}(a,b)) = 0$ for all $a < 0$ by direct computation.

- Step 2. Conclusion. Since $H \in D^2 = \langle \mathcal{O}(a,b) | a \geq 0 \rangle$, enough to show $\mathrm{Hom}(\mathcal{O}(a,b), q^*F) = 0$ for $a \geq 0$. This follows by computation.

$\square$

0.3 Mukai Flop

Setup: let $\mathbf{P}^n \subset X^{2n}$ a smooth variety and $N_{\mathbf{P}^n/X} = \Omega_{\mathbf{P}^n}$. Construction:

$$\begin{array}{ccc} E & \xhookrightarrow{i} & \widetilde{X} = \mathrm{Bl}_{\mathbf{P}^n} X \\ \big\downarrow {\scriptstyle \pi:\mathbf{P}^{n-1}\text{-bundle}} & & \big\downarrow {\scriptstyle p} \\ \mathbf{P}^n & \hookrightarrow & X \end{array}$$

with $E = \mathbf{P}\Omega_{\mathbf{P}^n}$. Fact: $\mathbf{P}\Omega_{\mathbf{P}^n} \subset \mathbf{P}^n \times (\mathbf{P}^n)^*$ is the incidence variety: pairs (p, H) such that $p \in H$.

Lemma 0.9. $\mathcal{O}_E(E) = \mathcal{O}(-1,-1)$ *is a universal family of hyperplanes.*

By contraction theorem,

$$\begin{array}{ccc} & \widetilde{X} & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & \overset{f}{\dashrightarrow} & X^+ \end{array} \qquad\qquad \begin{array}{ccc} & E & \\ \swarrow & & \searrow \\ \mathbf{P}^n & & \mathbf{P}^n \end{array}$$

is a Mukai flop.

But $\Phi = p_* q^* : D(X^+) \to D(X)$ is NOT fully faithful. To correct this, assume X and X^+ admit a common contraction to $\overline{X}$:

$$\begin{array}{ccc} & \widetilde{X} & \\ \swarrow & & \searrow \\ X & & X^+ \\ {\scriptstyle \varepsilon}\searrow & & \swarrow{\scriptstyle \varepsilon^+} \\ & \overline{X} & \end{array}$$

Let $Z := X \times_{\overline{X}} X^+$. In fact $Z = \widetilde{X} \cup (\mathbf{P}^n \times \mathbf{P}^n) \subset X \times X^+$ so this is well defined even without the common contraction. It turns out $\mathcal{O}_Z$ is the correct kernel of the equivalence.

Theorem 0.10 (Kawamata, Namikawa)**.** $\Phi = \Phi_{\mathcal{O}_Z} : D(X^+) \to D(X)$ *is an equivalence.*

Proof idea. Assume there is a curve C and deformations $\mathcal{X}, \mathcal{X}^+ \to C$ of X and X^+ that are isomorphic on $C \setminus \{0\}$. We can ensure that $\mathbf{P}^n \subset X$ has normal bundle $\mathcal{O}(-1)^{n+1}$ in $\mathcal{X}$. Thus, $\mathcal{X} \to \mathcal{X}^+$ is a standard flop and so a derived equivalence. This formally induces an equivalence between the fibers over $0 \in C$. $\square$