Semi-orthogonal Decompositions Seminar Notes

Notes taken by Amal Mattoo, who apologizes for any mistakes.

April 2

Yoonjoo Kim: Derived equivalence of standard and Mukai flops.

0.1 Backgrounds from birational geometry

We work over $k = \mathbf{C}$.

First, let's consider curves. If C_1 and C_2 are smooth projective curves and $C_1 \sim_{\text{bir}} C_2$, then $C_1 \cong C_2$ (by valuative criterion of properness).

For surfaces, birational geometry is already non-trivial. Let S be a smooth projective surface, $S \dashrightarrow S'$ birational. Then does that mean $S \cong S'$? No! If $x \in S$, then $\mathrm{Bl}_x S =: \widetilde{S} \xrightarrow{p} S$ is a birational morphism, but they are not isomorphic.

Fact: $p^{-1}(x) = E \cong \mathbf{P}^1$ and $E^2 = -1$. Then E is called a (-1)-curve.

Theorem 0.1 (Castelnuovo). If S is a smooth projective surface and $E \cong \mathbf{P}^1 \subset S$ with $E^2 = -1$, then there exists $S \to \overline{S}$ blowing down E.

So using this theorem, you can keep blowing down $S \to \overline{S}_1 \to \overline{S}_2 \to ... \to \overline{S}$ in a finite sequence of blowdowns of (-1)-curves until there are no more (-1)-curves; it terminates because each step decreases the Picard rank by 1. Call the smooth projective surface \overline{S} a minimal model.

Theorem 0.2. If \overline{S} is a minimal surface, then $\omega_{\overline{S}} = \mathcal{O}_{\overline{S}}(K_{\overline{S}})$ is a nef line bundle, or \overline{S} is \mathbf{P}^2 or \overline{S} is a \mathbf{P}^1 bundle over a smooth curve.

The first case means Kodaira dimension $K(\overline{S}) \geq 0$, and the latter two cases mean $K(\overline{S}) = -\infty$. This concludes the story of birational geometry for surfaces.

Now we turn to three-folds.

Definition 0.3. A smooth projective 3-fold X is a minimal model if K_X is nef.

Given an arbitrary X with $K(X) \ge 0$, can you develop a minimal model theory as with surfaces?

It turns out there exists a birational morphism $X \to \overline{X}_1$ that contracts a 2-dimensional subvariety that is covered by \mathbf{P}^1 's. Note \overline{X}_1 might not be smooth, but is a terminal \mathbf{Q} -factorial variety. Sometimes you can do so again with $\overline{X}_1 \to \overline{X}_2$. But you may not be able

to do it again, and instead take $\overline{X}_2 \to *$ contracting a rational curve. But this may not even be **Q**-Gorenstein, so you can't run the minimal model program further. But there is a birational map $\overline{X}_2 \dashrightarrow \overline{X}_2^+$ called a *flip* that commutes with a birational morphism \overline{X}_2^+ which regenerates a curve. Then you can flip again, or maybe now you can contract a divisor, and so on.

Note that flips do not change the Picard rank since contracting a curve does not change it. But it is not known in general that flips have to terminate.

For 3-folds, it has been proven that flips terminate: if X is a smooth projective 3-fold, then \overline{X} is a minimal 3-fold birational to X. But \overline{X} is usually not unique (and is usually quite singular).

If $\overline{X}_1 \dashrightarrow_{\text{bir}} \overline{X}_r$ are minimal models, then there exists a finite sequence of flops between them.

Now let's define flips and flops precisely.

Definition 0.4. Let $X^{\operatorname{sm}} \xrightarrow{\mathcal{E}} \overline{X}$ be a contraction of a rational curve $C \subset X$ to a point $x \in \overline{X}$. This is called a *flipping contraction* if $K_X \cdot C = 0$. It is called a *flopping contraction* if $K_X \cdot C = 0$.

Theorem 0.5. If $X \xrightarrow{\mathcal{E}} \overline{X}$ is a flipping (resp. flopping) contraction, exists a unique $X^+ \xrightarrow{\mathcal{E}^+} X$ such that \mathcal{E}^+ is a contraction of a curve C^+ to $x \in \overline{X}$ and $K_{X^+} \cdot C^+ > 0$ (resp. = 0).

The birational map $X \dashrightarrow X^+$ is a flip (resp. flop).

Example. Let X be a smooth projective 3-fold and

$$\begin{array}{c} X \xrightarrow{-f} X^+ \\ \varepsilon \downarrow \\ \overline{X} \end{array}$$

is a flopping diagram contracting a rational $C \subset X$. It is a theorem that actually C is smooth, so $C \cong \mathbf{P}^1$.

By the adjunction formula,

$$\mathcal{O}(-2) = \omega_C = \omega_X|_C \otimes \det N_{C/X}$$

where $\omega_X|_C$ is a line bundle with degree $K_X \cdot C = 0$, so det $N_{C/X} = \mathcal{O}(-2)$. Thus, $N_{C/X} = \mathcal{O}(-a) \oplus \mathcal{O}(a-2)$. Fact: either a = 1, 0, -1. If a = 1 so $N_{C/X} = \mathcal{O}(-1)^2$, then it is called a standard flop.

Example. Let X^{2n} be a smooth symplectic variety and a flopping diagram

$$\begin{array}{c} X \xrightarrow{-f} X^+ \\ \varepsilon \downarrow \\ \overline{X} \end{array}$$

contracting $Z \subset X$ of codimension ≥ 2 to $\overline{Z} \subset \overline{X}$. The fibers of $Z \to \overline{Z}$ are covered by \mathbf{P}^1 . Fact: dim $Z \geq n$. Moreover, if dim Z = n, then $Z = \mathbf{P}^n$ and $\overline{Z} = \mathrm{pt}$. The case of $Z = \mathbf{P}^n$ is called a Mukai flop. Fact: $N_{Z/X} \cong \Omega_{\mathbf{P}^n}$.

0.2 Standard flops

Setup: $\mathbf{P}^k \subset X^{k+\ell+1}$ with X a smooth variety and $k \geq \ell$ and $N_{\mathbf{P}^k/X} = \mathcal{O}_{\mathbf{P}^k}(-1)^{\oplus \ell+1}$ (threefold minimal model case: $k = \ell = 1$).

Construction:

$$E \stackrel{i}{\longleftrightarrow} \widetilde{X} = \operatorname{Bl}_{\mathbf{P}^{k}} X$$

$$\downarrow \qquad \qquad \downarrow^{p}$$

$$\mathbf{P}^{k} \stackrel{codim}{\longleftrightarrow} X$$

where $E = \mathbf{P}^k \times \mathbf{P}^\ell$. Note $\text{Pic}(E) = \mathbf{Z}^2$.

Lemma 0.6. $\mathcal{O}_E(E) = \mathcal{O}_{\widetilde{X}}(E)|_E \cong \mathcal{O}(-1, -1)$.

Proof. Adunction: $\omega_E = \omega_{\widetilde{X}}|_E \otimes \mathcal{O}_E(E)$. Adjunction: $\omega_{\mathbf{P}^k} = \omega_{X|_{\mathbf{P}^k}} \otimes \mathcal{O}(-\ell-1)$.

Because $E = \mathbf{P}^k \times \mathbf{P}^\ell$, have $\omega_E = \mathcal{O}(-k-1, -\ell-1)$. Fact: $\omega_{\widetilde{X}} = p^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(\ell E)$ for blowups. Then $\mathcal{O}_E((\ell+1)E) = \mathcal{O}(-\ell-1, -\ell-1)$ so $\mathcal{O}_E(E) = \mathcal{O}(-1, -1)$.

Theorem 0.7 (Fujiki-Nakano, Artin in the algebraic setting). Let Y be a smooth variety $E \subset Y$ a smooth divisor such that $E \to Z$ with \mathbf{P}^k -bundle with fiber $F = \mathbf{P}^k$. If $\mathcal{O}_F(E) \cong \mathcal{O}(-1)$, then there exists a smooth algebraic space \overline{Y} such that

$$Y \longrightarrow \overline{Y}$$

$$\subset \uparrow \qquad \subset \uparrow$$

$$E \stackrel{\mathbf{P}^k}{\longrightarrow} Z$$

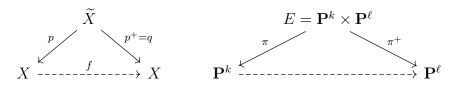
We showed $\mathcal{O}_E(E) = \mathcal{O}(-1, -1)$, and by the theorem we can contract E

$$E \xrightarrow{\subset} \widetilde{X}$$

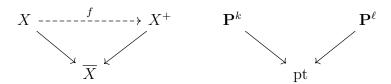
$$\mathbf{P}^{k} \downarrow \qquad \qquad \downarrow \text{Blowup}$$

$$\mathbf{P}^{\ell} \xrightarrow{\subset} X^{+}$$

Conclusion:



Remark. Suppose



Then $\deg(K_X|_{\mathbf{P}^k}) = \ell - k$. So if $k > \ell$ it is a flip, and if $k = \ell$ it is a flop. These are the standard flip and flop.

Theorem 0.8 (Bondal-Orlov 1995). $\Phi := p_*q^* : D^b(X^+) \to D^b(X)$ is fully faithful (for $k \ge \ell$). Moreover, if $k = \ell$, then Φ is an equivalence.

Proof. • Step 0. Recall

$$E \xrightarrow{i} \widetilde{X} = \text{Bl}_{Y}X$$

$$\downarrow_{\pi: \mathbf{P}^{\ell}\text{-bundle}} \qquad \downarrow_{p}$$

$$Y \xrightarrow{\text{codim } \ell+1} X$$

Then $\Phi_t := \Phi_{\mathcal{O}_E(tE)} : D(Y) \to D(\widetilde{X})$ is fully faithful for all $t \in \mathbf{Z}$. Also, $p^* : D(X) \to D(\widetilde{X})$ is same.

Setting $D(Y)_t = \operatorname{im}\Phi_t \subset D(\widetilde{X})$, exists a semi-orthogonal decomposition

$$D(\widetilde{X}) = \langle D(X)_{-\ell}, D(Y)_{-\ell+1}, ..., D(Y)_{-1}, p^*D(X) \rangle$$

Applying

$$\mathbf{P}^{k} \times \mathbf{P}^{\ell} \stackrel{i}{\longleftrightarrow} \widetilde{X} = \mathrm{Bl}_{Y} X$$

$$\downarrow_{\pi: \mathbf{P}^{\ell}\text{-bundle}} \qquad \downarrow_{p}$$

$$\mathbf{P}^{k} \stackrel{\mathrm{codim } \ell+1}{\longleftrightarrow} X$$

we have a semi-orthogonal $D(\widetilde{X}) = \langle \mathcal{O}(a,b), p^*D(X) \rangle$. We can write

$$D(\widetilde{X}) = \langle \mathcal{O}(-k, -\ell), \mathcal{O}(-k+1, -\ell), ..., \mathcal{O}(0, -\ell), \\ \mathcal{O}(-k+1), \mathcal{O}(-k+2, -\ell+1), \mathcal{O}(1, -\ell+1) \\ \vdots \\ \mathcal{O}(-k+\ell-1, -1), \mathcal{O}(-k+\ell-1), ..., \mathcal{O}(-\ell-1, -1), p^*D(X) \rangle$$

Let $D^1 := \langle \mathcal{O}(a,b) | a < 0 \rangle$ and $D^2 := \langle \mathcal{O}(a,b) | a \geq 0 \rangle$. We want $D(\widetilde{X}) = \langle D^1, D^2, p^*D(X) \rangle$.

• Step 1. Show $\Phi := p_*q^* : D(X^+) \to D(X)$ is fully faithful. For all $E, F \in D(X^+)$,

$$\operatorname{Hom}(E, F) = \operatorname{Hom}(p_*q^*E, p_*q^*E)$$

which are equal to both of

$$\operatorname{Hom}(q^*E, q^*F) \to \operatorname{Hom}(p^*p_*q^*E, q^*F)$$

Consider the exact triangle

$$p^*p_*q^*E \to q^*E \to H$$

It is enough to show $\operatorname{Hom}(H, q^*F) = 0$ for all $F \in D(X^+)$.

Claim: $H \in D^2$.

– $\operatorname{Hom}(p^*G,H)=\operatorname{Hom}(G,p_*H)=0$ for all $G\in D(X),$ since since p^* is fully faithful

$$p_*p^*p_*q^*E \xrightarrow{\cong} p_*q^*E \to p_*H$$

so $p_*H = 0$.

- $\operatorname{Hom}(H, \mathcal{O}(a, b)) = 0$ for all a < 0 by direct computation.
- Step 2. Conclusion. Since $H \in D^2 = \langle \mathcal{O}(a,b) | a \geq 0 \rangle$, enough to show $\text{Hom}(\mathcal{O}(a,b), q^*F) = 0$ for $a \geq 0$. This follows by computation.

0.3 Mukai Flop

Setup: let $\mathbf{P}^n \subset X^{2n}$ a smooth variety and $N_{\mathbf{P}^n/X} = \Omega_{\mathbf{P}^n}$. Construction:

$$E \stackrel{i}{\longleftrightarrow} \widetilde{X} = \mathrm{Bl}_{\mathbf{P}^n} X$$

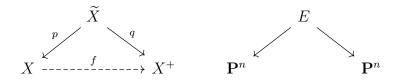
$$\downarrow_{\pi:\mathbf{P}^{n-1}\text{-bundle}} \qquad \downarrow_p$$

$$\mathbf{P}^n \stackrel{}{\longleftrightarrow} X$$

with $E = \mathbf{P}\Omega_{\mathbf{P}^n}$. Fact: $\mathbf{P}\Omega_{\mathbf{P}^n} \subset \mathbf{P}^n \times (\mathbf{P}^n)^*$ is the incidence variety: pairs (p, H) such that $p \in H$.

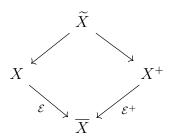
Lemma 0.9. $\mathcal{O}_E(E) = \mathcal{O}(-1, -1)$ is a universal family of hyperplanes.

By contraction theorem,



is a Mukai flop.

But $\Phi = p_*q^* : D(X^+) \to D(X)$ is NOT fully faithful. To correct this, assume X and X^+ admit a common contraction to \overline{X} :



Let $Z := X \times_{\overline{X}} X^+$. In fact $Z = \widetilde{X} \cup (\mathbf{P}^n \times \mathbf{P}^n) \subset X \times X^+$ so this is well defined even without the common contraction. It turns out \mathcal{O}_Z is the correct kernel of the equivalence.

Theorem 0.10 (Kawamata, Namikawa). $\Phi = \Phi_{\mathcal{O}_Z} : D(X^+) \to D(X)$ is an equivalence.

Proof idea. Assume there is a curve C and deformations $\mathcal{X}, \mathcal{X}^+ \to C$ of X and X^+ that are isomorphic on $C \setminus \{0\}$. We can ensure that $\mathbf{P}^n \subset X$ has normal bundle $\mathcal{O}(-1)^{n+1}$ in \mathcal{X} . Thus, $\mathcal{X} \to \mathcal{X}^+$ is a standard flop and so a derived equivalence. This formally induces an equivalence between the fibers over $0 \in C$.