Categorical Resolutions

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Main reference: Kuznetsov Lunts 2012. Additional references: Kaledin Kuznetsov 2015, Kuznetsov Shinder 2024.

1 Introduction

Let $\pi : X \to Y$ be a resolution of singularities of Y, assuming characteristic 0. We say Y has *rational singularities* if $\mathcal{O}_Y \xrightarrow{\cong} R\pi_*\mathcal{O}_X$. In this case, applying the projection formula,

$$R\pi_*(L\pi^*(-)) \cong - \otimes^{\mathbf{L}} R\pi_*\mathcal{O}_X \cong -$$

so $L\pi^* : D(Y) \to D(X)$ is fully faithful. But if Y does not have rational singularities this fails.

There is a notion of a categorical resolution that we want to achieve.

Definition 1.1 (1.3)**.** A *categorical resolution of a scheme* Y is a smooth cocomplete compactly generated triangulated category $\mathcal{T}$ with

$$\pi^* : D(Y) \to \mathcal{T}, \quad \pi_* : \mathcal{T} \to D(Y)$$

adjoints such that

1. $\pi_* \circ \pi^* = \mathrm{id}$,
2. Both π^* and π_* commute with arbitrary direct sums, and
3. $\pi_*(\mathcal{T}^c) \subset D^b_{\mathrm{Coh}}(Y)$.

Recall an object F is *compact* if $\mathrm{Hom}_{\mathcal{T}}(F, -)$ commutes with arbitrary direct sums.

Our goal is to show the following:

Theorem 1.2 (1.4)**.** *Any separated scheme of finite type Y over a field of characteristic 0 has a categorical resolution by $\mathcal{T}$ with a semi-orthogonal decomposition into $D(Z_i)$ for schemes Z_i. If Y is proper then so is $\mathcal{T}$.*

I have not yet defined what smooth and proper mean for triangulated categories! This and more will use the machinery of DG-categories, which we now describe.

2 DG-categories

This is an important topic that could have been a talk by itself, but for now I will say the bare minimum.

2.1 Basics

A *DG-category* $\mathcal{D}$ over a field k is a k-linear category enriched in complexes of k-vector spaces.

The *homotopy category* $[\mathcal{D}]$ has $\mathrm{Ob}([\mathcal{D}]) = \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Hom}_{[\mathcal{D}]}(X, Y) = H^0(\mathrm{Hom}_{\mathcal{D}}(X, Y))$.

A *right DG-module* over $\mathcal{D}$ is a DG-functor $\mathcal{D}^{\mathrm{op}} \to k\text{-dgm}$. The *Yoneda functor* is

$$Y : \mathcal{D} \to \mathcal{D}\text{-dgm}, \quad X \mapsto \mathrm{Hom}_{\mathcal{D}}(-, X)$$

The *diagonal bi-module* is

$$\mathcal{D}_1 \otimes \mathcal{D}_2^{\mathrm{op}} \to k\text{-dgm}, \quad \mathrm{Hom}_{\mathcal{D}}(-, -)$$

There is a construction of shifts and cones in a DG-category such that $[\mathcal{D}\text{-dgm}]$ is triangulated. A DG-subcategory $\mathcal{D}' \subset \mathcal{D}\text{-dgm}$ is *pre-triangulated* if $[\mathcal{D}'] \subset [\mathcal{D}\text{-dgm}]$ is triangulated. An *enhancement* for a triangulated category $\mathcal{T}$ is a pre-triangulated DG-category $\mathcal{D}$ with a triangulated equivalence $\mathcal{T} \cong \mathcal{D}$.

The minimal subcategory of $\mathcal{D}$-dgm containing all representable DG-modules and closed under shifts, cones, closed morphisms, and homotopy direct summands is the category of *perfect DG-modules*.

There is a construction of the *derived category* $D(\mathcal{D})$ as a Verdier quotient of $[\mathcal{D}\text{-dgm}]$. It is a triangulated category.

It is a theorem that compact objects of $D(\mathcal{D})$ are perfect DG-modules. If $\mathcal{D}$ is pre-triangulated and closed under homotopy direct summands, $D(\mathcal{D})^c = [\mathcal{D}]$.

Definition 2.1.

- A DG-category $\mathcal{D}$ is *smooth* if the diagonal bimodule $\mathcal{D}$ is a perfect bimodule. It is *proper* if for all $X, Y \in \mathcal{D}$ the cohomology of $\mathcal{D}(X, Y)$ is finite and bounded.
- A triangulated category $\mathcal{T}$ has these properties if $\mathcal{T} \cong D(\mathcal{D})$ for $\mathcal{D}$ with these properties.

A *partial categorical DG-resolution* of $\mathcal{D}$ is $\pi : \mathcal{D} \to \tilde{\mathcal{D}}$ with $[\mathcal{D}] \to [\tilde{\mathcal{D}}]$ fully faithful. If additionally $\tilde{\mathcal{D}}$ is smooth, it is a *categorical DG-resolution*.

With a bit of technical work we can see that categorical DG-resolutions induce categorical resolutions on the derived categories.

Finally we see that we can take enhancements for schemes.

Theorem 2.2 (3.11)**.** *There is a pseudofunctor* $\mathcal{D} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{sDG}$ *such that the following commutes:*

$$\begin{array}{ccc} \mathrm{Sch}^{\mathrm{op}} & \xrightarrow{\mathcal{D}} & \mathrm{sDG} \\ & \searrow^{D} & \downarrow D \\ & & \mathrm{Tria} \end{array}$$

The proof in the paper is quite technical, but I think this can be done more concretely with Cěch complexes.

2.2 Gluing

Given small DG-categories $\mathcal{D}_1$ and $\mathcal{D}_2$ and a bimodule $\varphi \in (\mathcal{D}_2^{\mathrm{op}} \otimes \mathcal{D}_1)$-dgm, there is a DG-category $\mathcal{D}_1 \times_\varphi \mathcal{D}_2$ called the *gluing along* φ.

I won't give the full definition but here is the idea:

Definition 2.3 (sketch)**.** The objects of $\mathcal{D}_1 \times_\varphi \mathcal{D}_2$ are triples (M_1, M_2, μ) with $M_i \in \mathcal{D}_i$ and $\mu \in \varphi(M_2, M_1)$ a closed element of degree 0. The morphisms and multiplication are defined such that there is a distinguished triangle

$$\mathrm{Hom}_{\mathcal{D}_1 \times_\varphi \mathcal{D}_2}(M, N) \to \mathrm{Hom}_{\mathcal{D}_1}(M_1, N_1) \oplus \mathrm{Hom}_{\mathcal{D}_2}(M_2, N_2) \to \varphi(N_2, M_1)$$

This gluing yields nice structure on the derived category:

Proposition 2.4 (4.6)**.** *There is a semi-orthogonal decomposition*

$$D(\mathcal{D}_1 \times_\varphi \mathcal{D}_2) = \langle D(\mathcal{D}_1), D(\mathcal{D}_2))\rangle$$

Importantly, we can relate smoothness and properness of the gluing to the components and the bimodule datum.

Proposition 2.5 (4.9)**.** *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be smooth. If $\varphi \in D(\mathcal{D}_2^{\mathrm{op}} \otimes \mathcal{D}_1)$ is perfect, then $\mathcal{D} = \mathcal{D}_1 \times_\varphi \mathcal{D}_2$ is smooth. If additionally $\mathcal{D}_1, \mathcal{D}_2$ are proper, then $\mathcal{D}$ is proper.*

One final useful point about gluings:

Proposition 2.6 (4.10)**.** *Assume $\mathcal{D}$ is a pretriangulated category with $[\mathcal{D}] = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$. Then $\mathcal{D}$ is quasi-equivalent to some $\mathcal{D}_1 \times_\varphi \mathcal{D}_2$ with $[\mathcal{D}_i] = \mathcal{T}_i$.*

3 Partial Categorical Resolution of a Reduced Scheme

We first look at the case of a reduced scheme, which will later help us build our general procedure. We introduce a new category of objects with which to build the resolution.

3.1 Auslander Algebras

Definition 3.1. Given S a separated *non-reduced* scheme of finite type over k, let $\mathfrak{r} \subset \mathcal{O}_S$ be a nilpotent sheaf of ideals with $\mathfrak{r}^n = 0$. Define a sheaf of non-commutative algebras

$$\mathcal{A}_{S,\mathfrak{r},n} := (\mathfrak{r}^{\max(j-i,0)}/\mathfrak{r}^{n+1-i}) = \begin{pmatrix} \mathcal{O}_S & \mathfrak{r} & \mathfrak{r}^2 & \dots & \mathfrak{r}^{n-1} \\ \mathcal{O}_S/\mathfrak{r}^{n-1} & \mathcal{O}_S/\mathfrak{r}^{n-1} & \mathfrak{r}/\mathfrak{r}^{n-1} & \dots & \mathfrak{r}^{n-2}/\mathfrak{r}^{n-1} \\ \mathcal{O}_S/\mathfrak{r}^{n-2} & \mathcal{O}_S/\mathfrak{r}^{n-2} & \mathcal{O}_S/\mathfrak{r}^{n-2} & \dots & \mathfrak{r}^{n-3}/\mathfrak{r}^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{O}_S/\mathfrak{r} & \mathcal{O}_S/\mathfrak{r} & \mathcal{O}_S/\mathfrak{r} & \dots & \mathcal{O}_S/\mathfrak{r} \end{pmatrix}$$
$$\subset \mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus \mathcal{O}_S/\mathfrak{r}^{n-1} \oplus ... \oplus \mathcal{O}_S/\mathfrak{r})$$

Let $\epsilon_i := 1 \in \mathcal{A}_{ii}$.

Note that $\sum_{i=1}^{n} \epsilon_i = 1$.

Example (5.2)**.** $\mathcal{A}_{S,0,1} = \mathcal{O}_S$.

Example (5.3)**.** Let $S = \text{Spec } k[t]/(t^2)$, $\mathfrak{r} = (t)$, $n = 2$. Then

$$\mathcal{A}_{S,\mathfrak{r},2} = \begin{pmatrix} k[t]/(t^2) & (t) \\ k & k \end{pmatrix}$$

This is the path algebra of $\alpha : \bullet \leftrightarrows \bullet : \beta$ subject to the relation $\beta\alpha = 0$. And $\text{QCoh}(\mathcal{A}_{s,\mathfrak{r},2})$ is the category of representations of this quiver.

An $\mathcal{A}$-module is (quasi-)coherent if it is such as an $\mathcal{O}_S$-module. There is some technical work necessary to construct the derived category as well as pushforward, pullback, induction, and restriction functors, but we take it on faith for now. Suffice it to say there is a category ASch of $\mathcal{A}$-spaces which globalize the notions we've defined.

3.2 Semi-orthogonal decompositions

Let S_0, S' be defined by $\mathcal{O}_{S_0} = \mathcal{O}_S/\mathfrak{r}$ and $\mathcal{O}_{S'} = \mathcal{O}_S/\mathfrak{r}^{n-1}$.

Let $\mathcal{I} = \mathcal{A}(1 - \epsilon_1)\mathcal{A} \subset \mathcal{A}$.

Lemma 3.2 (5.9)**.** $\mathcal{A}/\mathcal{I} \cong \mathcal{O}_S/\mathfrak{r} \cong \mathcal{O}_{S_0}$.

Definition 3.3. Let $i : \text{QCoh}(S_0) \to \text{QCoh}(\mathcal{A})$ be restriction of scalars using $\mathcal{A}$-algebra structure on $\mathcal{O}_{S_0}$.

Let $\mathcal{A}' = \mathcal{A}_{S',\mathfrak{r},n-1} = (1 - \epsilon_1)\mathcal{A}(1 - \epsilon_1)$. Define

$$e : \text{QCoh}(\mathcal{A}') \to \text{QCoh}(\mathcal{A}), \quad M \mapsto M \otimes_{\mathcal{A}'} (1 - \epsilon_1)\mathcal{A}$$

Doing some work to study the adjoints of these functors, we obtain:

Proposition 3.4 (5.14)**.** *There are semi-orthogonal decompositions*

$$D(\mathcal{A}) = \langle i(D(S_0)), e(D(\mathcal{A}'))\rangle \quad D^b_{\text{Coh}}(\mathcal{A}) = \langle i(D^b_{\text{Coh}}(S_0)), e(D^b_{\text{Coh}}(\mathcal{A}'))\rangle$$

Example (5.16)**.** Let $S = \text{Spec } k[t]/(t^2)$, $\mathfrak{r} = (t)$, $n = 2$. Then $S_0 = \text{Spec } k$.

$$\mathcal{A} = \begin{pmatrix} k[t]/(t^2) & (t) \\ k & k \end{pmatrix}, \mathcal{I} = \begin{pmatrix} (t) & (t) \\ k & k \end{pmatrix}, \mathcal{A}/\mathcal{I} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, (1 - \epsilon_1)\mathcal{A} = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, \mathcal{A}' = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

Then the semi-orthogonal decomposition $\langle E_1, \rangle E_2$ is

$$E_1 = i(D(S_0)) = i(\langle k \rangle) = \langle (k \leftrightarrows 0)\rangle$$

$$E_2 = e(D(\mathcal{A}')) = e(\langle k \rangle) = \langle (0 : k \leftrightarrow k : 1)\rangle$$

Iterating this decomposition on $\mathcal{A}'$ gives the following:

Corollary 3.5 (5.15)**.** *There are semi-orthogonal decompositions*

$$D(\mathcal{A}_{S,\mathfrak{r},n}) = \langle D(S_0), D(S_0), D(S_0)\rangle$$

$$D^b_{\text{Coh}}(\mathcal{A}_{S,\mathfrak{r},n}) = \langle D^b_{\text{Coh}}(S_0), D^b_{\text{Coh}}(S_0), ..., D^b_{\text{Coh}}(S_0)\rangle$$

3.3 Resolutions

Smoothness of S_0 gives nice categorical properties.

Proposition 3.6 (5.17, 5.18)**.** *If S_0 is smooth,then $D^b_{\mathrm{Coh}}(\mathcal{A}) = D(\mathcal{A})^c = D^{\mathrm{perf}}(\mathcal{A})$.*

This will help with boundedness results later. Furthermore, we can enhance the category of $\mathcal{A}$-spaces.

Theorem 3.7 (5.19, 5.20)**.** *There is a pseudofunctor $\mathcal{D} : \mathrm{ASch}^{\mathrm{op}} \to \mathrm{sDG}$ extending the pseudofunctor on schemes such that*

$$\begin{array}{ccc} \mathrm{ASch}^{\mathrm{op}} & \xrightarrow{\mathcal{D}} & \mathrm{sDG} \\ & \searrow^{D} & \downarrow D \\ & & \mathrm{Tria} \end{array}$$

If S_0 is smooth (resp. proper), then $\mathcal{D}(\mathcal{A})$ (resp. proper).

Proof idea. Induct on n using the semi-orthogonal decompositions to identify gluings. □

Definition 3.8. Let the morphism $\rho_S : \mathcal{A}_{(S,\mathfrak{r},n)} \to \mathcal{A}_{(S,0,1)}$ be induced by id_S.

Theorem 3.9 (5.23)**.** *If S_0 is smooth then*

- *$p_S^* : \mathcal{D}(S) \to \mathcal{D}(\mathcal{A}_S)$ is a categorical DG-resolution and $Lp_S^* : D(S) \to D(\mathcal{A}_S)$ is a categorical resolution,*
- *$D^b_{\mathrm{Coh}}(\mathcal{A}_S)$ restricts under ρ_S to $D^b_{\mathrm{Coh}}(S)$, and*
- *If S_0 is proper then so is the category $\mathcal{D}(\mathcal{A}_S)$.*

4 The Categorical Resolution Procedure

Definition 4.1. Let $f : X \to Y$ be a proper birational morphism. A subscheme $S \subset Y$ is a *non-rational center (NRC) of Y with respect to f* if the canonical morphism

$$I_S \to Rf_*(I_{f^{-1}(S)})$$

is an isomorphism, where $I_{f^{-1}(S)} := f^{-1}I_S \cdot \mathcal{O}_X$.

Remark. $S = \emptyset$ is a NRC means $\mathcal{O}_Y \xrightarrow{\cong} Rf_*\mathcal{O}_X$. And if X is smooth and S is a NRC then $Y \setminus S$ has rational singularities.

Theorem 4.2 (6.8)**.** *Let $F : X = Bl_I Y \to Y$ and $S = V(I^n) \subset Y$ be a nonrational center for f. Then there exists a gluing along φ for which*

$$\mathcal{D}(Y) \to \mathcal{D} = \mathcal{D}(\mathcal{A}_S) \times_\varphi \mathcal{D}(X)$$

is a partial categorical DG-resolution of singularities. Moreover restricting along π takes both $D^b_{\mathrm{Coh}}(\mathcal{A}_S)$ and $D^b_{\mathrm{Coh}}(X)$ to $D^b_{\mathrm{Coh}}(Y)$).

Let Y be a separated scheme of finite type over k with characteristic 0.

Let $Y_m \to Y_{m-1} \to ... \to Y_1 \to Y_0 = Y$ be a chain of blowups with smooth center $Z_i \subset Y_i$ with $(Y_m)_{\text{red}}$ smooth.

By the above theorem we have a partial categorical DG-resolution $\mathcal{D}(Y) \to \mathcal{D}(Y)_1 \times_\varphi \mathcal{D}(\mathcal{A}_S)$, where S is a thickening of Z_0. And $\mathcal{D}(\mathcal{A}_S)$ is smooth and φ is perfect, so all non-smoothess comes from Y_1. We can inductively reglue to replace $\mathcal{D}(Y_1)$ with its categorical resolution.

Theorem 4.3 (6.12)**.** *There is a pre-triangulated DG-category $\mathcal{D}$ glued from copies of $\mathcal{D}(Y_m)$ and $\mathcal{D}(Z_i)$ for $0 \leq i \leq m-1$ and a DG-functor $\pi : \mathcal{D}(Y) \to \mathcal{D}$ satisfying*

1. *$\mathcal{D}$ is a categorical DG-resolution of $\mathcal{D}(Y)$;*
2. *Restriction along π takes $[\mathcal{D}]$ to $D^b_{\text{Coh}}(Y)$.*
3. *If Y is proper then so is $\mathcal{D}$.*

5 Related Ideas

5.1 Refined blowups

Another approach to categorical resolutions is to use filtrations and graded objects in analogy with the Proj construction to get embeddings and decompositions.

Given a closed subscheme $Z \subset X$, we can use the ideal sheaf to obtain a filtration on coherent sheaves on X to obtain a filtered derived category $\mathcal{D}F^b(\mathcal{A})$ as well as the derived category with vanishing above a certain grading $\mathcal{D}F^b_{\text{tors}}(\mathcal{A})$.

The *n-refined blowup* of X in Z is

$$\text{Bl}_n(X, Z) = \mathcal{D}F^b(\mathcal{A})/\mathcal{D}F^b_{\text{tors}}(\mathcal{A})_n$$

Proposition 5.1. *For $n >> 0$, $\mathcal{D}^{perf}(X) \to Bl_n(X, Z)$ is fully faithful.*

There is a semi-orthogonal decomposition $\text{Bl}_n(X, Z) = \langle \mathcal{D}^b(Y), \mathcal{D}^b(Z), ..., \mathcal{D}^b(Z)\rangle$ with n copies of $\mathcal{D}^b(Z)$. So it is good that we know there is a finite n that works!

See Kaledin Kuznetsov 2015.

5.2 Categorical Absorption

This is a fundamentally different approach to dealing with singularities.

Definition 5.2. $\mathcal{P} \subset D^b(X)$ *absorbs singularities of* X if $\mathcal{P}$ is admissible and both $\mathcal{P}^\perp$ and ${}^\perp\mathcal{P}$ are smooth and proper.

Under some assumptions, the paper constructs a categorical absorption for a projective variety with isolated ordinary double points.

See Kuznetsov Shinder.