Semi-orthogonal Decompositions Seminar Notes

Notes taken by Amal Mattoo, who apologizes for any mistakes.

April 16

Fan Zhou: Bridgeland Stability and Highest Weight Categories.

Will say a bit about Bridgeland stability, then talk about highest weight categories, which is in a precise sense the same thing.

Definition 0.1. Given an abelian category $\mathcal{A}$, a *stability function* is a *central charge*

$$Z : K_0(\mathcal{A}) \xrightarrow{\text{grp}} \mathbf{C}$$

such that the image of any nonzero object is in the upper half plane, i.e., $\mathbf{R}_{>0}e^{i\pi\phi}$ or $0 < \phi \leq 1$.

Given $X \in \mathcal{A}$, call $\phi(X)$ the *phase* in $Z(X) \in \mathbf{C}$.

Say $X \in \mathcal{A}$ is *semi-stable* if for all $0 \neq A \subset X$ we have $\phi(A) \leq \phi(X)$. Say X is *stable* if all $\phi(A) < \phi(X)$.

Definition 0.2. Given Z on $\mathcal{A}$, a *Harder-Narasimhan filtration*

$$0 = X_0 \subset X_1 \subset ... \subset X_n = X$$

such that each X_i/X_{i-1} is semi-stable and

$$\phi(X_1/X_0) > ... > \phi(X_n/X_{n-1})$$

Harder-Narasimhan filtrations are unique.

We could equivalently define (semi-)stability by requiring for all $X \twoheadrightarrow B$ we have $\phi(X) \leq \phi(B)$. This is equivalent because given

$$0 \to A \to X \to B \to 0$$

we have $Z(X) = Z(A) + Z(B)$ so $\phi(X)$ is between $\phi(A)$ and $\phi(B)$.

If X and Y are semi-stable, then for any $0 \neq f : X \to Y$ we have $\phi(X) \leq \phi(Y)$, because

$$0 \to \ker f \to X \to \mathrm{Im}(f) \to 0$$

implies

$$\phi(\ker f) \leq \phi(X) \leq \phi(\mathrm{Im}(f)) \leq \phi(Y)$$

where the first two inequalities are by stability of X and the last is by stability of Y.

Proposition 0.3. *If a Z satisfies*

1. *There is no infinite sequence of subobjects*

$$... \subset X_{i+1} \subset X_i \subset ... \subset X_1$$

with $\phi(X_{i+1}) > \phi(X_i)$.

2. *There is no infinite sequence of quotients*

$$X_1 \twoheadrightarrow X_1 \twoheadrightarrow ... \twoheadrightarrow X_i \twoheadrightarrow X_{i+1} \twoheadrightarrow$$

with $\phi(X_i) > \phi(X_{i+1})$.

Then Z is Harder-Narasimhan.

Proof. Any $X \in \mathcal{A}$ has either X semi-stable or there exists $A \subset X$ with $\phi(A) > \phi(X)$.

Iterating this must terminate, so all X have some semi-stable subobject $A \subset X$ with $\phi(A) \geq \phi(X)$.

Similarly, all X are semi-stable or have a semi-stable quotient $X \twoheadrightarrow B$ with $\phi(X) \geq \phi(B)$.

Let a maximal destabilizing quotient of X be $X \twoheadrightarrow B$ with B semi-stable and $\phi(X) \geq \phi(B)$ such that all $X \twoheadrightarrow B'$ have $\phi(B') \geq \phi(B)$ with equality when

$$\begin{array}{ccc} X & \twoheadrightarrow & B' \\ \downarrow\!\!\downarrow & \nearrow & \\ B & & \end{array}$$

Claim: this exists. If X is semi-stable, then it's true. Otherwise, find

$$0 \to A \hookrightarrow X \to X' \to 0$$

with A semi-stable and $\phi(A) > \phi(X) > \phi(X')$. We want to show that if $X' \twoheadrightarrow B$ with $\phi(X') \geq \phi(B)$ is a maximal destabilizing quotient, then $X \to X' \twoheadrightarrow B$ works; this suffices, or else we could produce an infinite chain. This is because there exists $X \twoheadrightarrow B'$ semi-stable. Assume for the sake of contradiction $\phi(B') < \phi(B)$. Then

$$\phi(B') < \phi(B) \leq \phi(X) < \phi(A)$$

so there is no nonzero $A \to B'$, and so

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \twoheadrightarrow & X' \\ & & & & \downarrow\!\!\downarrow & \overset{\exists}{\swarrow} & \downarrow\!\!\downarrow \scriptstyle{mdq} \\ & & & & B' & & B \end{array}$$

which contradicts B having minimal phase. Repeating the argument with X' in $X \twoheadrightarrow X'$, $\phi(X) > \phi(X')$. This must terminate. Hence, maximal destabilizing quotients exist.

Next, if X is semi-stable, we have a Harder-Narasimhan filtration trivially. Otherwise, take

$$0 \to X' \hookrightarrow X \twoheadrightarrow B \to 0$$

with $X \twoheadrightarrow B$ a maximal destabilizing quotient, and $\phi(X') > \phi(X) > \phi(B)$.

Find a maximal destabilizing quotient $X' \twoheadrightarrow B'$ and build

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & X' & \longrightarrow\!\!\!\!\!\rightarrow & B' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B & \xrightarrow{=} & B & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

since B is a maximal destabilizing quotient, we have $\phi(Q) > \phi(B)$, so $\phi(B') > \phi(Q) > \phi(B)$. Letting $X'' = K$ we can iterate $... \subset X'' \subset X' \subset X$ with $... > \phi(X'') > \phi(X') > \phi(X)$ with $\phi(B') > \phi(B)$. This terminates by (2), giving a Harder-Narasimhan filtration. □

Definition 0.4. A stability condition on a triangulated category $\mathcal{D}$ is $(Z, \mathcal{P})$ with $Z : K_0(\mathcal{D}) \xrightarrow{\text{grp}} \mathbf{C}$ and $\mathcal{P}(\phi)$ is a full additive subcategory for $\phi \in \mathbf{R}$ such that

1. $Z : \mathcal{P}(\phi) \to \mathbf{R}_{>0} e^{i\pi\phi}$.
2. $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
3. $\mathrm{Hom}_{\mathcal{D}}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ for $\phi_1 > \phi_2$.
4. For all $X \neq 0$, exists $\phi_1 > \phi_2 > ... > \phi_n$ such that

$$0 = X_0 \to X_1 \to X_2 \to ... \to X_{n-1} \to X_{n-1} \to X_n$$

such that $\mathrm{Cone}(X_{i-1} \to X_i) \in \mathcal{P}(\phi_n)$.

Conditions (2), (3), (4) are called a *slicing*.

Can put a t-structure on a derived category $\mathcal{D}$ with a stability condition with $\mathcal{D}^{\leq 0} = \mathcal{P}(> \phi)$. The heart of $\mathcal{P}(> \phi)$ is $\mathcal{P}(\phi, \phi + 1]$.

Can similarly define a t-structure $\mathcal{D}^{\leq 0}\mathcal{P}(\geq \phi)$ with heart $\mathcal{P}[\phi, \phi + 1)$.

Lemma 0.5. *For $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$, have $\mathcal{P}(\phi)$ is abelian.*

Proposition 0.6. *Stability condition on $\mathcal{D}$ is equivalent to a bounded t-structure on $\mathcal{D}$ with a stability function on $\mathcal{D}^{\heartsuit}$ satisfying Harder-Narasimhan filtrations.*

Now we turn to highest weight categories.

Definition 0.7. A k-linear abelian category is "Deligne-finite" if

1. Hom, Ext-finite.
2. Jordan-Hölder category, i.e., Noetherian and Artinian.
3. Finite number of simples.
4. Enough projectives.

Such a category is Morita-equivalent to Mod-$\mathrm{End}_{\mathcal{A}}$(Proj-gen) endomorphisms of a projective generator.

Choose an ordering $L_1, ..., L_n$ of simples with projective covers P_i and injective hulls Q_i.

$$0 = \mathcal{A}_0 \subset \mathcal{A}_1 \subset ... \subset \mathcal{A}_n = \mathcal{A}$$

Then $\mathcal{A}_i = \mathrm{Filt}(L_1, ..., L_i)$ is a Serre subcategory, i.e., objects with Jordan-Hölder quotients L_i.

Definition 0.8. Let Δ_i be the maximal quotient of P_i lying in $\mathcal{A}_i$. This is the *standard module.*

Let ∇_i be the maximal subobject of Q_i lying in $\mathcal{A}_i$. This is the *costandard module.*

Then Δ_i is the projective cover of L_i in $\mathcal{A}_i$ and ∇_i is the injective hull of L_i in $\mathcal{A}_i$.

Definition 0.9. A *highest weight category* is a Deligne-finite category such that

1. $\mathrm{End}_{\mathcal{A}_i}(\Delta_i) \cong k$, and
2. $P_i \in \mathrm{Filt}(\Delta_i, .., \Delta_n)$.

Recall a recollement diagram consists of categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, with functors $\mathcal{A} \xrightarrow{i_*} \mathcal{B}$, $\mathcal{B} \xrightarrow{j^!} \mathcal{C}$, $\mathcal{B} \xrightarrow{i^*, i^!} \mathcal{A}$, $\mathcal{C} \xrightarrow{j_!, j_*} \mathcal{B}$ such that

1. i_*, j^* are exact with two adjoints given,
2. i_* is fully faithful with essential image the kernel of $j^!$, and
3. $j^* j_* \xrightarrow{\sim} [\varepsilon]\mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} [\eta] j^! j_!$.

An ideal $I \subset A$ is *hereditary* if

1. $I^2 = I$,
2. $I\mathfrak{j}I = 0$, where $\mathfrak{j}$ is the Jacobson radical, and
3. I is projective as left or right A-module.

And A is *quasi-hereditary* if there exists $0 = J_0 \subset J_1 \subset ... \subset J_n = A$ with J_i/J_{i-1} is hereditary in A/J_{i-1}.

Theorem 0.10. *The following are equivalent:*

- *$\mathcal{A}$ is a highest weight category,*
- *$\mathcal{A}_{i-1} \to \mathcal{A}_i \to \mathcal{A}_i/\mathcal{A}_{i-1}$ is a homological recollement,*
- *$(\Delta_1, ..., \Delta_n)$ is a strictly full exceptional sequence, and*
- *$\mathcal{A} \simeq \mathrm{Mod}\, A$ for A quasi-hereditary.*

Theorem 0.11. *If $\mathcal{A}$ be Deligne-finite category. Fix an ordering. The following are equivalent:*

1. *$\mathcal{A}$ is a highest weight category,*
2. *$[\Delta_i : L_i] = 1$, $[P_i] = \sum_{j=1}^n [\nabla_j : L_i][\Delta_j]$ in $K_0(\mathcal{A})$, and*
3. *Exists a stability function $Z : K_0(\mathcal{A}) \to \mathbf{C}$ such that*
    (a) *Δ_i is stable, $\phi(\Delta_1) < ... < \phi(\Delta_n)$, and*
    (b) *Harder-Narasimhan factors of P_i are direct sums of $\Delta_i, ..., \Delta_n$.*

Example. Recalling the Auslander construction from last week, letting $S = \mathrm{Spec}(k[t]/t^2)$, $I^n = 0$, we constructed an algebra $A_{S,I,n}$, e.g.,

$$A_{S,\langle t\rangle,2} = \begin{pmatrix} k[t]/t^2 & \langle t\rangle \\ k[t]/t & k[t]/t \end{pmatrix}$$

This corresponds to a quiver $e : \bullet \leftrightarrows \bullet : f$ with $ef = 0$.

Then L_i corresponds to the quiver representation with $\mathbf{C}$ on vertex i and 0 on the other vertex.

Then $\mathcal{A} = \mathrm{Filt}(\mathcal{L}_1, \mathcal{L}_2)$ while $\mathrm{Filt}(\mathcal{L}_1)$ is the quiver category of a single vertex.

So $P_2 = \Delta_2$ is the projective cover of L_2 in $\mathcal{A}_2 = \mathcal{A}$, which by general theory is $0 : \mathbf{C} \leftrightarrows \mathbf{C} : 1$.

And $\Delta_1 = L_1$, since it is projective in $\mathcal{A}_1 = \mathrm{Filt}(\mathcal{L}_1)$. And P_1 is the projective cover of L_1 in $\mathcal{A}$, which is again by general theory is $\mathbf{C}^2 \leftrightarrows \mathbf{C}$ with maps $1 : \mathbf{C} \to \mathbf{C}$ and $0 : \mathbf{C} \leftrightarrows \mathbf{C} : 1$.

The full exceptional collection is $\langle \Delta_1, \Delta_2 \rangle$, and we have exact sequences

$$0 \to L_1 \to \Delta_2 \to L_2 \to 0$$

$$0 \to \Delta_2 \to P_1 \to \Delta_1 \to 0$$