Semi-orthogonal Decompositions Seminar Notes

Notes taken by Amal Mattoo, who apologizes for any mistakes.

April 23

James Hotchkiss: Homological projective duality.

All due to Alexander Kuznetsov in Homological Projective Duality.

Let X be smooth projective over a field. Fix a map $X \to \mathbf{P}(V)$, e.g, an embedding — but not necessarily.

Given a linear subspace $L \subset V$, we can form $X_L = X \times_{\mathbf{P}(V)} \mathbf{P}(L)$. There is a nice class of semi-orthogonal decompositions that behave well under restriction.

Definition 0.1. A *(right) Lefschetz decomposition* is a semi-orthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(H), ..., \mathcal{A}_{m-1}((m-1)H) \rangle$$

for admissible subcategories $\mathcal{A}_0 \supset \mathcal{A}_1 \supset ... \supset \mathcal{A}_{m-1}$ and with H the pullback to X of $\mathcal{O}_{\mathbf{P}(V)}(1)$.

If $\mathcal{A}_0 = ... = \mathcal{A}_{m-1}$, we call this a *rectangular* Lefschetz decomposition.

Lemma 0.2. *Let $L \subset V$ be linear such that X_L has expected dimension $(*)$. If X has Lefschetz decomposition as above, then*

$$D^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_s(sH), .., \mathcal{A}_{m-1}((m-1)H) \rangle$$

is a semi-orthogonal decomposition for $s = \operatorname{codim}(L \subset V)$.

This is not, in general, a Lefschetz decomposition, as the Kuznetsov component $\mathcal{C}_L$ is mysterious for now.

Remark. The composition $\mathcal{A}_s(sH) \hookrightarrow D^b(X) \to D^b(X_L)$ is fully faithful.

Example. If $\mathbf{P}(W) \xrightarrow{|\mathcal{O}(d)|} \mathbf{P}(V)$, we get standard semi-orthogonal decomposition for a degree d hypersurface, for $d < \dim W$.

In this case, $\mathcal{A}_0 = \langle \mathcal{O}, ..., \mathcal{O}(d-1) \rangle$, $\mathcal{A}_1(d) = \langle \mathcal{O}(d), ..., \mathcal{O}(N) \rangle$ if $N < 2d$ for $N = \dim \mathbf{P}(W)$.

About $(*)$: ignore it if you use derived fiber product $X_L = X \times^L_{\mathbf{P}(V)} \mathbf{P}(L)$.

Remark. Exists a decomposition into $\mathcal{A}_0 = \langle a_0, ..., a_{m-1}\rangle$ such that $\mathcal{A}_i = \langle a_i, ..., a_{m-1}\rangle$. The categories a_i are called *primitive Lefschetz components.*

There is a notion of a left Lefschetz decomposition

$$D^b(X) = \langle \mathcal{A}_{1-m}((1-m)H), ..., \mathcal{A}_{-1}(-H), \mathcal{A}_0$$

for $\mathcal{A}_{1-m} \subset ... \subset \mathcal{A}_{-1} \subset \mathcal{A}_0$. The analogue of the lemma holds: $D^b(X_L) = \langle ..., \mathcal{D}_L\rangle$. Left and right are "equivalent": given $\mathcal{A}_0 \subset D^b(X)$,

$$\mathcal{A}_i := \begin{cases} \mathcal{A}_{i-1} \cap {}^\perp(\mathcal{A}_0(-iH)) & i \geq 1 \\ \mathcal{A}_{i+1} \cap (\mathcal{A}_0(iH))^\perp & i \leq -1 \end{cases}$$

And in fact: $\langle \mathcal{A}_0, ..., \mathcal{A}_{m-1}((m-1)H)\rangle$ is a right Lefschetz decomposition if and only if $\langle \mathcal{A}_{1-m}((1-m)H), ..., \mathcal{A}_0\rangle$ is a left Lefchetz decomposition.

Goal: let $L \subset V$ be a hyperplane. Consider $(\mathcal{C}_L)_{L \in \mathbf{P}(V^\vee)}$. Let

$$\mathcal{H} = \{(x, L) \in X \times \mathbf{P}(V^\vee) : x \text{ maps into } L \text{ under } X \to \mathbf{P}(V)\}$$

Lemma 0.3. *Exists a semi-orthogonal decomposition*

$$D^b(\mathcal{H}) = \langle \mathcal{C}, \mathcal{A}_1(H) \boxtimes D^b(\mathbf{P}(V)^\vee), ..., \mathcal{A}_{m-1}((m-1)H) \boxtimes D^b(\mathbf{P}(V^\vee))$$

where the embedding is via $\mathcal{A}_i(iH) \boxtimes D^b(\mathbf{P}(V^\vee)) \hookrightarrow D^b(X \times \mathbf{P}(V^\vee)) \to D^b(\mathcal{H})$.

Note that we need the above composition to be fully faithful, which it is.

The map $\mathcal{H} \to \mathbf{P}(V^\vee)$ implies there's an "action" of $D^b(\mathbf{P}(V^\vee))$ on $D^b(\mathcal{H})$, where $E \in D^b(\mathbf{P}(V^\vee))$ acts on $D^b(\mathcal{H})$ via $\mathrm{pr}_2^* E \otimes -$.

The action preserves the components of the semi-orthogonal decomposition of $D^b(\mathcal{H})$, i.e., the semi-orthogonal decomposition is $\mathbf{P}(V^\vee)$-linear. Can think of the semi-orthogonal decomposition of $D^b(X_L)$ as the semi-orthogonal decomposition of $D^b(\mathcal{H})$ base changed by:

$$\begin{array}{ccc} D^b(X_L) & \longrightarrow & D^b(\mathcal{H}) \\ \downarrow & & \downarrow \\ [L] & \longrightarrow & \mathbf{P}(V^\vee) \end{array}$$

The base change of $\mathcal{C}$ is $\mathcal{C}_L$.

Definition 0.4. The *homological projective dual (HPD)* of $D^b(X)$ is $\mathcal{C}$, denoted by $D^b(X)^\#$.

A variety $Y \to \mathbf{P}(V^\vee)$ is an *HPD variety* of X (denoted $X^\#$) if there is an equivalence $D^b(Y) \cong D^b(X)^\#$ linear over $D^b(\mathbf{P}(V^\vee))$.

Should think of $D^b(X)^\#$ as linear over $D^b(\mathbf{P}(V^\vee))$.

Theorem 0.5 (Main)**.** *Assume $X^\#$ exists.*

1. *$X^\#$ is smooth and proper over the base field, and*

$$\{\text{critical values of } X^\# \to \mathbf{P}(V^\vee)\} = \{L \in \mathbf{P}(V^\vee) \text{ such that } X_L \text{ is singular}\}$$

   The right hand side is the classical projective dual *$X^\vee$.*

2. *$X^\#$ has a left Lefschetz decomposition*

$$D^b(X^\#) = \langle \mathcal{B}_{1-n}((1-n)H'), ..., \mathcal{B}_0\rangle$$

   Moreover, $(X^\#)^\# = X$ in a natural way.

3. *Given $L \subset V$, let $L^\perp = \mathrm{Ann}(L) \subset V^\vee$. We have*

$$D^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_s(sH), ...\rangle$$

$$D^b(X^\#_{L^\perp}) = \langle ..., \mathcal{B}_{-r}(-rH'), \mathcal{D}_{L^\perp}\rangle$$

   where $s = \mathrm{codim}(L)$ and $r = \mathrm{codim}(L^\perp)$. Then $\mathcal{C}_L \simeq \mathcal{D}_{L^\perp}$.

In practice: let's say we have an embedding $X \hookrightarrow \mathbf{P}(V)$. The classical projective dual $X^\vee$ is usually singular. Then in practice a resolution $\widetilde{X^\vee} \to X^\vee$ has a semi-orthogonal component $D^b(X)^\# \subseteq D^b(\widetilde{X^\vee})$, and $D^b(X)^\#$ is a categorical resolution of $D^b(X^\vee)$.

How to look at it.

Example. Let $\dim V = 9$ and $\mathcal{A}_0 \supset \mathcal{A}_1 = \mathcal{A}_2 \supset \mathcal{A}_3 \neq 0$. It then follows that

$$\mathcal{B}_{-7} = \mathcal{B}_{-6} \subset \mathcal{B}_{-5} \subset \mathcal{B}_{-4} = \mathcal{B}_{-3} = \mathcal{B}_{-2} = \mathcal{B}_{-1} = \mathcal{B}_0$$

And it turns out $\mathcal{A}_0 = \mathcal{B}_0$.

Example. Let $X = \mathrm{Gr}(2,6) \hookrightarrow \mathbf{P}(V)$ where $V = \bigwedge^2 W$ and $\dim W = 6$. Then

$$D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), ..., \mathcal{A}_5(5)\rangle$$

where $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \langle \mathcal{O}, U^\vee, S^2U^\vee\rangle$ and $\mathcal{A}_3 = \mathcal{A}_4 = \mathcal{A}_5 = \langle \mathcal{O}, U^\vee\rangle$. We have

$$X^\vee = \mathrm{Pf}(4,6) = \{\omega \in \bigwedge^2 W^\vee : \mathrm{rk}(\omega) \leq 4\}$$

corresponds to a hypersurface of degree 3 (since the Pfaffian is the square root of the determinant). Then $X^\vee$ is singular along $\mathrm{Pf}(2,6) = \mathrm{Gr}(2, W^\vee)$, so singular locus has codimension 5.

Let $L \subset \bigwedge^2 W$, $L^\perp \subset \bigwedge^2 W^\vee$, $r = \dim L^\perp$. We have

$$\dim X_L = 8 - r, \quad \dim \mathrm{Pf}_L = r - 2, \quad \dim(\mathrm{Pf}^{\mathrm{sing}})_L = r - 7$$

Assume $r \leq 6$, so we can run HPD as if the Pfaffian were smooth.

- Let $r = 1$. Then $\mathrm{Pf}_L = \emptyset$, so $D^b(X_L)$ has a full exceptional collection. Here $X_L = \mathrm{LGr}(2,6)$, the Lagrangian Grassmannian, consisting of planes isotropic with respect to the two-form ω.

- Let $r = 2$. Then Pf_L has 3 points, so X_L has a full exceptional collection.
- Let $r = 3$. Then Pf_L is an elliptic curve.
- Let $r = 4$. Then Pf_L is a cubic surface, so X_L has a full exceptional collection.
- Let $r = 5$. Then Pf_L is a cubic 3-fold. And $X_L = V_{14}$ a Fano 3-fold.
- $r = 6$. Then Pf_L is a cubic 4-fold and X_L is a K3 surface. This verifies the conjecture that cubic fourfolds are rational if and only if their Kuznetsov component is isomorphic to the derived category of a K3 surface.