Semi-orthogonal Decompositions Seminar Notes

Notes taken by Amal Mattoo, who apologizes for any mistakes.

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Andrés Ibáñez Núñez: Windows and magic windows.

If X is a scheme, $i : S \hookrightarrow X$ closed, $j : U := X \setminus S \to X$ open, it is natural to ask if we can understand $D^b(X)$ in terms of $D^b(U)$ and $D^b(S)$. This leads to local cohomology: for $E \in D^b(X)$, have fiber sequence

$$\Gamma_E(E) \to E \to j_* j^* E$$

But the issue is that $\Gamma_S(E) \in \mathrm{QC}(X)$ but not in $D^b(X)$. We do get a semi-orthogonal decomposition of

$$\mathrm{QC}(X) = \langle \mathrm{QC}(U), \mathrm{QC}_S(X) \rangle$$

where $\mathrm{QC}(X)$ is the unbounded derived category of quasi-coherent sheaves. But it does not restrict to a semi-orthogonal decomposition of $D^b(X)$.

Sometimes we can fix this bug, but need equivariance.

Let $\mathbf{G}_m$ act on $\mathbf{Q}^1$ over $\mathbf{C}$ by $t * x = t^{-1}x$. Have $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[x])$, and x has weight 1. Let $\mathcal{X} = \mathbf{A}^1/\mathbf{G}_m$. Then set $S = 0/\mathbf{G}_m \hookrightarrow X \hookleftarrow U = X \setminus S$.

Then $\mathrm{QCoh}(\mathcal{X})$ consists of $\mathbf{Z}$-graded $\mathbf{C}[x]$-modules. We have an exact sequence

$$0 \to \mathcal{O}_X = \mathbf{C}[x] \to j_* j^* \mathcal{O}_X \to x^{-1}\mathbf{C}[x^{-1}] \to 0$$

Here $x^{-1}\mathbf{C}[x^{-1}][-1] = \Gamma_S(\mathcal{O}_X) = \mathrm{colim}\ \mathbf{C}[x]/(x^{-j})\langle j \rangle[-1]$.

For $E \in \mathrm{QC}(\mathcal{X})$, we have $j_* j^* E = E \otimes \mathbf{C}[x, x^{-1}]$,

$$\Gamma_S(E) = E \otimes \Gamma_S(\mathcal{O}_\mathcal{X}) = \mathrm{colim}_j E \otimes \mathbf{C}[x]/(x^{-j})\langle j \rangle[-1]$$

- $x^{-1}\mathbf{C}[x^{-1}]$ is not finitely generated as a $\mathbf{C}[x]$-module.
- When I take the submodule of $x^{-1}\mathbf{C}[x^{-1}]$ of weights $\geq w$, then that is finitely generated.
- $\mathbf{C}[x]/x^{-j}\langle j \rangle$ is an extension of $\mathbf{C}\langle i \rangle$ ($\mathbf{C}$ at weight i).

Let $D^b_S(\mathcal{X})^{\geq w}$ be the subcategory of $D^b(\mathcal{X})$ generated by $\mathbf{C}\langle j \rangle$ for $j \geq w$. Define $D^b_S(\mathcal{X})^{<w}$ correspondingly.

Observation. *$D^b_S(\mathcal{X})^{\geq w} \hookrightarrow D^b(\mathcal{X})$ has a right adjoint $\beta_S^{\geq w}$ taking subobjects of weights $\geq w$.*

$$\beta^{\geq w}\Gamma_S(E) = \mathrm{colim}_{j>0}\beta^{\geq w}(E \otimes \mathbf{C}[x]/(x^j)\langle -j\rangle[-1])$$

The colimit stabilizes for $j >> 0$.

Similarly, $D^b_S(\mathcal{X})^{<w} \hookrightarrow D^b(\mathcal{X})$ has a left adjoint $\beta_S^{<w}$ taking maximal quotient of weight $< w$:

$$D_S^{<w}(E) = \beta^{<w}(E \otimes \mathbf{C}[x]/(x^j)), \quad j >> 0$$

Example. $\beta_S^{<0}(\mathbf{C}[x]\langle -j\rangle) = \mathbf{C}\langle -j\rangle \oplus \mathbf{C}x\langle -j+1\rangle \oplus ... \oplus \mathbf{C}x^{j-1}\langle -1\rangle$.

We get a semi-orthogonal decomposition

$$D^b(\mathcal{X}) = \langle D^b_S(X)^{<w}, \mathbf{G}^w, D^b_S(\mathcal{X})^{\geq w}\rangle$$

where

$$\mathbf{G}^w = \langle E \in D^b(X) : \beta_S^{\geq w}(E) = \beta_S^{<w}(E) = 0\rangle$$

$$D^b_S(\mathcal{X}) = \langle E \in D^b(\mathcal{X}) : E|_U\rangle = \langle \mathbf{C}\langle j\rangle, j \in \mathbf{Z}\rangle = \langle D^b_S(X)^{<w}, D^b_S(X)^{\geq w}\rangle$$

and

$$\mathbf{G}^w \cong D^b(U) = D^b(\mathrm{pt}) = \langle \mathcal{O}_{\mathcal{X}}\langle w\rangle\rangle$$

where $U = (\mathbf{A}^1 \setminus 0)/\mathbf{G}_m = \mathrm{pt}$.

- $\mathrm{Hom}(\mathbf{C}\langle j\rangle, \mathcal{O}_{\mathcal{X}}\langle w\rangle) = 0$ if $j \geq w$.
- $\mathrm{Hom}(\mathcal{O}_{\mathcal{X}}\langle w\rangle, \mathbf{C}\langle j\rangle) = 0$ if $j < w$.
- Triangle $\mathcal{O}_{\mathcal{X}}\langle j+1\rangle \to \mathcal{O}_{\mathcal{X}}\langle j\rangle \to \mathbf{C}\langle j\rangle$ generation.

This semi-orthogonal decomposition can be generalized. Let $\mathcal{X}$ be an algebraic stack (derived). Let $\theta = \mathbf{A}^1/\mathbf{G}_m$ (usual scaling action, x weight -1). Define

$$\mathrm{Filt}(\mathcal{X}) = \mathrm{Map}(\theta, \mathcal{X}) \xrightarrow{\{1\}\hookrightarrow\theta} \mathcal{X}$$

A *θ-stratum* is an open and closed substack $S \subset \mathrm{Filt}(\mathcal{X})$ such that $S \to \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$ is a closed immersion.

Example. Let $\mathbf{G}_m$ act on $\mathrm{Spec}(A)$, corresponding to the grading $A = \bigoplus_{n\in\mathbf{Z}} A_n$. Let $\mathcal{X} = \mathrm{Spec}(A)/\mathbf{G}_m$. The *canonical* θ-stratum is

$$\mathrm{Spec}\left(\bigoplus_{n\leq 0} A_n\right)/\mathbf{G}_m \hookrightarrow \mathcal{X}$$

And

$$\mathrm{Spec}(\bigoplus_{n\leq 0} A_n) = \{x \in \mathrm{Spec}(A) : \exists \lim_{t\to 0} t\cdot x\}$$

This $\lim_{t\to 0} t\cdot x$ generalizes to a map $S \xrightarrow{\text{gr}} \mathcal{Z}$, where $\mathcal{Z}$ is the center of the θ-stratum. Have $\mathcal{Z}$ closed open in $\text{Grad}(\mathcal{X}) = \text{Map}(B\mathbf{G}_m, \mathcal{X})$.

Theorem 0.1 (Halpern-Leistner)**.** *Suppose S is a θ-stratum of $\mathcal{X}$ and $i : S \hookrightarrow \mathcal{X}$ is regular.*

$$\text{Perf}_S(\mathcal{X}) = \{E \in \text{Perf}(\mathcal{X}) : E|_{\mathcal{X}\setminus S} = 0\}$$

$$\text{Perf}_S(\mathcal{X})^{\geq w} = \{E \in \text{Perf}_S(\mathcal{X}) : \sigma^* i^*(E) \in \text{Perf}(\mathcal{Z})^{\geq w}\}$$

The action of $B\mathbf{G}_m$ on $\mathcal{Z}$ induces $\text{Perf}(\mathcal{Z}) = \bigoplus_{w\in\mathbf{Z}} \text{Perf}(\mathcal{Z})^w$, $\text{Perf}(\mathcal{Z})^{\geq w} = \bigoplus_{j\geq w} \text{Perf}(Z)^j$. Have

$$\begin{aligned}\text{Perf}_S(\mathcal{X})^{<w} &= \{E \in \text{Perf}_S(\mathcal{X}) : \sigma^* i^!(E) \in \text{Perf}(\mathcal{Z})^{<w}\}\\ &= \{E \in \text{Perf}_S(\mathcal{X}) : \sigma^* i^*(E) \in \text{Perf}(\mathcal{Z})^{w+i}\}\end{aligned}$$

where $i^! E = i^* E \otimes \det \mathcal{N}^\vee_{S/\mathcal{Z}}[-c]$, with $\det \mathcal{N}^\vee_{S/\mathcal{X}} = 1$. And

$$\mathbf{G}^w = \{E \in \text{Perf}(\mathcal{X}) : \sigma^* i^*(E) \in \text{Perf}(\mathcal{Z})^{[w,w+i)}\}$$

Then we have a semi-orthogonal decomposition

$$\text{Perf}(\mathcal{X}) = \langle \text{Perf}_S(\mathcal{X})^{<w}, \mathbf{G}^w, \text{Perf}_S(\mathcal{X})\rangle^{\geq w}$$

Restriction induces an equivalence $\mathbf{G}^w \xrightarrow{\simeq} \text{Perf}(\mathcal{X}\setminus S)$.

Example. $\mathbf{P}^n \subset \mathbf{A}^{n+1}/\mathbf{G}_m = \mathcal{X} \hookleftarrow S = 0/\mathbf{G}_m$.

$$D^b(\mathbf{P}^n) = \mathbf{G}^w = \{E \in \text{Perf}(\mathbf{A}^{n+1}/\mathbf{G}_m) : E|_{0/\mathbf{G}_m} \in \text{Perf}^{[w,w+n+1)}(0/\mathbf{G}_m)\}$$

and $\text{Perf}(\mathcal{X})$ is generated by $\mathcal{O}_{\mathcal{X}}\langle j\rangle$, $j \in \mathbf{Z}$. Here $\mathcal{O}_{\mathcal{X}}\langle j\rangle|_{0/\mathbf{G}_m} = \mathbf{C}\langle j\rangle$. and

$$\mathbf{G}^w = \langle \mathcal{O}_{\mathcal{X}}\langle w\rangle, \mathcal{O}_{\mathcal{X}}\langle w+1\rangle, ..., \mathcal{O}_{\mathcal{X}}\langle w+n\rangle\rangle$$

Restricting to $\mathbf{P}^n$ we recover the Beilinson exceptional collection.