Introducing Semi-orthogonal Decompositions

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February 5, 2025

1 Basics of Semi-orthogonal Decompositions

For X/k a variety, we want to understand $D^b_{\mathrm{Coh}}(X)$, by breaking it into pieces. First, let's understand what the pieces look like. The following applies in the generality of triangulated categories.

Definition 1.1. A subcategory $i : \mathcal{A} \hookrightarrow \mathcal{D}$ of a triangulated category $\mathcal{D}$ is *left (rep. right) admissible* if the inclusion i has a left (resp. right) adjoint i^* (resp. $i^!$). It is *admissible* if it is left and right admissible.

The following is a key example for generating admissible subcategories.

Definition 1.2. An object E is *exceptional* if $\mathrm{Hom}^*(E, E) = k[0]$.

That is, E behaves homologically like a point, so $\langle E \rangle = D^b_{\mathrm{coh}}(k)$. This is the simplest form a component of a triangulated category can take.

Proposition 1.3. *If E is exceptional, then $\langle E \rangle$ is admissible.*

Proof. We will construct the adjoints to $i : \langle E \rangle \hookrightarrow \mathcal{D}$ directly. Define for $A \in \mathcal{D}$,

$$i^*(A) = \mathrm{Hom}^*(A, E)^\vee \otimes E, \quad i^!(A) = \mathrm{Hom}^*(E, A) \otimes E$$

We have an evaluation map $A \to \mathrm{Hom}^*(A, E)^\vee \otimes E$. Applying $\mathrm{Hom}^*(-, E)^\vee$ yields an isomorphism, so $\mathrm{Hom}^*(i^*(A), E) \cong \mathrm{Hom}^*(A, i(E))$. Every element of $\langle E \rangle$ is $\bigoplus E$, so we have adjunction. The case of $i^!$ follows symmetrically. $\square$

Remark. $\mathrm{Hom}^*(A, B) = \bigoplus_i \mathrm{Hom}^i(A, B)[-i]$.

Now, let's understand how to decompose a triangulated category into admissible subcategories.

Definition 1.4. A *semi-orthogonal decomposition* of a triangulated category $\mathcal{D}$, denoted

$$\mathcal{D} = \langle \mathcal{D}_1, ..., \mathcal{D}_m \rangle$$

is a sequence of full triangulated subcategories such that

1. $\mathcal{D}$ is the smallest triangulated category containing all $\mathcal{D}_i$.

2. If $A \in \mathcal{D}_i, B \in \mathcal{D}_j$ with $i > j$, then $\mathrm{Hom}(A, B) = 0$.

3. Any $A \in \mathcal{D}$ has a "filtration"

$$0 = A_m \to A_{m-1} \to ... \to A_1 \to A_0 = A$$

with $\pi_i(A) = \mathrm{Cone}(A_i \to A_{i-1}) \in \mathcal{D}_i$

Remark. Can check that the projections π_i are unique and functorial (using semi-orthogonality to diagram chase and applying Yoneda). And given $\langle \mathcal{A}, \mathcal{B} \rangle$ we can check $\mathcal{A}$ is left-admissible and $\mathcal{B}$ is right-admissible.

Definition 1.5. Given an admissible subcategory $\mathcal{A} \hookrightarrow \mathcal{D}$, define the *left-orthogonal* and *right-orthogonal* subcategories to be

$${}^{\perp}\mathcal{A} := \{B \in \mathcal{D} : \mathrm{Hom}^*(B, A) = 0, \forall A \in \mathcal{A}\} \text{ and } \mathcal{A}^{\perp} := \{B \in \mathcal{D} : \mathrm{Hom}^*(A, B) = 0, \forall A \in \mathcal{A}\}$$

Thus, we have semi-orthogonal decompositions

$$\mathcal{D} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$$

with projections given by taking cones.

There are some technical results about sufficient conditions for admissibility which have nice consequences in the case of $D^b_{\mathrm{Coh}}(X)$.

Proposition 1.6. *For X smooth and projective, in $D^b_{Coh}(X)$, left and right admissibility are equivalent, and the left or right orthogonal of an admissible category is admissible.*

Proof idea. A triangulated category $\mathcal{D}$ is *left (resp. right) saturated* if every covariant (resp. contravariant) functor $\mathcal{D} \to D^b_{\mathrm{Coh}}(k)$ is representable.

- A left (resp. right) admissible subcategory of a saturated category is saturated: get representing object by applying $i^*/i^{-1}/i^!$.

- Saturated implies admissible: construct the adjoint point-wise via representing objects.

- $D^b_{\mathrm{Coh}}(X)$ is saturated: follows from strong generation.

See Section 2.3 for details. □

2 Fourier-Mukai Kernels

Let $X, Y/k$ be smooth projective varieties, and let $p_1, p_2 : X \times Y \to X, Y$ be the projections.

Definition 2.1. Given $\mathcal{E} \in D^b_{\mathrm{Coh}}(X)$, the *Fourier-Mukai functor with kernel $\mathcal{E}$* is the functor $D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$

$$\Phi_{\mathcal{E}}(-) := p_{2*}(\mathcal{E} \otimes p_1^*(-))$$

And let $\Phi'_{\mathcal{E}}$ respectively swap X and Y.

Can think of the kernel as the graph of the functor. This allows us to convert information about functors into information about objects of $D^b_{\text{Coh}}(X)$.

Example.

$$\begin{aligned}\Phi_{\Delta_*\mathcal{O}_X}(-) &= p_{2*}(\Delta_*\mathcal{O}_X \otimes p_1^*(-)) \\ &= p_{2*}(\Delta_*(\mathcal{O}_X \otimes \Delta^* p_1^*(-))) \\ &= (p_2 \circ \Delta)_* \circ (\pi_1 \circ \Delta)^*(-) \\ &= \text{id}\end{aligned}$$

by the projection formula (projection maps are flat so we're lazily omitting derived pullback notation). Likewise, $\Phi_{\Delta_*\mathcal{O}_X} = \text{id}$, $\Phi_{\Gamma_f} = Rf_*$, $\Phi_{\Gamma'_f} = Lf^*$, $\Phi_{\mathcal{O}_X} = H^*(X, -)$.

Remark. Orlov showed that every fully faithful exact functor with an adjoint between D^b_{Coh} of smooth projective varieties is Fourier-Mukai.

Theorem 2.2 (Kuznetsov)**.** *Projection functors of semi-orthogonal decompositions are Fourier-Mukai.*

Proof idea. Can lift $D^b_{\text{Coh}}(X) = \langle \mathcal{A}_0, ..., \mathcal{A}_m \rangle$ to $D^b_{\text{Coh}}(X \times X) = \langle \mathcal{A}_{0X}, ..., \mathcal{A}_{mX} \rangle$, where $\mathcal{A}_{iX}$ is generated by $A \boxtimes F$ for $A \in \mathcal{A}_i$ and $F \in D^b_{\text{Coh}}(X)$. Then taking the cones of the filtration of $\Delta_*\mathcal{O}_X$ give the Fourier-Mukai kernels. Requires some technical considerations with base change. □

The filtration of $\Delta_*\mathcal{O}_X$ can be thought of as a *universal filtration.*

Theorem 2.3. $D^b_{\text{Coh}}(\mathbf{P}^n) = \langle \mathcal{O}, \mathcal{O}(1) ... \mathcal{O}(n) \rangle$.

Proof sketch. Since $H^*(\mathbf{P}^n, \mathcal{O}) = \mathbf{C}[0]$ and $H^*(\mathbf{P}^n, \mathcal{O}(-i)) = 0$ for $1 \leq i \leq n$, the line bundles are exceptional and we have a semi-orthogonal decomposition, so it remains to check that it generates $D^b_{\text{Coh}}(\mathbf{P}^n)$.

Can show that $\Omega^1(1) \boxtimes \mathcal{O}(-1)$ is the ideal sheaf of $\Delta \subset \mathbf{P}^n \times \mathbf{P}^n$, which gives Koszul resolution

$$\Omega^n(n) \boxtimes \mathcal{O}(-n) \to ... \to \Omega^1(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbf{P}^n \times \mathbf{P}^n} \to \mathcal{O}_\Delta \to 0$$

So for any $F \in D^b_{\text{Coh}}(\mathbf{P}^n)$, we get a resolution for $\Phi_{\mathcal{O}_\Delta}(F) = F$ whose terms are

$$\begin{aligned}p_{2*}(p_1^*\Omega^i(i) \otimes p_2^*\mathcal{O}(-i) \otimes p_1^* F) &= p_{2*}p_1^*(\Omega^i(i) \otimes F) \otimes \mathcal{O}(-i) \\ &= H^*(\mathbf{P}^n, \Omega^i(i) \otimes F) \otimes_k \mathcal{O}(-i)\end{aligned}$$

by the projection formula and flat base change. Thus, $F \in \langle \mathcal{O}(-n), ..., \mathcal{O}(-1), \mathcal{O} \rangle$, and we can twist by n to get the desired decomposition.

□

Remark. If we use Φ' instead, we get another full exceptional collection $D^b_{\text{Coh}}(\mathbf{P}^n) = \langle \mathcal{O}, \Omega^1(1), ..., \Omega^n(n) \rangle$.

Definition 2.4. A semi-orthogonal decomposition with each component generated by exceptional objects is a *(full) exceptional collection.*

3 Splitting Functors

Another source of semi-orthogonal decompositions is mapping one triangulated category into another and taking kernels/images.

Let's look at a concrete example first, then give the general results.

Proposition 3.1. *If $f : X \to Y$ is a smooth proper map between varieties, then there is a semi-orthogonal decomposition $D^b_{\mathrm{Coh}}(X) = \langle \ker Rf_*, Lf^*(D^b_{\mathrm{Coh}}(Y))\rangle$.*

Proof. By adjunction, $\mathrm{Hom}(Lf^*B, \ker Rf_*) = \mathrm{Hom}(B, Rf_* \ker Rf_*) = 0$, so we need to show admissibility and fullness. We do these at the same time by giving a decomposition triangle for arbitrary $F \in D^b_{\mathrm{Coh}}(X)$:

$$Lf^*Rf_*F \xrightarrow{\varphi} F \to \mathrm{Cone}(\varphi)$$

Clearly the object on the left is in the image of Lf^*, so it suffices to show $Rf_*(\varphi)$ is an isomorphism. We have $Rf_*\mathcal{O}_X \cong \mathcal{O}_Y$, so by the projection formula:

$$\begin{aligned} Rf_*Lf^*Rf_*F &= Rf_*(Lf^*(\mathcal{O}_Y \otimes Rf_*F)) \\ &= Rf_*(\mathcal{O}_Y) \otimes Rf_*F \\ &= Rf_*F \end{aligned}$$

So the natural map is the identity. □

Definition 3.2. An exact functor $\Phi : \mathcal{B} \to \mathcal{A}$ is *right (resp. left) splitting* if $\ker \Phi$ is right (resp. left) admissible in $\mathcal{B}$, Im Φ is right (resp. left) admissible in $\mathcal{A}$, and $\Phi|_{(\ker \Phi)^\perp}$ (resp. $\Phi|_{^\perp(\ker \Phi)}$) is fully faithful.

Theorem 3.3. *An exact functor Φ is a left splitting iff the natural map $\Phi \to \Phi\Phi^*\Phi \cong \Phi$ is an isomorphism, in which case there are semi-orthogonal decompositions*

$$\langle \ker \Phi, \mathrm{Im}\ \Phi^*\rangle, \quad \langle \mathrm{Im}\ , \ker \Phi^*\rangle$$

See Theorem 3.3 for the full statement and proof.

Remark. Can think of splitting functors as a generalization of fully faithful functors with an adjoint: it is fully faithful away from its kernel. Kuznetsov conjectures that any splitting functor between D^b_{Coh} of smooth projective varieties is Fourier Mukai. Conversely, every Fourier-Mukai functor is a splitting functor.

4 Serre Functors

The notion of left vs. right has come quite a bit, and those two are symmetric in our cases thanks to Serre duality.

Definition 4.1. A *Serre functor* S on a triangulated category $\mathcal{D}$ is an autoequivalence such that for all $A, B \in \mathcal{D}$, there is a natural isomorphism

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, S(A))^\vee$$

By general category theory, this functor is unique up to isomorphism.

Proposition 4.2. *For X a smooth and projective variety, any admissible subcategory $i : \mathcal{D} \hookrightarrow D^b_{Coh}(X)$ has a Serre functor $S_{\mathcal{D}}$.*

Proof sketch. First note $S_X(-) := (-) \otimes \omega_X[\dim X]$ by Serre duality. Then can check

$$S_{\mathcal{D}} \cong i^! \circ S_X \circ i$$

satisfies the universal property. □

The Serre functor of an admissible subcategory is not always easy to compute explicitly — one of the later talks may examine this. This also means that $i^!$ (or dually, in some cases, i^*) may be hard to compute.

Remark. $i^! = S_{\mathcal{D}} \circ i^* \circ S_X^{-1}$ and $i^* = S_{\mathcal{D}}^{-1} \circ i^! \circ S_X$.

The Serre functor is often one of the only tools available. For instance, consider the following big theorem.

Theorem 4.3 (Bondal-Orlov)**.** *Let X be a smooth projective variety with $\pm K_X$ ample. If there exist X' smooth projective variety with $D^b_{\mathrm{Coh}}(X) \cong D^b_{\mathrm{Coh}}(X')$, then $X \cong X'$.*

Proof idea. Call P a *point object* of codimension n if $S(P) \simeq P[s]$, $\mathrm{Hom}^{<0}(P,P) = 0$, and $\mathrm{Hom}^0(P,P) = k$.

Point objects in $D^b_{\mathrm{Coh}}(X)$ must map to those of $D^b_{\mathrm{Coh}}(X')$, which we use to construct a map $X \to X'$. But for this you need to show $P \cong \mathcal{O}_{X,x}[k]$. We have $\mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P)$, which if $\omega_X^{\pm 1}$ is ample means $\mathcal{H}^i(P)$ has finite support, and with a bit more argument must be as desired. □

Remark. This argument obviously fails for abelian varieties, where $\omega_X \cong \mathcal{O}_X$. Indeed, Mukai showed for an abelian variety A that $D^b_{\mathrm{Coh}}(A) \cong D^b_{\mathrm{Coh}}(A^\vee)$ using the Fourier-Mukai transform.