SOD FOR BLOW-UPS

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1. Setup

We start by recalling the following proposition:

Proposition 1.1. *Suppose $f : S \to T$ is a projective morphism of smooth projective varieties such that $f_* : D^b(S) \to D^b(T)$ sends $\mathcal{O}_S$ to $\mathcal{O}_T$. Then $f^* : D^b(T) \to D^b(S)$ is fully faithful and gives an equivalence of $D^b(T)$ with an admissible subcategory of $D^b(S)$.*

Proof. Adjunction gives a natural map

$$F \to f_* f^*(F) \cong f_*(f^* F \otimes \mathcal{O}_S) \cong F \otimes f_*(\mathcal{O}_S) \cong F.$$

So $f_* f^*$ is naturally isomorphic to the identity functor, so f^* is fully faithful. ∎

Let $\mathcal{N}$ be a vector bundle on a smooth projective variety Y. There is a natural projective morphism $\pi : \mathbf{P}(\mathcal{N}) \to Y$ that satisfies the condition in the proposition, so it induces a fully faith functor $\pi^* : D^b(Y) \to D^b(\mathbf{P}(\mathcal{N}))$ whose image is an admissible subcategory.

In fact, if $\mathcal{N}$ has rank r, we have a semi-orthogonal decomposition of $D^b(\mathbf{P}(\mathcal{N}))$:

$$\pi^* D^b(Y) \otimes \mathcal{O}(a), \cdots, \pi^* D^b(Y) \otimes \mathcal{O}(a + r - 1).$$

This is a relative version of the semi-orthogonal decomposition $\mathbf{P}^r = \langle \mathcal{O}(a), \cdots, \mathcal{O}(a + r - 1) \rangle$.

The second situation where the condition in the proposition is satisfied is blow-ups. Let $q : \widetilde{X} \to X$ be the blow-up of a smooth projective variety X along some smooth closed $Y \subset X$. Then q is projective and $q_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$. So $q^* : D^b(X) \to D^b(\widetilde{X})$ gives an admissible subcategory of $D^b(\widetilde{X})$.

For blow-ups, let $i : E \hookrightarrow \widetilde{X}$ be the exceptional divisor, and let $\mathcal{N}$ be the normal sheaf of Y, which is the dual of $\mathcal{I}/\mathcal{I}^2$. Then $E = \mathbf{P}(\mathcal{N})$ is a projective bundle $\pi : \mathbf{P}(\mathcal{N}) \to Y$ over Y. This is summarized by the following diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\quad q \quad} & X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ E = \mathbf{P}(\mathcal{N}) & \xrightarrow{\quad \pi \quad} & Y \end{array}$$

2. Closed Immersion

A first step is to understand the closed immersion $j : Y \hookrightarrow X$. Suppose first Y is the zero subvariety of a section s of a locally free sheaf $\mathcal{E}$ of rank c (so Y is of codimension c in X). Then we have the Koszul resolution

$$0 \to \bigwedge^c \mathcal{E}^\vee \to \cdots \to \mathcal{E}^\vee \to \mathcal{O}_X \to j_* \mathcal{O}_Y \to 0.$$

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(Affine locally, given a element s of some free R-module M, we obtain a map $M^\vee \to R$ by evaluating against s. The differential is

$$d(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{n} (-1)^i e_i(s) e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_n$$

where $e_1, \cdots, e_n$ are the dual basis.) In other words, we have a quasi-isomorphism $\bigwedge^\bullet \mathcal{E}^\vee \to j_*\mathcal{O}_Y$. Also, in this case $\mathcal{N} \cong j^*\mathcal{E}$.

Proposition 2.1. *In the situation above, we have a canonical isomorphism*

$$j^* j_* \mathcal{O}_Y \cong \bigoplus_k \bigwedge^k \mathcal{N}^\vee[k].$$

Moreover, for any object $\mathcal{F}^\bullet \in D^b(Y)$, we have

$$j_* j^* j_* \mathcal{F}^\bullet \cong j_*\mathcal{O}_Y \otimes j_*\mathcal{F}^\bullet \cong j_* \left(\bigoplus \bigwedge^k \mathcal{N}^\vee[k] \otimes \mathcal{F}^\bullet \right)$$

and

$$\mathscr{H}om_X(j_*\mathcal{O}_Y, j_*\mathcal{F}^\bullet) \cong j_* \left(\bigoplus \bigwedge^k \mathcal{N}[-k] \otimes \mathcal{F}^\bullet \right)$$

Proof. Since we have the quasi-isomorphism $\bigwedge^\bullet \mathcal{E}^\vee \to j_*\mathcal{O}_Y$, we can compute $j^*j_*\mathcal{O}_Y$ by $\bigwedge^\bullet j^*\mathcal{E}^\vee$. The definition of the Koszul complex tells us the differentials of $\bigwedge^\bullet j^*\mathcal{E}^\vee$ are all zero, because the are given by evaluating against s but Y is exactly the subscheme where s is 0. Therefore

$$j^* j_* \mathcal{O}_Y \cong \bigwedge^\bullet j^*\mathcal{E}^\vee \cong \bigoplus_k \bigwedge^k j^*\mathcal{E}^\vee[k] \cong \bigoplus_k \bigwedge^k \mathcal{N}^\vee[k].$$

To prove the other two isomorphisms, we split the resolution $\bigwedge^\bullet \mathcal{E}^\vee \to j_*\mathcal{O}_Y$ into short exact sequences, and tensor each one with $j_*\mathcal{F}^\bullet$. Since $j_*\mathcal{F}^\bullet$ is supported on Y and the differentials vanish on Y, we obtain splitting triangles. Then taking the direct sum of all of those gives the desired results. ∎

Corollary 2.2. *In the same situation, for any $\mathcal{F}^\bullet \in D^b(Y)$ we have*

$$\mathcal{H}^l(j^* j_* \mathcal{F}^\bullet) \cong \bigoplus_{s-r=l} \bigwedge^r \mathcal{N}^\vee \otimes \mathcal{H}^s(\mathcal{F}^\bullet)$$

and

$$\mathcal{E}xt^l_X(j_*\mathcal{O}_Y, j_*\mathcal{F}^\bullet) \cong j_* \left(\bigoplus_{r+s=l} \bigwedge^r \mathcal{N} \otimes \mathcal{H}^s(\mathcal{F}^\bullet) \right).$$

Proof. Since j is a closed immersion, j_* is exact, so it commutes with taking cohomology. Also, $\mathcal{N}$ is locally free, so tensoring by $\mathcal{N}$ and its exterior powers also commutes with taking cohomology. Using the previous proposition and these two fact to push symbols around and conclude. ∎

Corollary 2.3. *Now assume further that Y has codimension 1 in X. Then*

$$j^*j_*\mathcal{O}_Y \cong \mathcal{O}_Y \oplus j^*\mathcal{O}_X(-Y)[1].$$

Moreover, for any object $\mathcal{F}^\bullet \in D^b(Y)$, we have a distinguished triangle

$$\mathcal{F}^\bullet \otimes \mathcal{O}_Y(-Y)[1] \to j^*j_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$$

and an isomorphism

$$j_*j^*j_*\mathcal{F}^\bullet \cong j_*\mathcal{F}^\bullet \oplus j_*(\mathcal{F}^\bullet \otimes \mathcal{O}_Y(-Y))[1].$$

Proof. From the Proposition we get $j^*j_*\mathcal{O}_Y \cong \mathcal{O}_Y \oplus (\mathcal{I}/\mathcal{I}^2)[1]$, and $\mathcal{I} = \mathcal{O}_X(-Y)$. The third isomorphism is true for the same reason. Proof of the distinguished triangle is omitted. ■

We can upgrade to a more global situation.

Proposition 2.4. *Let $j : Y \to X$ be an arbitrary closed immersion of smooth varieties. Then there are isomorphisms*

$$\mathcal{H}^i(j^*j_*\mathcal{O}_Y) \cong \bigwedge^{-i} \mathcal{N}^\vee$$

and

$$\mathcal{E}xt^i_X(j_*\mathcal{O}_Y, j_*\mathcal{O}_Y) \cong \bigwedge^{i} \mathcal{N}$$

Proof. Choose a global locally free resolution $\mathcal{G}^\bullet \to \mathcal{O}_Y$, and glue the local results proved above. The local isomorphisms are canonical so they glue. ■

3. Blow-ups

We return to the situation

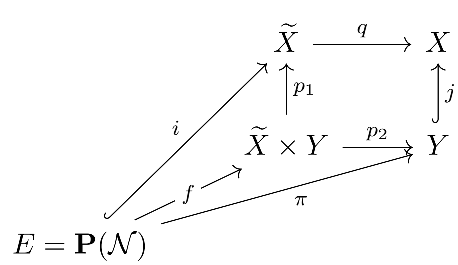
$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{q} & X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ E = \mathbf{P}(\mathcal{N}) & \xrightarrow{\pi} & Y \end{array}.$$

Proposition 3.1. *Suppose $Y \subset X$ is of codimension $c \geq 2$. The functor*

$$\Phi_k : D^b(Y) \to D^b(\widetilde{X})$$
$$\mathcal{F}^\bullet \mapsto i_*(\mathcal{O}_E(kE) \otimes \pi^*(\mathcal{F}^\bullet))$$

is fully faithful for any k, and admits a right adjoint.

The functor Φ_k is a Fourier-Mukai transform with kernel $\mathcal{O}_E(kE)$ considered as an object in $D^b(Y \times \widetilde{X})$ (remember that $\mathcal{O}_E(kE)$ means $i^*\mathcal{O}_{\widetilde{X}}(kE)$). More precisely, we have the diagram



The Fourier-Mukai transform with kernel $f_*\mathcal{O}_E(kE)$ is

$$\begin{aligned}(p_1)_*(p_2^*\mathcal{F}^\bullet \otimes f_*\mathcal{O}_E(kE)) &\cong (p_1)_*f_*(f^*p_2^*\mathcal{F}^\bullet \otimes \mathcal{O}_E(kE))\\ &\cong i_*(\pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_E(kE)).\end{aligned}$$

Bondal and Orlov has the following result which gives a criterion for whether a Fourier-Mukai transform is fully faithful:

Theorem 3.2. *A Fourier-Mukai transform $\Phi_\mathcal{P} : D^b(X) \to D^b(Y)$ with kernel $\mathcal{P}$ is fully faithful if and only if for any two closed points $x, y \in X$,*

$$\mathrm{Hom}(\Phi_\mathcal{P}(k(x)), \Phi_\mathcal{P}(k(y))[i]) = \begin{cases} k, & x = y \text{ and } i = 0\\ 0, & x \neq y \text{ or } i < 0 \text{ or } i > \dim(X).\end{cases}$$

Proof of Proposition 3.1. To use the criterion we first need to compute $\Phi_k(k(y))$ for $y \in Y$. We have $\pi^*k(x) = \mathcal{O}_{\pi^{-1}(y)}$, and $\mathcal{O}_E(E)$ restricted to any fiber is $\mathcal{O}(-1)$ of that fiber, so we end up with $i_*\mathcal{O}_{\pi^{-1}(y)}(-k)$. Hence if $x \neq y$, then $\Phi_k(k(x))$ and $\Phi_k(k(y))$ have disjoint supports, so there is no non-trivial morphism between them.

We are left with the case $x = y$. Namely, we want to compute

$$\mathrm{Hom}_{\widetilde{X}}(\Phi_k(k(x)), \Phi_k(k(x))[i]) = \mathrm{Ext}^i_{\widetilde{X}}(\Phi_k(k(x)), \Phi_k(k(x))) = \mathrm{Ext}^i_{\widetilde{X}}(i_*\mathcal{O}_{\pi^{-1}(x)}(-k), i_*\mathcal{O}_{\pi^{-1}(x)}(-k))$$

This is zero for $i < 0$. We have the spectral sequence

$$E_2^{p,q} = H^p(\widetilde{X}, \mathcal{E}xt^q(i_*\mathcal{O}_{\pi^{-1}(x)}, i_*\mathcal{O}_{\pi^{-1}(x)})) \Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(i_*\mathcal{O}_{\pi^{-1}(x)}, i_*\mathcal{O}_{\pi^{-1}(x)})$$

We have just seen in Proposition 2.4 how to compute this. Namely,

$$\mathcal{E}xt^q(i_*\mathcal{O}_{\pi^{-1}(x)}, i_*\mathcal{O}_{\pi^{-1}(x)}) = \bigwedge^q \mathcal{N}_{\pi^{-1}(x)/\widetilde{X}}.$$

So

$$E_2^{p,q} = H^p(\pi^{-1}(x), \bigwedge^q \mathcal{N}_{\pi^{-1}(x)/\widetilde{X}}) \Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(i_*\mathcal{O}_{\pi^{-1}(x)}, i_*\mathcal{O}_{\pi^{-1}(x)})$$

We use the short exact sequence

$$0 \to \mathcal{N}_{\pi^{-1}(x)/E} \to \mathcal{N}_{\pi^{-1}(x)/\widetilde{X}} \to \mathcal{N}_{E/\widetilde{X}}|_{\pi^{-1}(x)} \to 0.$$

General theory of blow-ups tells us $\mathcal{N}_{\pi^{-1}(x)/E} \cong \mathcal{O}^{\oplus d}_{\pi^{-1}(x)}$ and $\mathcal{N}_{E/\widetilde{X}}|_{\pi^{-1}(x)} \cong \mathcal{O}_{\pi^{-1}(x)}(-1)$. Also,

$$\mathrm{Ext}^1(\mathcal{O}_{\pi^{-1}(x)}(-1), \mathcal{O}^{\oplus d}_{\pi^{-1}(x)}) = H^1(\pi^{-1}(x), \mathcal{O}_{\pi^{-1}(x)}(1)) = 0$$

because $\pi^{-1}(x) \cong \mathbf{P}^{c-1}$. So the short exact sequence splits. This allows use to conclude $E_2^{p,q} = 0$ for all $p > 0$ or $p = 0$ and $q > d$. Hence $\mathrm{Ext}^q_{\widetilde{X}}(i_*\mathcal{O}_{\pi^{-1}(x)}(-k), i_*\mathcal{O}_{\pi^{-1}(x)}(-k)) = E^{0,q} = 0$ for $q > d$ and $\mathrm{Ext}^0 = H^0(\mathcal{O}_{\pi^{-1}(x)}) = k$. ■

Thus, for each $k = -c+1, \cdots, -1$, the image

$$\mathcal{D}_k = \mathrm{Im}\left(\Phi_{-k} : D^b(Y) \to D^b(\widetilde{X})\right)$$

is an admissible subcategory of $D^b(\widetilde{X})$ that is equivalent to $D^b(Y)$. We will denote by $\mathcal{D}_0$ the full subcategory $q^*D^b(X)$.

Theorem 3.3 (Orlov)**.** *The admissible subcategories*

$$\mathcal{D}_{-c+1}, \cdots, \mathcal{D}_{-1}, \mathcal{D}_0$$

form a semi-orthogonal decomposition of $D^b(\widetilde{X})$.

Proof. There are always two steps: check they are orthogonal, and check they generate $D^b(X)$.

First let $-c+1 \leq l < k < 0$. We will show $\mathcal{D}_l \subset \mathcal{D}_k^{\perp}$. Let $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(Y)$. Also write $\mathcal{O}_\pi(m)$ for $\mathcal{O}_E(-mE)$. We have

$$\mathrm{Hom}(i_*(\pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_\pi(k)), i_*(\pi^*\mathcal{E}^\bullet \otimes \mathcal{O}_\pi(l))) \cong \mathrm{Hom}(i^*i_*\pi^*\mathcal{F}^\bullet, \pi^*\mathcal{E}^\bullet \otimes \mathcal{O}_\pi(l-k))$$

We have a distinguished triangle by Corollary 2.3, which applies since E has codimension 1 in $\widetilde{X}$

$$\pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_\pi(1)[1] \to i^*i_*\pi^*\mathcal{F}^\bullet \to \pi^*\mathcal{F}^\bullet \to \pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_\pi(1)[2]$$

So it suffices to show

$$\mathrm{Hom}(\pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_\pi(1), \pi^*\mathcal{E}^\bullet \otimes \mathcal{O}_\pi(l-k)) = 0$$
$$\mathrm{Hom}(\pi^*\mathcal{F}^\bullet, \pi^*\mathcal{E}^\bullet \otimes \mathcal{O}_\pi(l-k)) = 0$$

For example the second one is

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \pi_*\mathcal{O}_\pi(l-k)) = 0$$

because $\pi_*\mathcal{O}_\pi(l-k)) = 0$ for $l-k < 0$ (the fibers are global section of $\mathcal{O}(l-k)$ on $\mathbf{P}^{c-1}$).

Now we show these categories generate $D^b(\widetilde{X})$. The plan is to take an object $\mathcal{E}^\bullet$ that is contained in $\mathcal{D}_l^{\perp}$ for all $-c+1 \leq l < 0$, and show it is can't also be in $\mathcal{D}_0^{\perp}$. This assumption means

$$\mathrm{Hom}(i_*(\pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_\pi(l)), \mathcal{E}^\bullet) = 0$$

for all $-c+1 \leq l < 0$. Applying the adjoint $i^!\mathcal{E} \cong i^*\mathcal{E} \otimes \mathcal{O}_E(E)[-1]$ and twising by 1 gives

$$\mathrm{Hom}(\pi^*\mathcal{F}^\bullet \otimes \mathcal{O}_\pi(l), i^*\mathcal{E}^\bullet) = 0$$

for all $-c+2 \leq l < 1$ and all $\mathcal{F}^\bullet \in D^b(Y)$. This means $i^*\mathcal{E}^\bullet$ is contained in orthogonal complement of $\pi^*D^b(Y) \otimes \mathcal{O}_\pi(l), -c+2 \leq l \leq 0$ inside $D^b(E)$. But in the very beginning we saw a semi-orthogonal decomposition of $D^b(E)$ with these terms, so we conclude that $i^*\mathcal{E}^\bullet \in \pi^*D^b(Y) \otimes \mathcal{O}_\pi(-c+1)$. In other words, there exists some $\mathcal{G}^\bullet \in D^b(Y)$ such that $i^*\mathcal{E} \otimes \mathcal{O}_\pi(c-1) \cong \pi^*\mathcal{G}$.

Note that is $\mathcal{G} \cong 0$, then $i^*\mathcal{E} \otimes \mathcal{O}_\pi(c-1)$ has support outside the exceptional divisor E so it is in $\mathcal{D}_0$, so we win. If $\mathcal{G}$ is non-zero we can find a point $x \in X$ such that there is a non-zero map from $i^*\mathcal{E} \otimes \mathcal{O}_\pi(c-1)$ to $q^*k(x)$ after some shift, but this requires a bit more theory in the closed immersion situation than we covered.

■

Let's look an example. Let $X = \mathbf{P}^2$ and we blow it up at two points to obtain $\pi : \widetilde{X} \to X$. Then the blow-up $\widetilde{X}$ has Picard group isomorphic to $\mathbf{Z}^3$, generated by a line in the original $\mathbf{P}^2$, and the two exceptional curves E_1, E_2 which are both isomorphic to $\mathbf{P}^1$. The functor Φ_1 in this case has image generated by

$$i_*\mathcal{O}_{E_1}(-1), i_*\mathcal{O}_{E_2}(-1).$$

and we have a semi-orthogonal decomposition

$$\langle i_*\mathcal{O}_{E_1}(-1), i_*\mathcal{O}_{E_2}(-1), \pi^*\mathcal{D}^b(\mathbf{P}^2)\rangle$$

We know $\mathcal{D}^b(\mathbf{P}^2) = \langle \mathcal{O}_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}(1), \mathcal{O}_{\mathbf{P}^2}(2)\rangle$, so we get a exceptional collection

$$\langle i_*\mathcal{O}_{E_1}(-1), i_*\mathcal{O}_{E_2}(-1), \mathcal{O}_{\widetilde{X}}, \pi^*\mathcal{O}_{\mathbf{P}^2}(1), \pi^*\mathcal{O}_{\mathbf{P}^2}(2)\rangle$$

It is well-known that the same surface $\widetilde{X}$ can also by realized as a blow up of $\mathbf{P}^1 \times \mathbf{P}^1$: The Castelnovo criterion says there is a blow-up map $\beta : \widetilde{X} \to \mathbf{P}^1 \times \mathbf{P}^1$ that contracts the (-1)-curve on $\widetilde{X}$ which is the strict transform of the line that passes through the two points on $\mathbf{P}^2$. Let the exceptional curve form β be F. Then we obtain a semi-orthogonal decomposition

$$\langle \mathcal{O}_F(-1), \beta^* D^b(\mathbf{P}^1 \times \mathbf{P}^1) \rangle.$$

We know that $\mathbf{P}^1 \times \mathbf{P}^1$ has a strong full exceptional collection

$$\langle \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1) \rangle$$

So we get another full exceptional collection

$$\langle \mathcal{O}_F(-1), \mathcal{O}_{\widetilde{X}}, \beta^*\mathcal{O}(1,0), \beta^*\mathcal{O}(0,1), \beta^*\mathcal{O}(1,1) \rangle$$

of $D^b(\widetilde{X})$.