Semi-orthogonal Decompositions Seminar Notes

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Kuan-Wen Chen: More examples of SOD's (Grassmannians, quadrics).

Definition 0.1. Given a k-linear triangulated category D, and $E_1, ..., E_n \in \mathrm{Ob}(D)$,

- $E_1, ..., E_n$ is an exceptional sequence if

$$\mathrm{Hom}(E_i, E_j[\ell]) = \begin{cases} k & i = j, \ell = 0 \\ 0 & i > j \text{ or } i = j, \ell \neq 0 \end{cases}$$

- An exceptional sequence is full if $D = \langle E_1, ..., E_n \rangle$.

A full exceptional collection defines a semi-orthogonal decomposition.
Let's review the full exceptional collection on $\mathbf{P}^n$, due to Beilinson.

$$D^b(\mathbf{P}^n) = \langle \mathcal{O}(a), ..., \mathcal{O}(a+n) \rangle$$

Strategy to prove this:

1. Exceptionalness is just a cohomology computation, following from $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) = H^\ell(\mathbf{P}^n, \mathcal{O}(j-i))$
2. For fullness, consider the diagonal $\Delta \subseteq \mathbf{P}^n \times \mathbf{P}^n$, for which $Rp_{2*}(Lp_1^*\mathcal{F} \otimes \mathcal{O}_\Delta) = \mathcal{F}$ for all $\mathcal{F}$, and construct a resolution of $\mathcal{O}_\Delta$

$$0 \to \bigwedge^n (\Omega(1) \boxtimes \mathcal{O}(-1)) \to \bigwedge^{n-1} (\Omega(1) \boxtimes \mathcal{O}(-1)) \to ... \to \Omega(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbf{P}^n \times \mathbf{P}^n} \to \mathcal{O}_\Delta \to 0$$

which follows from the Koszul resolution from

$$0 \to \Omega(1) \to \mathcal{O}^{\oplus n+1} \to \mathcal{O}(1) \to 0.$$

For each term we have a quasi-isomorphism

$$Rp_{2*}(p_1^*\mathcal{F} \otimes (\Omega^k(k) \boxtimes \mathcal{O}(-k))) \simeq \bigoplus_{i \in \mathbf{Z}} \mathbf{H}^i(\mathcal{F} \otimes \Omega^k(k)) \otimes \mathcal{O}(-k)[-i] =: T_k,$$

so for the whole resolution we have a quasi-isomorphism

$$\mathcal{F} = Rp_{2*}(p_1^*\mathcal{F} \otimes \mathcal{O}_\Delta) \simeq \mathrm{Cone}(...\mathrm{Cone}(\mathrm{Cone}(T_n \to T_{n-1}) \to T_{n-2})),$$

which is in $\langle \mathcal{O}(n), ..., \mathcal{O}(-1) \rangle$.

Now, we will do something similar for the Grassmannian. Fix $G := \mathrm{Gr}(k, V)$ for $\dim V = n$.

Question 1: What is the resolution of $\mathcal{O}_\Delta$?

Let $\Delta \subseteq G \times G$ be the diagonal, and let $S \subseteq V$ be the tautological sub-bundle (k-dimensional). We have

$$0 \to S \to V \to V/S \to 0$$

or, dualizing

$$0 \to (V/S)^* \to V^* \to S^*$$

define $S^\perp := (V/S)^*$.

Fact 1: $H^0(G, S^*) = V^*$, $H^0(G, V/S) = V$.

Then

$$H^0(G \times G, S^* \boxtimes V/S) = V^* \otimes V = \mathrm{End}(V)$$

Let $\xi \in H^0(G \times G, S^* \boxtimes V/S)$ correspond to $\mathrm{Id} \in \mathrm{End}(V)$, and note that it corresponds to $\mathcal{O}_{G\times G} \to S^* \boxtimes (V/S)$, or dually to

$$S \boxtimes S^\perp \xrightarrow{\lrcorner\xi} \mathcal{O}_{G\times G} \to \mathcal{O}_\Delta \to 0.$$

Taking the Koszul complex yilds a resolution

$$\ldots \to \bigwedge^k (S \boxtimes S^\perp) \to \bigwedge^{k-1} (S \boxtimes S^\perp) \to \ldots \to S \boxtimes S^\perp \to \mathcal{O}_{G\times G} \to \mathcal{O}_\Delta \to 0.$$

Fact 2: For W_1, W_2 vector spaces, we have

$$\bigwedge^p (W_1 \otimes W_2) \cong \bigoplus_{\alpha \geq 0, \sum \alpha_i = p} \Sigma^\alpha W_1 \otimes \Sigma^{\alpha^*} W_2$$

as a $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ module. Here α is a Young diagram and Σ^α is a Schur functor associated to α, and α^* is the dual diagram, which reverses rows and columns.

Definition 0.2. For $\alpha = (\alpha_1, ..., \alpha_n)$ a decreasing sequence of integers and W a vector space of $\dim W = n$, the irreducible representation of $\mathrm{GL}(W)$ with highest weight α is called $\Sigma^\alpha W$. If $\alpha \geq 0$, then α corresponds to a Young diagram.

The representation is constructed by tensor operations.

Example. $\Sigma^{(1,1,\ldots,1)} W = \bigwedge^n W$, and $\Sigma^{(n)} W = \mathrm{Sym}^n W$.

Fact 2 implies

$$\bigwedge^p (S \boxtimes S^\perp) = \bigoplus_{\alpha \geq 0, \sum \alpha_i = p} \Sigma^\alpha S \boxtimes \Sigma^{\alpha^*} S^\perp$$

Proposition 0.3 (Kapranov)**.**

$$D^b(\mathrm{Gr}(k, V)) \cong \langle \Sigma^\alpha S : \Sigma^{\alpha^*} S^\perp \neq 0 \rangle$$

This corresponds to $\Sigma^\alpha S$ for α such that the number of rows of α is $\leq k$ an the number of columns of α is $\leq n - k$.

Question 2: Are $\Sigma^\alpha S$ an exceptional sequence?

We will show that they are, taking the objects in any order preserving inclusion of Young diagrams.

$$\begin{aligned}\mathrm{Hom}(\Sigma^\alpha S, \Sigma^\beta S[\ell]) &= \mathrm{Ext}^\ell(\Sigma^\alpha S, \Sigma^\beta S) \\ &= H^\ell(G, \Sigma^{(-\alpha_k, -\alpha_{k-1}, \ldots, -\alpha_1)} S \otimes \Sigma^\beta S) \\ &= H^\ell(G, \Sigma^{(-\alpha_k, -\alpha_{k-1}, \ldots, -\alpha_1)^*} S^\perp \otimes \Sigma^\beta S)\end{aligned}$$

where the last step requires $(-\alpha_k, \ldots, -\alpha_1) \geq 0$.

Proposition 0.4.

- *If $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq -(n-k)$, then*

$$H^i(G, \Sigma^\alpha S) = \begin{cases} 0 & i > 0 \text{ or } i = 0 \text{ for some } \alpha_i \geq 0 \\ \Sigma^{-\alpha_k \ldots, -\alpha_1} V^* & \text{all } \alpha_i < 0 \end{cases}$$

- *If $n-k \geq \alpha_1 \geq \ldots \geq \alpha_k \geq 0$ and $k \geq \beta_1 \geq \ldots \geq \beta_{n-k} \geq 0$, then $\Sigma^\alpha S \boxtimes \Sigma^\beta S^\perp$ can have non-zero cohomology only when $\beta = \alpha^*$. In this case, $H^{\Sigma \alpha_i} = \mathbf{C}$, $H^j = 0$ for $j \neq \sum \alpha_i$.*

Proof idea. Use Bott–Borel–Weil theorem in representation theory. Consider

$$F = \mathrm{Flag}(1, 2, \ldots, n-1, V) = \mathrm{GL}(V)/B$$

where B is the Borel subgroup consisting of upper triangular matrices. Line bundles on F correspond to characters χ of maximal torus $\chi = (a_1, \ldots, a_n)$. Denote the line bundle corresponding to χ as $\mathcal{O}(\chi)$.

Let π_i be the i-dimensional tautological bundle, so

$$\pi_1 \subset \ldots \subset \pi_{n-2} \subset \pi_{n-1} \subset V \otimes \mathcal{O}_F.$$

Then the line bundle π_{i+1}/π_i corresponds to the character $(0, 0, 0, \ldots, -1, 0, \ldots)$ where only the $i+1$ entry is nonzero.

Define the positive chamber of the character space $C^+ = \{(a_1, \ldots, a_n) : a_1 \geq a_2 \geq \ldots \geq a_n\}$.

Theorem 0.5 (BBW).

- *If $\chi \in C^+$, then*

$$H^i(F, \mathcal{O}(\chi)) = \begin{cases} \Sigma^\chi V & i = 0 \\ 0 & i > 0 \end{cases}$$

- *If $\alpha \notin C^+$, take a $\sigma \in S_n$ such that $\sigma(\alpha) \in C^+$. Let $\rho = (n, n-1, \ldots, 1)$. Then:*
  - *if $\sigma(\alpha + \rho) - \rho \notin C^+$, then $H^i(\mathcal{O}(\chi)) = 0$ for all i.*
  - *if $\sigma(\alpha + \rho) - \rho \in C^+$, then*

$$H^i(\mathcal{O}(\chi)) = \begin{cases} \Sigma^{\sigma(\chi+\rho)-\rho} V & i = \ell(\sigma) \\ 0 & \text{otherwise} \end{cases},$$

where $\ell(\sigma)$ is the length of σ.

In our case, let $F_1 = \text{Flag}(1, 2, 3, .., k, V)$ and $p_1 : F_1 \to G$ the natural map. Let

$$L_1 = \pi_1^{\otimes \alpha_k} \otimes (\pi_2/\pi_1)^{\alpha_{k-1}} \otimes ... \otimes (\pi_k/\pi_{k-1})^{\alpha_1}$$

be the line bundle corresponding to $(-\alpha_k, -\alpha_{k-1}, ..., -\alpha_1)$. Then using the BBW theorem, $Rp_{1*}L_1 = \Sigma^\alpha S$.

Similarly, let $F_2 = \text{Flag}(k, k+1, ..., n-1, V) \xrightarrow{p_2} G$ and

$$L_2 = (\pi_{k+1}/\pi_k)^{\otimes 1-\beta_1} \otimes ... \otimes (V/\pi_{n-1})^{-\beta_{n-k}}.$$

Then $Rp_{2*}L_2 = \Sigma^\beta S^\perp$.

Thus, letting $F_1 \times F_2 \xrightarrow{p=(p_1,p_2)} G$, we have

$$H^i(G, \Sigma^\alpha S \boxtimes \Sigma^\beta S^\perp) = H^i(F_1 \times F_2, L_1 \boxtimes L_2).$$

□

Finally, we turn to the case of quadrics. Let $Q \subset \mathbf{P}(E)$, for E an N-dimensional vector space.

Question 1: can we find a resolution of $\mathcal{O}_\Delta \subseteq Q \times Q$?

A *Koszul algebra* is a quadratic algebra with a linear minimal free resolution, i.e., exists

$$... \xrightarrow{\phi_2} B(-1) \xrightarrow{\phi_1} B \to k \to 0$$

requiring that entries of ϕ_i are either 0 or linear.

Fact: any complete intersection quadratic algebra is Koszul.

Koszul dual (Priddy dual):

- If B is a Koszul algebra $T(V)/R$, let $A = \text{Ext}_B(k, k) = T(V^*)/R^*$, where R and R^* are dual relations. Then $\text{Ext}_A(k, k) = B$.
- There is a generalized Koszul complex

$$\to A_1^* \otimes B \to A_0^* \otimes B \to \mathbf{C} \to 0$$

exact as a B-module, where the maps are $\Sigma_i(r_{\xi_i} \otimes \ell_{\chi_i})$, where ξ_i and χ_i are dual bases.

For our quadric Q, let $B = \bigoplus_i H^i(Q, \mathcal{O}(i))$ and A be the Koszul dual of B. Exists an exact sequence

$$L(A^*) = \{\to ... \to \widetilde{A_i^*}(-i) \to ... \to A_0^*\}$$

Set $\Psi_i = \ker(\widetilde{A_i^*} \to \widetilde{A_{i-1}(1)})$ for $i \geq 0$.

Proposition 0.6. *Exists an exact sequence*

$$... \to \Psi_2 \boxtimes \mathcal{O}(-2) \to \Psi_1 \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0$$

Proposition 0.7.

$$\ker(C^{-N+3} \to C^{-N+4}) = \Sigma(-1) \otimes \Sigma(-N+2)$$

when N is odd. Here Σ is a Spinor bundle on Q (involves Σ_+ and Σ_- when N is even).

Theorem 0.8 (Kapranov)**.** *$D^b(Q) = \langle \Sigma, \mathcal{O}, \mathcal{O}(1), ..., \mathcal{O}(N+3)\rangle$ when N is odd (slightly different when N is even).*

Example. $\mathrm{Gr}(2,4) \hookrightarrow \mathbf{P}^5$ is a quadric under the Plücker embedding. Then

$$D^b(\mathrm{Gr}(2,4)) = \langle \mathcal{O}, S, \mathrm{Sym}^2 S, \mathcal{O}(-1), S(-1), \mathcal{O}(-2)\rangle,$$

where the terms correspond to the Young diagrams $\emptyset, (1), (1,1), (2), (2,1), (2,2)$.

Here $\Sigma_+ = S^*$ and $\Sigma_- = \mathbf{C}^4/S$, so we have a decomposition

$$\langle S^*, \mathbf{C}^4/S, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)\rangle$$

which is mutation equivalent.