MUTATIONS AND SERRE FUNCTORS

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Abstract. Notes prepared for the *Semi-orthogonal decompositions seminar* (Spring 2025), organized by Amal Mattoo and myself at Columbia University.

Contents

1. Serre functors

Let us start by recalling the following notion, which was already introduced a couple of weeks ago.

Definition 1 ([Huy06, 1.28]). Let $\mathcal{D}$ be a k-linear triangulated category. A *Serre functor* is a k-linear equivalence $S\colon \mathcal{D} \to \mathcal{D}$ together with natural isomorphisms

$$\operatorname{Hom}(A,B) \to \operatorname{Hom}(B, S(A))^\vee.$$

This notion was originally introduced by Bondal and Kapranov in [BK89]. The name is motivated by the following prototypical example: if X is a smooth, proper variety of dimension n, then $S(A) = A \otimes \omega_X[n]$ is a Serre functor on the derived category $\mathrm{D}^b(X)$. We will see more examples later on. In the meantime, there are a couple of basic properties worth knowing about.

Lemma 2 ([Huy06, 1.30–1]).
(1) The Serre functor is unique up to unique isomorphism, provided that it exists.
(2) Assume that $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a triangulated functor, and that $\mathcal{D}_1, \mathcal{D}_2$ have Serre functors S_1, S_2. Then F has a left adjoint if and only if it has a right adjoint. Moreover

$$G \dashv F \Leftrightarrow F \dashv S_1 \circ G \circ S_2^{-1}.$$

(3) Assume that $i\colon \mathcal{A} \to \mathcal{D}$ is a (one-sided) triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ has a Serre functor, then $\mathcal{A}$ has one; thus, it is admissible.

Proof. (1) This is direct from Yoneda: the image of the Serre functor $S(A)$ is characterized by the contravariant functor $\operatorname{Hom}(-, S(A)) \cong \operatorname{Hom}(A, -)^\vee$.

(2) The trick here is that Serre duality allows us to swap the two factors in the Hom functor. In fact, assume that F has a left adjoint G. We have:

$$\begin{aligned}\mathrm{Hom}_{D_2}(F(A), B) &= \mathrm{Hom}_{D_2}(B, S_2 \circ F(A))^\vee = \mathrm{Hom}_{D_2}(S_2^{-1}(B), F(A))^\vee \\ &= \mathrm{Hom}_{D_1}(G \circ S_2^{-1}(B), A)^\vee = \mathrm{Hom}_{D_1}(G \circ S_2^{-1}(B), S_1 \circ S_1^{-1}(A))^\vee \\ &= \mathrm{Hom}_{D_1}(S_1^{-1}(A), G \circ S_2^{-1}(B)) = \mathrm{Hom}_{D_1}(A, S_1 \circ G \circ S_2^{-1}(B)).\end{aligned}$$

(3) Say that $i\colon \mathcal{A} \to \mathcal{D}$ has a right adjoint $i^!$. Using the previous ideas, the Serre functor of $\mathcal{A}$ is given by $i^! \circ S \circ i$. Similarly, if $i^* \dashv i$, then the Serre functor is given by the inverse of $i^* \circ S \circ i$. □

Example 3. Let $f\colon X \to Y$ be a morphism between smooth, projective varieties of dimensions m and n, respectively. We know that $Rf_*\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ admits a left adjoint Lf^*. Thus, by the lemma we get a right adjoint

$$S_X \circ Lf^* \circ S_Y^{-1}(-) = Lf^*(-) \otimes \omega_f[m-n],$$

where $\omega_f = \omega_X \otimes f^*\omega_Y^{-1}$. This adjoint is denoted by $f^!$, the *upper shriek*.

We point out that $f^!$ exists in much more generality, but its definition and basic properties are much more delicate. The basic idea is as follows: if $f\colon X \to Y$ is a morphism between finite type $\mathbb{C}$-schemes, we pick a relative compactification $X \hookrightarrow \overline{X} \xrightarrow{\overline{f}} Y$, which exists by a theorem of Nagata. The functor $Rf_*\colon \mathrm{D}_{\mathrm{QCoh}}(X) \to \mathrm{D}_{\mathrm{QCoh}}(Y)$ admits a right adjoint using Brown's criterion, which we denote by $\overline{a}\colon \mathrm{D}_{\mathrm{QCoh}}(Y) \to \mathrm{D}_{\mathrm{QCoh}}(X)$. One declares

$$f^!\colon \mathrm{D}^+_{\mathrm{QCoh}}(Y) \to \mathrm{D}^+_{\mathrm{QCoh}}(X), \qquad f^!(K) = \overline{a}(K)|_X.$$

This approach for $f^!$ is relatively clean; however, it does *not* say anything about how to compute $f^!$. This argument is essentially due to Neeman.

Alternatively, one can construct $f^!$ directly if f is an embedding or if it is of the form $\mathbb{P}^n_A \to A$. One can then construct $f^!$ by working locally and "gluing" these two models. In any case, there is a lot of work to be done. We refer the interested reader to [Stacks, Tag 0DWF] and the references therein.

2. Mutations

Assume that we are given a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$. In particular, $\mathcal{A}$ is left admissible and the inclusion map $i\colon \mathcal{A} \to \mathcal{B}$ admits a left adjoint i^*. If $\mathcal{D}$ admits a Serre functor, the previous section shows that i also has a right adjoint $i^!$. This way, we can produce a new semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$. This is what we call a *mutation* of a semi-orthogonal decomposition. Note that $\mathcal{A}$ is preserved, while $\mathcal{B}$ is replaced by the new subcategory $\mathcal{A}^\perp$.

Lemma 4. *The subcategories $\mathcal{A}^\perp$ and $\mathcal{B}$ are equivalent.*

Proof. Note that the inclusion $j\colon \mathcal{B} \to \mathcal{D}$ admits a right adjoint $j^!$. Using this functor, we assemble the composition $\mathcal{A}^\perp \to \mathcal{D} \to \mathcal{B}$. One quickly checks that this is an equivalence. □

The same idea can be applied with more than two pieces. For example, if $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \rangle$, we can move $\mathcal{A}_2$ and $\mathcal{A}_3$ around. Let us introduce some notation.

Definition 5 ([BK89, §4], cf. [Kuz09, §2.3])**.** Let $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_\ell \rangle$ be a semi-orthogonal decomposition. Given $i = 2, \dots, \ell$, the *ith right mutation* is

$$\mathbb{R}_i(\mathcal{A}_\bullet) = \{\mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i, *, \mathcal{A}_{i+1}, \dots, \mathcal{A}_\ell\},$$

where $* = {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i \rangle \cap \langle \mathcal{A}_{i+1}, \dots, \mathcal{A}_\ell \rangle^\perp$. Similarly, for $i = 1, \dots, \ell - 1$, the *ith left mutation* is

$$\mathbb{L}_i(\mathcal{A}^\bullet) = \{\mathcal{A}_1, \dots, \mathcal{A}_{i-1}, *, \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_\ell\},$$

where $* = {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1} \rangle \cap \langle \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_\ell \rangle^\perp$.

Let us make some comments on the notation. First of all, note that the i is the position of the "mutated" entry, and the left/right is where it is coming from. Second, we think of $\mathbb{R}_i, \mathbb{L}_i$ as acting on the collection of all semi-orthogonal decompositions.

Proposition 6. *The operators $\mathbb{R}_i, \mathbb{L}_i$ satisfy the braid relations: for $i \geq 2$, we have $\mathbb{R}_i\mathbb{R}_{i+1}\mathbb{R}_i = \mathbb{R}_{i+1}\mathbb{R}_i\mathbb{R}_{i+1}$ (resp. with $\mathbb{L}$).*

Let us point out that the definition of a mutation is pretty messy, as it requires computing two orthogonal complements. However, note that

$$\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle \implies \mathcal{D} = \langle S_{\mathcal{D}}(\mathcal{B}), \mathcal{A} \rangle.$$

This way, if we want to mutate a block "to the other side" (i.e. apply $\mathbb{L}_1 \dots \mathbb{L}_{\ell-1}$). Something similar applies in the other direction.

3. Mutations of exceptional collections

Let us recall that an *exceptional collection* of a triangulated category $\mathcal{D}$ is a collection of objects $E_1, \dots, E_r$ with no Homs from right to left, and with $\operatorname{Hom}^\bullet(E_i, E_i) = \mathbb{C}[0]$. Using the same ideas of the previous section, we can mutate these sequences. However, in this case there is an explicit way of performing these mutations.

Proposition 7. *Let $\mathcal{D} = \langle A_1, \dots, A_\ell \rangle$ be a full exceptional collection, with $\mathcal{A}_\bullet$ the corresponding semi-orthogonal decomposition. Let L_i and R_i be given by the triangles*

$$A_i \otimes R\operatorname{Hom}(A_i, A_{i+1}) \xrightarrow{ev} A_{i+1} \to L_i \to *,$$

$$R_i \to A_{i-1} \xrightarrow{ev^*} A_i \otimes R\operatorname{Hom}(A_{i-1}, A_i)^* \to *.$$

where ev and ev^* are the canonical (co)evaluation maps. Then we have full exceptional collections

$$\begin{aligned} \mathcal{D} &= \langle A_1, \dots, A_{i-2}, A_i, R_i, A_{i+1}, \dots, A_\ell \rangle \\ &= \langle A_1, \dots, A_{i-1}, L_i, A_i, A_{i+2}, \dots, A_\ell \rangle. \end{aligned}$$

Moreover, the corresponding semi-orthogonal decompositions correspond to $\mathbb{R}_i(\mathcal{A}_\bullet)$ and $\mathbb{L}_i(\mathcal{A}_\bullet)$ respectively.

Proof. (Idea) Let us focus[1] on the case $\ell = 2$, i.e. $\mathcal{D} = \langle A, B \rangle$. Consider the triangle

$$A \otimes R\operatorname{Hom}(A, B) \xrightarrow{ev} B \to C \to *.$$

We claim that $\mathcal{D} = \langle C, A \rangle$. To show this, start by applying $\operatorname{Hom}(A, -)$ to the triangle above, which shows that $\operatorname{Hom}^i(A, C) = 0$ holds for all i. Now, apply

[1]The general case is a formal reduction to this one!

$\mathrm{Hom}^i(B,-)$ to get that $\mathrm{Hom}^\bullet(B,C)=\mathbb{C}[0]$. At last, we apply $\mathrm{Hom}(-,C)$ to get that $\mathrm{Hom}^\bullet(C,C)=\mathbb{C}[0]$. In other words, this shows that C is exceptional.

At last, note that the triangle above allows us to recover B from A and C. This shows that C and A span $\mathcal{D}$. □

We will use this result to produce various examples.

3.1. **On Beilinson SOD.** Recall from a couple of lectures ago that $\mathrm{D}^b(\mathbb{P}^n)$ admits a full exceptional collection $\langle \mathscr{O}, \mathscr{O}(1), \dots, \mathscr{O}(n)\rangle$, thanks to Beilinson. Let's see what happens if we mutate (over and over)

First, let us focus on $n=1$, i.e. $\mathrm{D}^b(\mathbb{P}^n)=\langle \mathscr{O}, \mathscr{O}(1)\rangle$. To compute $\mathbb{L}_1$ we need to look at the triangle

$$\mathscr{O}\otimes R\,\mathrm{Hom}(\mathscr{O},\mathscr{O}(1))\to \mathscr{O}(1)\to L_1\to *.$$

Here $L_1=\mathscr{O}(-1)[1]$, hence we get the SOD $\mathrm{D}^b(\mathbb{P}^n)=\langle \mathscr{O}(-1),\mathscr{O}\rangle$.

Let us look now at the case $n=2$, starting with $\mathrm{D}^b(\mathbb{P}^2)=\langle \mathscr{O},\mathscr{O}(1),\mathscr{O}(2)\rangle$. We apply $\mathbb{L}_1$ to get:

$$\mathscr{O}\otimes R\,\mathrm{Hom}(\mathscr{O},\mathscr{O}(1))\to \mathscr{O}(1)\to C\to *.$$

Here $C\cong \Omega^1(1)[1]$, by the Euler exact sequence. Thus, we get

$$\mathbb{L}_1\colon \langle \Omega^1(1),\mathscr{O},\mathscr{O}(2)\rangle.$$

We now apply $\mathbb{L}_1\circ\mathbb{L}_2$ simultaneously. We can use our previous discussion to get $S_{\mathbb{P}^2}(\mathscr{O}(2))=\mathscr{O}(-1)[1]$. This way, we get

$$\langle \mathscr{O},\mathscr{O}(1),\mathscr{O}(2)\rangle \xrightarrow{\mathbb{L}_1} \langle \Omega^1(1),\mathscr{O},\mathscr{O}(2)\rangle \xrightarrow{\mathbb{L}_1\circ\mathbb{L}_2} \langle \mathscr{O}(-1),\Omega^1(1),\mathscr{O}\rangle.$$

In general, one can show that $\mathrm{D}^b(\mathbb{P}^n)=\langle \Omega^n(n),\dots,\Omega^1(1),\mathscr{O}\rangle$, and that this is obtained by mutation as above.

3.2. **The two SOD.** Let us go back to an example discussed a couple of weeks ago. Let $\alpha\colon S\to\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at two points. Here $\mathrm{Pic}(S)=\mathbb{Z}E_1\oplus \mathbb{Z}E_2\oplus\mathbb{Z}\alpha^*L$, where E_1,E_2 are the two exceptional divisors and L is the class of a line in $\mathbb{P}^2$. Using the blow-up semi-orthogonal decomposition we get

$$\mathrm{D}^b(S)=\langle \mathscr{O}_{E_1}(-1),\mathscr{O}_{E_2}(-1),\mathscr{O}_S,\mathscr{O}_S(\alpha^*L),\mathscr{O}_S(\alpha^*2L)\rangle.$$

On the other hand, we can blow-down the strict transform of the curve joining both points in $\mathbb{P}^2$. This gives us a blow-down $\beta\colon S\to\mathbb{P}^1\times\mathbb{P}^1$. If M_1,M_2 are the two rulings of $\mathbb{P}^1\times\mathbb{P}^1$, and F is the exceptional divisor, we get an SOD

$$\mathrm{D}^b(S)=\langle \mathscr{O}_F(-1),\mathscr{O}_S,\mathscr{O}_S(\beta^*M_1),\mathscr{O}_S(\beta^*M_2),\mathscr{O}_S(\beta^*M_1+\beta^*M_2)\rangle.$$

Let's see if we can relate these two decompositions via blow-downs. We point out that

$$\beta M_1^*=E_1+F,\quad \beta M_2^*=E_2+F,\quad \alpha^*L=E_1+E_2+F,$$

which will be useful to compare these two decompositions. We have depicted this situation in Figure 1.

This way, our previous semi-orthogonal decompositions can be written as

$$\mathrm{D}^b(S)=\langle \mathscr{O}_{E_1}(-1),\mathscr{O}_{E_2}(-1),\mathscr{O}_S,\mathscr{O}_S(E_1+E_2+F),\mathscr{O}_S(2E_1+2E_2+2F)\rangle, \tag{1}$$

$$\mathrm{D}^b(S)=\langle \mathscr{O}_F(-1),\mathscr{O}_S,\mathscr{O}_S(E_1+F),\mathscr{O}_S(E_1+F),\mathscr{O}_S(E_1+E_2+2F)\rangle. \tag{2}$$

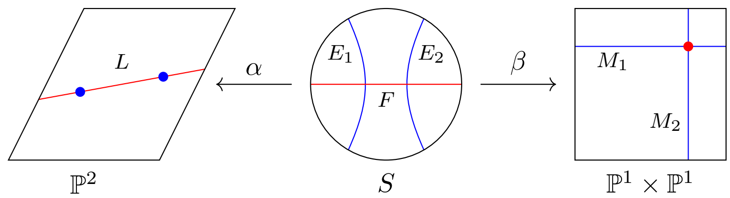

FIGURE 1. The blow-up of $\mathbb{P}^2$ at two points, and the blow-up pf $\mathbb{P}^1 \times \mathbb{P}^1$ at a point.

We start by applying $\mathbb{R}_3$ to (1). Here, note that

$$\begin{aligned}\operatorname{Hom}^\bullet(\mathscr{O}_{E_2}(-1), \mathscr{O}_S) &= \operatorname{Hom}^{2-\bullet}(\mathscr{O}_S, \mathscr{O}_{E_2}(-1) \otimes \omega_S)^\vee \\ &= H^{2-\bullet}(S, \mathscr{O}_E(-2))^\bullet = \mathbb{C}[-1].\end{aligned}$$

This way, the mutated object A fits into the triangle

$$\mathscr{O}_{E_1}(-1) \to \mathscr{O}_S[1] \to A \to \mathscr{O}_{E_1}(-1)[1].$$

It follows that $A = \mathscr{O}_S(E_1)$, and we get the SOD

$$\mathrm{D}^b(S) = \langle \mathscr{O}_{E_1}(-1), \mathscr{O}_S, \mathscr{O}_S(E_2), \mathscr{O}_S(E_1 + E_2 + F), \mathscr{O}_S(2E_1 + 2E_2 + 2F)\rangle$$

The same argument applies after applying $\mathbb{R}_2$, which yields

$$\mathrm{D}^b(S) = \langle \mathscr{O}_S, \mathscr{O}_S(E_1), \mathscr{O}_S(E_2), \mathscr{O}_S(E_1 + E_2 + F), \mathscr{O}_S(2E_1 + 2E_2 + 2F)\rangle.$$

Our next step is to apply $\mathbb{L}_1 \circ \mathbb{L}_2 \circ \mathbb{L}_3 \circ \mathbb{L}_4$. In other words, we are mutating the last object all the way to the first position. The effect is the same as tensoring it by the canonical bundle (which is $\mathscr{O}_S(-2E_1 - 2E_2 - 3F)$). We get:

$$\mathrm{D}^b(S) = \langle \mathscr{O}_S(-F), \mathscr{O}_S, \mathscr{O}_S(E_1), \mathscr{O}_S(E_2), \mathscr{O}_S(E_1 + E_2 + F)\rangle.$$

At last, apply $\mathbb{L}_1$. The same computations we did previously apply (after tensoring by $\mathscr{O}_S(F)$), yielding

$$\mathrm{D}^b(S) = \langle \mathscr{O}_F, \mathscr{O}_S(-F), \mathscr{O}_S(E_1), \mathscr{O}_S(E_2), \mathscr{O}_S(E_1 + E_2 + f)\rangle.$$

This gives us (2) up to twisting by $\mathscr{O}_S(F)$.

4. Applications on families

We will finish today's talk by giving some justification to the idea of mutations. In fact: why would want to produce more semi-orthogonal decompositions?

One way of justifying this is to find a semi-orthogonal decomposition where the components have some geometric meaning, or some useful form. For example, we showed that the blow-up of $\mathbb{P}^2$ at two points admit a semi-orthogonal decomposition whose components are all line bundles. More generally, the argument of the previous section shows:

Proposition 8. *Let S be a smooth, projective rational surface. Then S admits a semi-orthogonal decomposition consisting only on line bundles.*

Proof. (Sketch) Pick a sequence of blow-dows $S \to S_1 \to \cdots \to S_r$, where S_r is either $\mathbb{P}^2$ or a Hirzebruch surface. The proof proceeds by induction on r: the case $r = 0$ is clear (recall that Hirzebruch surfaces are $\mathbb{P}^1$-bundles!), and the induction step follows from Orlov's semi-orthogonal decomposition together with the previous computation. □

For another example, we follow [Kuz21]. A surface S is a *sextic du Val del Pezzo* if S is normal, integral, with at worst du Val singularities and ω_S^{-1} ample with $\omega_S^2 = 6$. The smooth ones are easy to characterize: they are blow-ups $S \to \mathbb{P}^2$ at three general points. However, if the points are not chosen generally, we will get (-2)-curves which need to be contracted.

So, producing semi-orthogonal decompositions, at least for the smooth ones, is not difficult. If $S \to \mathbb{P}^2$ has exceptional divisors E_1, E_2, E_3, Orlov's SOD gives us

$$\mathrm{D}^b(S) = \langle \mathscr{O}_{E_1}(-1), \mathscr{O}_{E_2}(-1), \mathscr{O}_{E_3}(-1), \mathscr{O}_S, \mathscr{O}_S(L), \mathscr{O}_S(2L) \rangle,$$

where L is the (pullback of the) class of a line in $\mathbb{P}^2$. One can mutate this family to get:

Proposition 9 ([Kuz21, 3.1, 3.13])**.** *There is a semi-orthogonal decomposition*

$$\mathrm{D}^b(S) = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle,$$

where

$$\begin{aligned} \mathcal{A}_1 &= \langle \mathscr{O}_S \rangle \\ \mathcal{A}_2 &= \langle \mathscr{O}_S(L - E_1), \mathscr{O}_S(L - E_2), \mathscr{O}_S(L - E_3) \rangle \\ \mathcal{A}_3 &= \langle \mathscr{O}_S(L), \mathscr{O}_S(2L - E_1 - E_2 - E_3) \rangle. \end{aligned}$$

From here one can interpret the semi-orthogonal decompositions in an interesting way. Note that ω_S^{-1} embeds S into $\mathbb{P}^6$. We let Z_d be the Hilbert scheme with polynomial $dt+1$ with respect to this polarization. It turns out that each component $\mathcal{A}_d^\vee$ can be interpreted as coming from the universal family in Z_d. Moreover, one can use this interpretation to extend the results for du Val del Pezzo surfaces.

Theorem 10 ([Kuz21, 5.2])**.** *Assume that $\mathscr{X} \to S$ is a family of du Val del Pezzo surfaces. Then there is an S-linear semi-orthogonal decomposition*

$$\mathrm{D}^b(\mathscr{X}) = \langle \mathrm{D}^b(S), \mathrm{D}^b(\mathscr{X}_2, \beta_2), \mathrm{D}^b(\mathscr{X}_3, \beta_3) \rangle,$$

where $\mathscr{X}_2, \mathscr{X}_3 \to S$ are finite flat of degree $3, 2$ respectively, and $\beta_{\mathscr{X}_2}, \beta_{\mathscr{X}_3}$ are Brauer classes of order 2 and 3.

At the end, we get a result where mutations are not mentioned anywhere; however, these are a useful tool to get the correct ending result.

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