A FAST INTRODUCTION TO DERIVED CATEGORIES

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Abstract. Notes prepared for the *Semi-orthogonal decompositions seminar* (Spring 2025), organized by Amal Mattoo and myself at Columbia University.

Contents

0. Introduction

First, a word of warning. This is a *fast* introduction; as such, we will omit various technical details, some of which might actually be important. We will focus on how to work with derived categories in practice. We refer to [Har66] and especially [Huy06] for a more in-depth discussion.

Let us start by recalling the construction of sheaf cohomology, say as in [Har77, §III.2]. Let X be a topological space, and let $\mathscr{E}$ be a sheaf of abelian groups on $\mathscr{E}$. The sheaf cohomology $H^i(X,\mathscr{E})$ is defined as follows.

(1) Start by taking an injective resolution $0 \to \mathscr{E} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \dots$, where each $\mathscr{I}^n$ is an injective sheaf.
(2) Drop $\mathscr{E}$, and apply the global sections functor. We get a complex
$$\Gamma(X,\mathscr{I}^0) \to \Gamma(X,\mathscr{I}^1) \to \dots.$$
(3) The ith cohomology of this complex is set as $H^i(X,\mathscr{E})$.

As one quickly realizes, this is a delicate construction. The big issue comes from the first point: injective resolutions are not unique. Instead, any two of them *homotopic*. This is, if $\mathscr{I}^\bullet$ and $\mathscr{J}^\bullet$ are two injective resolutions, there are maps $\phi\colon \mathscr{I}^\bullet \to \mathscr{J}^\bullet$ and $\psi\colon \mathscr{J}^\bullet \to \mathscr{I}^\bullet$ such that the compositions $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to the identity.

So, our first instinct would be to consider morphisms of complexes up to homotopy. This still is not good enough, as in general $\mathscr{E}$ is not homotopic to its injective resolutions. Instead, we need to formally declare these maps to be invertible; something called a *quasi-isomorphism*.

1. The homotopy category

We will start by discussing the homotopy category of an abelian category. This is an intermediate step to construct the derived category. We follow [Huy06, pp. 27–31].

Let $\mathcal{A}$ be an abelian category. (There are a couple of good examples to keep in mind: Mod_R for a ring R, $\mathrm{Coh}(X)$ and $\mathrm{QCoh}(X)$ for a variety X.) Recall that a *complex* $A^\bullet$ is a collection $\{A^i\}$ of elements of $\mathcal{A}$, and maps $d^i\colon A^i \to A^{i+1}$ (called *differentials*) satisfying $d^{i+1} \circ d^i = 0$. A map of complexes $\phi\colon A^\bullet \to B^\bullet$ is a collection of maps $\phi^i\colon A^i \to B^i$ satisfying $\phi^{i+1} \circ d^i_A = d^i_B \circ \phi^i$, i.e. that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^i & \xrightarrow{d^i_A} & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi^i} & & \downarrow{\scriptstyle \phi^{i+1}} & & \\ \cdots & \longrightarrow & B^i & \xrightarrow[d^i_B]{} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

commutes. This defines an abelian category, denoted $\mathrm{Kom}(\mathcal{A})$.

Definition 1. Let $\phi, \psi\colon A^\bullet \to B^\bullet$ be two morphisms in $\mathrm{Kom}(\mathcal{A})$. An *homotopy* between ϕ and ψ is a collection of maps $h^i\colon A^i \to B^{i-1}$ (note the change of degree!) satisfying $d^{i-1}_B h^i + h^{i+1} d^i_A = \phi^i - \psi^i$.

One quickly verifies that (i) being homotopic is an equivalence relation, and (ii) it is compatible with composition. This way, we define the *homotopy category* $\mathrm{K}(\mathcal{A})$ to be the category whose objects are the same as in $\mathrm{Kom}(\mathcal{A})$, and $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A, B)/\text{homotopy}$.

Example 2. Let $R = k[x]$, and let $A = k[x] \xrightarrow{\cdot x} k[x]$. Let us compute $\mathrm{Hom}(A, A)$ in the homotopy category. To do so, note that a map of complexes $\phi = (\phi^0, \phi^1)$ must satisfy $\phi^1 \cdot x = x \cdot \phi^0$, i.e. $\phi^0 = \phi^1$. Moreover, two maps ϕ, ψ are homotopic if there exists some h such that $h \cdot x = \phi^0 - \psi^0$. In other words, two maps are homotopic if and only if $\phi^0(0) = \psi^0(0)$, and so $\mathrm{Hom}_{\mathrm{K}}(A, A)$ is one-dimensional.

2. The triangulated structure

Our next stop is to describe the triangulated structure of $\mathrm{Kom}(\mathcal{A})$. The idea here is that $\mathrm{Kom}(\mathcal{A})$ is *not* an abelian category, as kernels and cokernels fail to exist. However, the homotopy category "remembers" the short exact sequences — sort of. Let us introduce the key objects.

To start, given a complex $A^\bullet$, we let $A^\bullet[n]$ be the *shift*. This is defined as

$$(A^\bullet[n])^i = A^{n+i}, \qquad d^k_{A^\bullet[n]} = (-1)^n d^{n+k}_{A^\bullet}.$$

The sign here is a bit annoying, but we cannot avoid it: a sign needs to appear somewhere.

Now, let us introduce the *cone* of a map of complexes. Given a morphism $\phi\colon A^\bullet \to B^\bullet$ of complexes, we set $C(\phi)$ to be the complex

$$C(\phi)^n = A^{n+1} \oplus B^n, \qquad d_{C(\phi)^n} = \begin{pmatrix} -d^{n+1}_A & 0 \\ \phi^{n+1} & d^n_B \end{pmatrix}.$$

Note that by construction the natural maps $B^\bullet \to C(\phi)^\bullet \to A^\bullet[1]$ are morphisms of complexes. Moreover, we have the following cool result.

Lemma 3. *Assume that* $\phi\colon A^\bullet \to B^\bullet$ *is an injective morphism in* $\mathrm{Kom}(\mathcal{A})$*, and let* $\psi\colon B^\bullet \to C^\bullet$ *be the image. We have that the map* $C(\phi)^\bullet \to C^\bullet$ *induces isomorphisms in cohomology.*

Note however that this might *not* admit an "inverse": witness $\phi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\times 2$. This is part of the reason why we want to allow inverses of quasi-isomorphisms.

In any case, we will introduce the notion of a *distinguished triangle*. These are a collection of three complexes $A^\bullet, B^\bullet, C^\bullet$ and maps $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ that are isomorphic (in $\mathrm{K}(\mathcal{A})$) to the "cone construction"

$$A^\bullet \xrightarrow{\phi} B^\bullet \to C(\phi) \to A^\bullet[1].$$

Theorem 4. *The collection of distinguished triangles in* $\mathrm{K}(\mathcal{A})$ *satisfies the following axioms:*

TR1. *(Existence of triangles)*

(i) For any A*, the triangle*

$$A \xrightarrow{\mathrm{id}} A \xrightarrow{0} 0 \to A[1]$$

is distinguished.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished

(iii) Any morphism $f\colon A \to B$ *can be extended to a triangle*

$$A \xrightarrow{f} B \to C \to A[1].$$

TR2. *(Shifts of triangles) The triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished.

TR3. *(Extension of morphisms) Suppose we have a diagram with arrows* f, g *and* $f[1]$*, as follows.*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1]. \end{array}$$

Assume that the left square commutes. Then there exists a (non-unique!) $h\colon C \to C'$ *making the diagram commute.*

TR4. *(Octahedral axiom) Assume that we have triangles* $A^\bullet \to B^\bullet \to X^\bullet \to A^\bullet[1]$, $B^\bullet \to C^\bullet \to Y^\bullet \to B^\bullet[1]$, $A^\bullet \to C^\bullet \to Z^\bullet \to A^\bullet[1]$*. Then there is a triangle*

$$X^\bullet \to Z^\bullet \to Y^\bullet \to X^\bullet[1]$$

satisfying some compatibilities (that we will omit).

In general, a *triangulated category* is an additive category endowed with a shift functor and a collection of "distinguished triangles" satisfying the axioms TR1–TR4 above. One should think on the triangulated axioms in the same way we think about abelian categories: many "basic properties" (like the existence of long exact sequences and such) follow directly from the triangulated axioms, but it is a good idea to focus on the "concrete" setups at the beginning. (Let us point out that in the seminar most of the triangulated categories will be subcategories of the derived category, so this is really the "main" example to keep in mind.)

3. The derived category

As we have hinted before, the homotopy category is not the object we need for our purposes, as sometimes we get quasi-isomorphisms that are not isomorphisms (in $\mathrm{K}(\mathcal{A})$). Instead, our replacement will be to formally take "inverses" of these maps. Let us start by introducing the notion of a quasi-isomorphism.

Definition 5. A map $\phi\colon A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if it induced isomorphisms in cohomology. Equivalently, if $C(\phi)^\bullet$ is *acyclic* (i.e. it has zero cohomology objects).

It is worth pointing out that homotopic maps induce the *same* map in cohomology. Thus, the notion of quasi-isomorphism is well defined in $\mathrm{K}(\mathcal{A})$.

Definition 6. The *derived category* $\mathrm{D}(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $\mathrm{K}(\mathcal{A})$, and morphisms from $A^\bullet \to B^\bullet$ are diagrams,

$$A^\bullet \xleftarrow[qis]{f} C^\bullet \xrightarrow{g} B^\bullet$$

where the two maps are in $\mathrm{Kom}(\mathcal{A})$. (We should think of this as a "fraction gf^{-1}".) Two maps $A^\bullet \leftarrow C_i^\bullet \to B^\bullet$ are equivalent if they can be dominated (with a quasi-isomorphism) by a third diagram.

This definition has many subtle details. First of all, we have not said anything about composition — it is a tricky construction, see [Huy06, 2.17]. Second, it is impossible from the definition itself to compute anything, as we need to consider all possible quasi-isomorphisms $C^\bullet \to A^\bullet$. Third, there is a hidden set-theoretic issue that we will ignore completely.

What can we do in the derived category? First of all, we still have a triangulated structure: we take all triangles that are isomorphic (in $\mathrm{D}(\mathcal{A})$!) to the cone triangle. Second, note that we still have well-defined maps $H^i\colon \mathrm{D}(\mathcal{A}) \to \mathcal{A}$. In particular[1] one realizes that any triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ induces a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \ldots.$$

There are many variants of the construction of the derived category. For example, we could have consider the *bounded* derived category $\mathrm{D}^b(\mathcal{A})$, where complexes are only allowed to have $A^i \neq 0$ for finitely many indices. Or the *left/right bounded* categories.

Similarly, say that we are considering the category of coherent sheaves on X. Instead of working directly in $\mathrm{D}^b(\mathrm{Coh}(X))$, we could look at the subobjects of $\mathrm{D}(\mathscr{O}_X)$ that have bounded and coherent cohomology. (There are technical reasons why we would want this, e.g. usually injective resolutions will be infinite.) It turns out that these details will not make a difference in the cases we are interested, so we will ignore them.

[1] Which is something that also holds in $\mathrm{K}(\mathcal{A})$.

4. Derived functors

Let us finish up by discussing two related problems that $\mathrm{D}(\mathcal{A})$ carries, which (a bit surprisingly at first) have a related solution.

- How do we compute maps $\mathrm{Hom}(A, B)$ in the derived category?
- Is there a way to extend the functors $f_*, f^*, \otimes$ and so on?

We will focus on the second part for now. Assume that $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. It is clear that F is compatible with homotopies, and so it induces a functor $\mathrm{K}(F)\colon \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$. This functor is compatible with cones, and it will map triangles to triangles. (Such functors are called *triangulated*). However, functors usually will *not* preserve quasi-isomorphisms.

Example 7. Consider the complex $A = [\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}]$ (in degrees -1 and 0) and $B = \mathbb{Z}/2\mathbb{Z}$ (in degree zero). There is a quasi-isomorphism $A \to B$. However, if we apply the functor $\mathbb{Z}/2\mathbb{Z} \otimes -$, the resulting map is *not* a quasi-isomorphism.

Now, there is a trick we can do. Instead of trying to apply the functor $- \otimes \mathbb{Z}/2\mathbb{Z}$, we will try to apply it only to finite complexes of free $\mathbb{Z}$-modules. The key point is that $- \otimes \mathbb{Z}/2$ is exact if we apply it only to complexes of free modules. Thus, we will do a two-step process to extend $- \otimes \mathbb{Z}/2$ to $\mathrm{D}^b(\mathbb{Z})$: first, replace the complex by free $\mathbb{Z}$-modules, and then apply $- \otimes \mathbb{Z}/2$ in the usual way. Let us introduce this in full generality.

Definition 8. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories, where $\mathcal{A}$ has enough projective objects. The *left derived* functor $LF\colon \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$ is defined as

$$LF(A) := \mathrm{K}(F)(\tilde{A}),$$

where $\tilde{A} \to A$ is any quasi-isomorphism from a complex $\tilde{A}$ of projective objects.

One shows that this does not depend on the choice of a projective resolution, and so we get a triangulated functor LF. Moreover, by using a version of the horseshoe lemma, we quickly verify that this is a triangulated functor.

Remark 9. Note that in the examples we are interested in (e.g. in $\mathrm{D}^b(\mathrm{Coh}(X))$), there are no projective objects in general. In any case, all right exact functors we will be interested about (tensor products and pullbacks) can be computed with locally free resolutions. The key point is that these are acyclic objects for the tensor product, cf. [Har66, p. 53].

A dual version applies for left-exact functors: we take an injective resolution and apply the functor directly. Of course, this is only useful for proofs — in practice, one can use Čech covers to compute the derived functors of $\Gamma(X, -)$ and f_*.

At last, the same recipe applies to compute $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B)$: we replace A by projective objects (or B by injective objects), apply $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}$, and take the zeroth degree piece.

5. Computations

Let us use the remaining time to discuss how these computations are performed in practice.

Example 10. Let $X = \mathbb{A}^2$, and let $\mathscr{O}_{(0,0)}$ be the skyscraper sheaf at the origin. Let us compute the derived tensor product $\mathscr{O}_{(0,0)} \overset{L}{\otimes} \mathscr{O}_{(0,0)}$.

To start, we need a locally free resolution. In this case this is quite direct: we take the Koszul resolution

$$\mathscr{O}_{\mathbb{A}^2} \xrightarrow{(y,-x)^t} \mathscr{O}_{\mathbb{A}^2}^{\oplus 2} \xrightarrow{(x,y)} \mathscr{O}_{\mathbb{A}^2},$$

and we apply $- \otimes \mathscr{O}_{(0,0)}$. Note that this kills the differentials, and so the complex is given by

$$\mathscr{O}_{(0,0)} \overset{L}{\otimes} \mathscr{O}_{(0,0)} = [\mathscr{O}_{(0,0)} \xrightarrow{0} \mathscr{O}_{0,0}^{\oplus 2} \xrightarrow{0} \mathscr{O}_{(0,0)}^{\oplus 2}].$$

We can write

$$\mathscr{O}_{(0,0)} \overset{L}{\otimes} \mathscr{O}_{(0,0)} = \mathscr{O}_{(0,0)} \oplus \mathscr{O}_{(0,0)}^{\oplus 2}[1] \oplus \mathscr{O}_{(0,0)}[2],$$

thanks to the fact that the differentials are zero.

This is a good opportunity to mention that the tensor product can be computed by resolving *either* of its factors. We also point out that the usual properties (associativity, commutativity and such) are still present. At last, the ith cohomology of the derived tensor product $H^i(A \overset{L}{\otimes} B)$ is known as the $(-i)$ *Tor functor*.

Warning! Just knowing the cohomology $H^i(A \overset{L}{\otimes} B)$ is not enough to recover $A \overset{L}{\otimes} B$. The differential here makes a big difference!

Example 11. Let us try to compute $R\Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$. Here, our definition instruct us to take an injective resolution on $\mathscr{O}_{\mathbb{P}^2}(1)$. But this is almost impossible! There are two options here.

- Recall that the sheaf cohomology $H^i(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$ can be computed by taking a Čech cover, assembling the complex

$$\bigoplus_{|I|=1} \Gamma(U_I, \mathscr{O}_{\mathbb{P}^2}(1)) \to \bigoplus_{|I|=2} \Gamma(U_I, \mathscr{O}_{\mathbb{P}^2}(1)) \to \dots$$

  and computing its cohomology objects. This complex (without taking cohomology) recovers $R\Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$.
- The complex $R\Gamma(X, F)$ is equal to $\bigoplus_n H^n(X, F)[-n]$. This works only because $R\Gamma(X, F)$ is a complex of vector spaces, and so it does *not* work in general! (In particular, if we wish to compute Rf_* instead, we will almost certainly need to take a Čech cover.)

As we hinted in the example, the ith cohomology object of $R\Gamma(X, F)$ is known as the *ith sheaf cohomology* $H^i(X, F)$. It is also worth mentioning that if X is proper, then the $R\Gamma(X, F)$ are finite dimensional.

Example 12. Let us go back to $\mathbb{P}^2$. Pick a smooth curve C of degree 3, so that it has genus 1. How do we compute $\mathrm{Hom}(\mathscr{O}_C, \mathscr{O}_C[1])$ in the derived category?

Our recipe tells us to take a resolution of $\mathscr{O}_C[1]$ by injective objects, apply $\mathrm{Hom}(\mathscr{O}_C, -)$, and take the zeroth degree piece. But this is the same as resolving $\mathscr{O}_C$, applying $\mathrm{Hom}(\mathscr{O}_C, -)$ and taking the first degree piece. This equals $\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C)$.

In fact, one verifies that $\mathrm{Hom}_{\mathrm{D}}(F, G[i]) = \mathrm{Ext}^i(F, G)$ if F, G are sheaves.

It is worth pointing out that we *cannot* compute $\mathrm{Hom}_{\mathrm{D}}(F, G)$ by taking locally free resolutions of F. The issue is that locally free sheaves might have cohomology! Instead, we have that $R\Gamma(X, R\mathcal{H}om(F, G)) = R\,\mathrm{Hom}(F, G)$ (the "composition of functors" theorem). If we are only interested in $R^i\,\mathrm{Hom}(F, G)$, these can be computed by the *Grothendieck spectral sequence*

$$E_2^{p,q} = R^p\Gamma(X, R^q\mathcal{H}om(F, g)) \Rightarrow R^{p+q}\,\mathrm{Hom}(F, G) = \mathrm{Hom}(F, G[p + q]).$$

We have hinted at various compatibilities between derived functors. The basic philosophy here is that we can replace objects by injective resolutions or locally free resolutions, which allows us to extend "classical" properties to the derived world. For example, if $f : X \to Y$ is a morphism of schemes, then

$$Rf_*(A \overset{L}{\otimes}_X Lf^*(B)) \cong Rf_*(A) \overset{L}{\otimes}_Y B$$

holds provided that B can be represented by a complex of locally free sheaves. (This holds in more generality though, compare [Stacks, Tag 08EU].)

Example 13. Let us prove the following fact. Let S be a smooth surface and let $p \in S$ be a point. If $\pi : \tilde{S} \to S$ is the blow-up of S at p, we claim that $R\pi_*\mathscr{O}_{\tilde{S}}$ is isomorphic to $\mathscr{O}_S$. Here it is worth pointing out that

$$\mathrm{Hom}_S(\mathscr{O}_S, R\pi_*\mathscr{O}_{\tilde{S}}) = \mathrm{Hom}_{\tilde{S}}(L\pi^*\mathscr{O}_S, \mathscr{O}_{\tilde{S}}) = \mathrm{Hom}_{\tilde{S}}(\mathscr{O}_{\tilde{S}}, \mathscr{O}_{\tilde{S}}),$$

so there is a canonical map $\mathscr{O}_S \to Rf_*\mathscr{O}_{\tilde{S}}$.

To start, let us focus on a really concrete example: can we do this for the plane? For sure! We put coordinates and compute $R\pi_*\mathscr{O}_{\tilde{S}}$ via a Čech cocycle. In fact, this strategy works provided that S is affine and p is a complete intersection (in which case we can compute the blow-up explicitly).

How do we tackle the global case? It should be "obvious" now, as π is an isomorphism outside of p. In fact, we let $i : U \to S$ be an affine open subset containing p such that $p \in U$ is a complete intersection, and we consider the diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{j} & \tilde{S} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow[i]{} & S. \end{array}$$

One quickly verifies that[2] $i^*R\pi_*E = R\phi_*j^*E$: replace E by a complex of injective modules, and use that the restriction to an open subset is still injective[3]

At last, the argument should make intuitive that this works if we blow-up any smooth subvariety of a variety.

References

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[2] Note that the functors i^* and j^* are exact, as i, j are open immersions. Thus, they extend directly to the derived category.

[3] Results of these flavor hold in greater generality: if i is flat, or if π is flat (!), or more generally if the square is Tor-independent; see [Stacks, Tag 08IB] for example.

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