Semi-orthogonal Decompositions Seminar Notes

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Anna Abasheva: Kuznetsov components.

0.1 Review

Recall given $i : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ an admissible subcategory, we can define orthogonals

$$\langle \mathcal{D}_0, {}^\perp\mathcal{D}_0 \rangle \quad \langle \mathcal{D}_0^\perp, \mathcal{D}_0 \rangle$$

And we have mutation functors

$$\mathbf{R}_{\mathcal{D}_0} := k^! : \mathcal{D} \to {}^\perp\mathcal{D}_0, \quad \mathbf{L}_{\mathcal{D}_0} = j^* : \mathcal{D} \to \mathcal{D}^\perp$$

Given a decomposition $\mathcal{D}_0 = \langle \mathcal{D}_1, ..., \mathcal{D}_m \rangle$, the mutation functors compose as $\mathbf{L}_{\mathcal{D}_0} = \mathbf{L}_{\mathcal{D}_1} \circ ... \circ \mathbf{L}_{\mathcal{D}_m}$.

Serre functors satisfy $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, S_{\mathcal{D}}E)^*$.

Proposition 0.1. *If $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ is admissible, and $\mathcal{D}$ has a Serre functor, then $\mathcal{D}_0$ has a Serre functor as do its orthogonals. In fact,*

$$S_{\mathcal{D}_0} \simeq i^! \circ S_{\mathcal{D}} \circ i, \quad S^{-1}_{\mathcal{D}_0^\perp} \simeq \mathbf{L}_{\mathcal{D}_0} \circ S_{\mathcal{D}}^{-1}$$

Proof. For $F, E \in \mathcal{D}_0^\perp$, we have

$$\mathrm{Hom}(F, E) \simeq \mathrm{Hom}_{\mathcal{D}}(S_{\mathcal{D}}^{-1}E, F)^* \simeq \mathrm{Hom}_{\mathcal{D}_0^\perp}(\mathbf{L}_{\mathcal{D}_0} \circ S_{\mathcal{D}}^{-1}E, F)^*$$

□

Definition 0.2. $\mathcal{D}$ is *Calabi-Yau* if $S_{\mathcal{D}} = [n]$. It is *fractional Calabi-Yau* if $S_{\mathcal{D}}^q = [n]$.

If $\mathcal{D} = D^b(X)$, then $S_{\mathcal{D}} = - \otimes \omega_X[n]$ for $n = \dim x$. So if $\omega_X = \mathcal{O}_X$ then $\mathcal{D}$ is Calabi-Yau, and if $\omega_X^q = \mathcal{O}_X$, then $\mathcal{D}$ is fractional Calabi-Yau.

0.2 Kuznetsov components

Let $X \subset \mathbf{P}^{n+1}$ be a hypersurface of degree d. Then $S_X : E \mapsto E \otimes \mathcal{O}_X(d-(n+2))[n]$.

Observe:

- $\mathcal{O}_X(i)$ is exceptional if $d \leq n = 1$. This follows from $H^m(\mathcal{O}_X) = 0$ for all $m > 0$.
- $\mathcal{O}_X(i) \in \mathcal{O}_X(j)^{\perp}$. This follows from $H^m(\mathcal{O}_X(j-i)) = 0$ for $0 < j-1 \leq n+1-d$.

Now we can produce a semi-orthogonal decomposition:

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, ..., \mathcal{O}_X(n+1-d)\rangle$$

Definition 0.3. $\mathcal{A}_X$ is the *Kuznetsov component.*

Example. Let $d = 2$. Then we saw $\mathcal{A}_X = \langle \Sigma \rangle$ or $\langle \Sigma^+, \Sigma^- \rangle$, where these objects are spinor bundles.

Kuznetsov components are often Calabi-Yau categories.

Theorem 0.4. *If $n+1 < 2d$, then $\mathcal{A}_X$ is fractional Calabi-Yau.*

Example. For $d = 3$, can prove for any line $L \subset X$, the ideal sheaf $\mathcal{I}_L$ is the right orthogonal to $\langle \mathcal{O}_X, \mathcal{O}_X(1)\rangle$. To check this, see

$$0 \to \mathcal{I}_L \to \mathcal{I}_{L/\mathbf{P}^{n+1}} \to \mathcal{O}_X(-3) \to 0$$

and use that the middle term is $\mathcal{O}(-1)^{\oplus n+1}$.

So if $n = 3$, we have a semi-orthogonal decomposition $\langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1)\rangle$ with $\mathcal{I}_L \in \mathcal{A}_X$.

If $n = 4$, we have a decomposition $\langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle$. So $\mathcal{I}_L(1) \in \langle \mathcal{A}_X, \mathcal{O}_X\rangle$, and so $\mathbf{L}_{\mathcal{O}_X}(\mathcal{I}_L(1)) \in \mathcal{A}_X$.

Proof of Theorem 0.4. To prove the theorem, we need to understand

$$S_{\mathcal{A}_X}^{-1} = \mathbf{L}_{\langle \mathcal{O}_X, ..., \mathcal{O}_X(n+1-d)} \circ S_{D^b(X)}^{-1}$$

Proposition 0.5. $\mathbf{L} = T^{n+2-d}[-n]$ *for* $T = \mathbf{L}_{\mathcal{O}_X} \circ (\mathcal{O}_X(1) \otimes -)|_{\mathcal{A}_X}$.

Lemma 0.6. *Given an auto-equivalence, $\varphi : \mathcal{D} \to \mathcal{D}$ and $\mathcal{D}_0 \subset \mathcal{D}$ admissible, we have*

$$\varphi \circ \mathbf{L}_{\mathcal{D}_0} = \mathbf{L}_{\varphi(\mathcal{D}_0)} \circ \varphi$$

So then

$$\begin{aligned} T^{n+2-d} &= \mathbf{L}_{\mathcal{O}_X} \circ (\mathcal{O}_X(1) \otimes -) \circ ... \circ \mathbf{L}_{\mathcal{O}_X(1)\otimes -} \\ &= \mathbf{L}_{\mathcal{O}_X} \circ ... \circ \mathbf{L}_{\mathcal{O}_X(n+1-d)} \circ (\mathcal{O}_X(n+2-d) \otimes -) \\ &= S_{D^b(X)}^{-1}[n] \end{aligned}$$

And finally we have:

Proposition 0.7. $T^d = [2]$.

□

Letting $c = \gcd(d, n+2)$, the fractional dimension of $\mathcal{A}_X$ is $\left(\frac{(n+2)(d-2)}{c} / \frac{d}{c}\right)$-Calabi-Yau.

Example. For cubic surfaces, $\mathcal{A}_X$ is (4/3)-Calabi Yau. For cubic threefolds, $\mathcal{A}_X$ is (5/3)-Calabi-Yau. For cubic fourfolds, $\mathcal{A}_X$ is 2-Calabi-Yau.

Proof of Proposition 0.7. Let us find the Fourier-Mukai kernel of T.

Lemma 0.8. *The Fourier-Mukai kernel of* $\mathbf{L}_{\mathcal{O}_X(i)}$ *is*

$$[\mathcal{O}_X(i) \boxtimes \mathcal{O}_X(-i) \to \mathcal{O}_{\Delta_X}]$$

Proof of Lemma 0.8. For $E \in D^b(X)$,

$$\mathcal{O}_X(i) \boxtimes \mathcal{O}_X(-i) \to \mathcal{O}_{\Delta_X} \to \mathcal{C}$$
$$E(i) \boxtimes \mathcal{O}_X(i) \to E_\Delta \to \mathcal{C} \otimes E$$

giving an exact triangle

$$\bigoplus H^m(E(-i)) \otimes \mathcal{O}_X(i)[-m] \to E \to \mathbf{L}_{\mathcal{O}_X(i)} E$$

□

Then we have that the Fourier-Mukai kernel of T is

$$[\mathcal{O}_X \boxtimes \mathcal{O}_X(1) \to \mathcal{O}_{\Delta_X}(1)]$$

Lemma 0.9. *If* $K_1, K_2 \in D^b(X \times X)$ *are Fourier-Mukai kernels, the compositions of their transforms has Fourier-Mukai kernel*

$$p_{13*}(p_{12}^* K_1 \otimes p_{23}^* K_2)$$

So the Fourier-Mukai kernel of T^2 is

$$\begin{aligned} &p_{13*}[\mathcal{O}_X \boxtimes \mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1) \to \mathcal{O}_{\Delta_{12}}(2) \boxtimes \mathcal{O}_X \oplus \mathcal{O}_X(1) \boxtimes \mathcal{O}_{\Delta_{23}}(1) \to \mathcal{O}_{\Delta_{123}}(2)] \\ &= [\mathcal{O}(1) \otimes \Omega^1_{\mathbf{P}^n}(1)|_X \to \mathcal{O}(2,0) \to \mathcal{O}_\Delta(-2)] \end{aligned}$$

And the Fourier-Mukai kernel of T^i is

$$K_i := [\mathcal{O}_X(1) \otimes \Omega^{i-1}_{\mathbf{P}}(i-1)|_X \to ... \to \mathcal{O}_X(i-1) \boxtimes \Omega^1_{\mathbf{P}}(1)|_X \to \mathcal{O}_X(i) \boxtimes \mathcal{O}_X \to \mathcal{O}_X(i)]$$

Observe that it is similar to the resolution of $\mathcal{O}_{\Delta_{\mathbf{P}^n}}(i)$.

So we have an exact triangles

$$\begin{array}{ccccc} \mathcal{O}_{\Delta_{\mathbf{P}^n}} \otimes \mathcal{O}_{X \times X}(d) & \longrightarrow & \mathcal{O}_{\Delta_X}(d) & \longrightarrow & \mathcal{O}_{\Delta_X}[2] \\ \uparrow & & \uparrow = & & \uparrow \\ K_{d'} & \longrightarrow & \mathcal{O}_{\Delta_X}(d) & \longrightarrow & K_d \end{array}$$

where $K_{d'} = [\mathcal{O}_X(1) \boxtimes ...\mathcal{O}_X(d) \boxtimes \mathcal{O}_X]$.

Claim: $K_{d'}$ and $\mathcal{O}_{\Delta_{\mathbf{P}^n}} \otimes \mathcal{O}_{X \times X}(d)$ induce the same Fourier-Mukai transform on $\mathcal{A}_X$.

Thus K_d, the Fourier-Mukai kernel of T^d, induces the same Fourier-Mukai transform as $\mathcal{O}_{\Delta_X}[2]$, which is [2] as desired. □

0.3 Cubic fourfolds

Theorem 0.10 (Bernardara-Macri, Mehrotra-Stellari). *Let Y and Y' be (smooth) cubic threefolds. Then $\mathcal{A}_Y \simeq \mathcal{A}_{Y'}$ implies $Y \simeq Y'$.*

Theorem 0.11 (Huybrechts). *Let X and X' be cubic fourfolds. If X X' are not special, then $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ implies $X \simeq X'$.*

Not true without generality assumption.

Conjecture 0.12. *$\mathcal{A}_X \simeq \mathcal{A}_{X'}$ implies $X \sim_{\text{bir}} X'$.*

Conjecture 0.13 (Kuznetsov). *A cubic 4-fold is rational if and only if $\mathcal{A}_X \simeq D^b(S)$ for S a $K3$ surface.*

Now we do some setup for a conjecture of Hassett. For a cubic fourfold, $\dim H^{3,1} = \dim H^{1,3} = 1$,

$$h^2 \in H^{2,2}(X, \mathbf{Z}) = H^{2,2}(X) \cap H^4(X, \mathbf{Z})$$

where h is a hyperplane section.

Definition 0.14. X is special if $\operatorname{rank}(H^{2,2}(X, \mathbf{Z})) > 1$.

A labeled special cubic fourfold is an assignment of a primitive sublattice $K \subset H^{2,2}(X, \mathbf{Z})$ of rank 2 with $K \ni h^2$. Let $d = \operatorname{discr}(K)$. If $d \geq 8$ and $d \equiv 0, 2 \bmod 6$, then C_d is non-empty and irreducible, where C_d is the moduli space of labeled cubic 4-folds.

Next, $\operatorname{rk}(K^\perp) = 21$, where the orthogonal is taken in $H^4(X, \mathbf{Z})$.

Definition 0.15. A cubic 4-fold has an associated $K3$, if there exists (S, ℓ) with $\ell \in H^2(S, \mathbf{Z})$ an ample class such that $K^\perp \subset H^4(X, \mathbf{Z})$ is isomorphic to $\ell^\perp \subset H^2(S, \mathbf{Z})$.

Proposition 0.16. *X has an associated $K3$ if and only if d is not divisible by 4, 9, or any prime number that is $\equiv 2$ mod 3.*

Conjecture 0.17 (Hassett). *X is rational if and only if it has an associated $K3$.*

So we have two perspectives on conjectured rationality: Hodge theoretic, and derived categorical. The following theorem shows that these conjectures are equivalent.

Theorem 0.18 (Addington-Thomas). *X has an associated $K3$ if and only if $\mathcal{A}_X \simeq D^b(S)$.*

Now let's look at the example of $d = 8$. It turns out in this case (and only in this case), the corresponding cubic fourfold X contains a plane $\mathbf{P}^2 \subset X$.

Let $P \subset X$ be a plane. We construct a rational projection $X \dashrightarrow \mathbf{P}^2$ undefined along P, giving a diagram:

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{P}^2 \\ \uparrow & \nearrow & \\ \mathrm{Bl}_P X & & \end{array}$$

where $\mathrm{Bl}_P X \to \mathbf{P}^2$ is a $\dim 2$ quadric bundle.

Lemma 0.19. *The locus in* $\mathbf{P}^2$ *where the fibers of* $\mathrm{Bl}_P X \to \mathbf{P}^2$ *is singular is a sextic.*

Now, consider the relative Fano scheme of lines $F(X) \to \mathbf{P}^2$ on this quadric bundle. The smooth fibers are smooth quadrics $\mathbf{P}^1 \times \mathbf{P}^1$, for which the space of lines is $\mathbf{P}^1 \sqcup \mathbf{P}^1$, and the singular fibers are quadric cones for which the space of lines is $\mathbf{P}^1$. Taking the Stein factorization, we have

$$\begin{array}{ccc} F(X) & \longrightarrow & \mathbf{P}^2 \\ {\scriptstyle \mathbf{P}^1}\big\downarrow & \overset{2:1}{\nearrow} & \\ S & & \end{array}$$

where S is a $K3$ surface, $F(X) \to S$ is a $\mathbf{P}^1$ bundle, and $S \to \mathbf{P}^2$ is a double cover ramified in the sextic.

From the $\mathbf{P}^1$-bundle over S, we get a Brauer class $\alpha \in \mathrm{Br}(S)$. It turns out:

Theorem 0.20. $\mathcal{A}_X \simeq D^b(S, \alpha)$.

Now, suppose there are disjoint planes $P_1, P_2 \subset X$. Then we deduce:

1. X is rational, as we can define $P_1 \times P_2 \dashrightarrow X$ by sending any (x, y) to $\ell \cap X$ where ℓ is the line between x and y.

2. From a section of $\mathrm{Bl}_{P_1} X \to \mathbf{P}^2$ we get a section of $F(X) \to S$, which means the Brauer class is trivial, i.e., $\alpha = 0$.

This confirms Hassett's conjecture in this case!