Semi-orthogonal Decompositions Seminar Notes

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Let X be a normal rational (projective) surface with rational singularities.

We want to understand semi-orthogonal decompositions

$$D^b_{\mathrm{Coh}}(X) = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$$

We would like for $\mathcal{A}_i$ to be generated by exceptional objects, but that might be too much. So instead we hope $\mathcal{A}_i \sim D^b(K_i\text{-mod})$ for K_i a finite local algebra, e.g., $k[x]/(x^2)$.

If X has cyclic quotient singularities, we can do this.

Let $\pi : Y \to X$ be a resolution of singularities, a proper birational map. Because X is rational, we have $\pi_* \mathcal{O}_Y \simeq \mathcal{O}_X$ (all functors are derived).

Let D be the exceptional locus. We have D is a disjoint union of trees of smooth rational curves.

Take a semi-orthogonal decomposition

$$D^b(Y) = \langle \widetilde{\mathcal{A}}_1, ..., \widetilde{\mathcal{A}}_n \rangle$$

compatible with π. This means that if E_i is an irreducible component of D, then $\mathcal{O}_{E_i}(-1) \in \widetilde{\mathcal{A}}_j$. Then we will get

$$\widetilde{A}_j / \langle \mathcal{O}_{E_i}(-1) \rangle_{i : \mathcal{O}_{E_i}(-1) \in \widetilde{A}_j} \simeq \mathcal{A}_i \subset D^b(X)$$

giving a semi-orthogonal decomposition on X.

We will go about proving this. Preliminaries. Since X is singular, π^* does not necessarily preserve boundedness. Instead, we have

$$\pi^* : D^-(X) \to D^-(Y), \quad \pi_* : D^-(Y) \to D^-(X)$$

From $\pi_* \circ \pi^* \simeq \mathrm{id}_{D^-(X)}$ we get a semi-orthogonal decomposition

$$D^-(Y) = \langle \ker \pi_*, \pi^* D^-(X) \rangle$$

We assume we have a semi-orthogonal decomposition $D^-(Y) = \langle \widetilde{\mathcal{A}}_1^-, ..., \widetilde{\mathcal{A}}_n^- \rangle$ compatible with π. We want to combine these decompositions and then bring them down to $D^b(X)$.

Lemma 0.1. *Let E, E' be irreducible components of the exceptional locus D, and $\mathcal{O}_E(-1) \in \widetilde{\mathcal{A}}_i$ and $\mathcal{O}_{E'}(-1) \in \widetilde{\mathcal{A}}_j$. Then $E \cap E' = \emptyset$.*

Proof. For $j > i$, we have $\mathrm{Ext}^\bullet(\mathcal{O}_{E'}(-1), \mathcal{O}_E(-1)) = 0$. By the Riemann Roch formula for the intersection, $E \cdot E'$ equals $\chi(\mathcal{O}_{E'}(-1), \mathcal{O}_E(-1))$, and the latter is zero. $\square$

Now, partition the exceptional locus $D = \bigcup D_i$ where $D_i = \{E_i : \mathcal{O}_{E_i}(-1) \in \mathcal{O}_{E'}(-1), \widetilde{\mathcal{A}}_i^-\}$. Note that there is a way to extend a semi-orthogonal decomposition $D^b(Y) = \langle \widetilde{\mathcal{A}}_1, ..., \widetilde{\mathcal{A}}_n \rangle$ to $D^-(Y) = \langle \widetilde{\mathcal{A}}_1^-, ..., \widetilde{\mathcal{A}}_n^- \rangle$ so that $\widetilde{\mathcal{A}}_i^- \cap D^b(Y) = \widetilde{\mathcal{A}}_i$.

Lemma 0.2.

1. *$\widetilde{\mathcal{A}}_i^- \cap \ker(\pi_*) = \langle \mathcal{O}_{E_i} \rangle_{E_i \in D_i}$ allowing infinite direct sums.*
2. *$\mathcal{F} \in \ker(\pi_*)$ satisfies $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ with $\mathcal{F}_i \in \widetilde{\mathcal{A}}_i \cap \ker \pi_*$.*
3. *$\mathrm{Ext}^\bullet(\widetilde{\mathcal{A}}_i^-, \widetilde{\mathcal{A}}_j^- \cap \ker(\pi_*)) = 0$ if $i < j$ and π is crepant along D_j or $i > j$.*

Brief aside on crepancy. Say $Y \to X$ is a crepant resolution if $\pi^* K_X = K_Y$, i.e., $K_Y \cdot E = 0$. Say π is crepant along D_j if $K_Y \cdot E_i = 0$ for all $E_i \in D_j$.

We have projection functors $\widetilde{\alpha}_i : D^-(Y) \to D^-(Y)$ whose essential image is $\widetilde{\mathcal{A}}_i^-$. It follows formally $\widetilde{\alpha}_i|_{D^b(Y)} = \widetilde{\alpha}_i$, where the right hand side is the projection functor for $D^b(Y)$.

We have $\pi_* \widetilde{\mathcal{A}}_i^- = \widetilde{\mathcal{A}}_i^- \subset D^-(X)$.

Theorem 0.3.

1. *We have a semi-orthogonal decomposition*
$$D^-(X) = \langle \mathcal{A}_1^-, ..., \mathcal{A}_n^- \rangle$$
with projection functors $\alpha_i = \pi_* \circ \widetilde{\alpha}_i \circ \pi^*$.
2. *$\pi^*(\mathcal{A}_i^-) \subset \langle \widetilde{\mathcal{A}}_i^-, \widetilde{\mathcal{A}}_{i+1}^- \cap \ker \pi_*, ..., \widetilde{\mathcal{A}}_n^- \cap \ker \pi_* \rangle$.*
3. *If in addition π is crepant along D_j for $j > i$, then $\pi^*(\widetilde{\mathcal{A}}_i^-) \subset \widetilde{\mathcal{A}}_i^-$.*

Proof sketch. Let $\mathcal{F} \in \widetilde{\mathcal{A}}_i^-$. Define $\mathcal{F}'$ by

$$\pi^* \pi_* \mathcal{F} \to \mathcal{F} \to \mathcal{F}'$$

Since $\pi_* \pi^* = \mathrm{id}$, applying π^* to this triangle yields $\pi_* \mathcal{F}' = 0$, so $\mathcal{F}' \in \ker \pi_*$.

By the lemma we have $\mathcal{F}' = \bigoplus_{m=1}^n \mathcal{F}_m'$ with $\mathcal{F}_i' \in \widetilde{\mathcal{A}}_i \cap \ker \pi_*$. We have $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F}_j') = 0$ for $j < i$ and $\mathrm{Ext}^\bullet(\pi^* \pi_* \mathcal{F}, \mathcal{F}_j') = 0$ for $j < i$. Thus, $\mathrm{Hom}(\mathcal{F}', \mathcal{F}_j') = 0$ for $j < i$ and so $\mathcal{F}_j' = 0$ for $j < i$.

Thus, $\mathcal{F}' \subset \langle \widetilde{\mathcal{A}}_i^- \cap \ker \pi_*, ..., \widetilde{\mathcal{A}}_n \cap \ker \pi_* \rangle$. And by assumption $\mathcal{F} \subset \widetilde{\mathcal{A}}_i^-$, so $\pi^* \pi_* \mathcal{F} \subset \langle \widetilde{\mathcal{A}}_i^- \cap \ker \pi_*, ..., \widetilde{\mathcal{A}}_n \cap \ker \pi_* \rangle$. Then any element of $\widetilde{\mathcal{A}}_i^-$ is $\pi^* g$ for $g \in \mathcal{A}_i^-$.

To see that $\mathcal{A}_i^-$ is triangulated, for any $g \in \mathcal{A}_i^-$ we can take

$$g' \to \pi^* g \to \widetilde{\alpha}_i(\pi^* g)$$

Applying π_* we get $\pi_* g' = 0$ and $g \cong \pi_* \pi^* g \cong (\pi_* \circ \widetilde{\alpha}_i \circ \pi^*)(g)$ which we call $\alpha_i(g)$.

Semi-orthogonality follows formally, and pushing forward filtrations works as well. $\square$

Now we want to return to the bounded derived category.

Theorem 0.4. *1. $\mathcal{A}_i := \mathcal{A}_i^- \cap D^b(X)$ gives a semi-orthogonal decomposition with projection functors α_i.*

2. α_i preserves boundedness.

3. $\mathcal{A}_i \simeq \pi_*(\widetilde{A}_i)$ and π_* induces equivalence of triangulated categories.

Proof sketch. We need to check boundedness of α_i. Let $\mathcal{F} \in D^{[k_-,k_+]}(X)$, so $\pi^*\mathcal{F} \in D^{(-\infty,k_+]}(Y)$. Take truncation

$$\tau^{\leq k-2}(\pi^*\mathcal{F}) \to \pi^*\mathcal{F} \to \tau^{k-1}(\pi^*\mathcal{F})$$

Then apply $\widetilde{\alpha}_i$

$$\widetilde{\alpha}_i(\tau^{\leq k-2}(\pi^*\mathcal{F})) \to \widetilde{\alpha}_i\pi^*\mathcal{F} \to \widetilde{\alpha}_i\tau^{k-1}(\pi^*\mathcal{F})$$

Then $\widetilde{\alpha}_j$ sends $\ker \pi_*$ to itself. So $\pi_*(\tau^{k-2}\pi^*\mathcal{F}) = 0$. □

Now let's apply this to cyclic quotient singularities. Consider weighted projective space $\mathbf{A}^n/\mu^r$, or $\mathbf{P}(1,1,d)$ for $d > 1$. Let $\pi : Y \to X$ be a resolution with E the exceptional locus.

There exists a semi-orthogonal decomposition

$$D^b(Y) = \langle \widetilde{\mathcal{A}}_1, ..., \widetilde{\mathcal{A}}_n \rangle$$

with each $\widetilde{\mathcal{A}}_i = D^b(\mathrm{End}(\bigoplus \text{vector bundles}))$.

By Hille-Ploog we have $\pi_*\widetilde{\mathcal{A}}_i = D^b(K_i\text{-alg})$ for K_i a finite local algebra.

For $X = \mathbf{P}(1,1,d)$, there is a resolution by $Y = \mathbf{F}_d = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O}(d))$. Let $E, C, F \in \mathrm{Pic}(\mathbf{F}_d)$ with $E \cap C = \emptyset$, $E^2 = -d$, $C^2 = d$, $F^2 = 0$, and $K_{\mathbf{F}_d} = -E - C - 2H$. Then there is a semi-orthogonal decomposition

$$D^b(\mathbf{F}_d) = \langle \mathcal{O}(-H-E), \mathcal{O}(-H), \mathcal{O}, \mathcal{O}(C) \rangle$$

Let $\widetilde{\mathcal{A}}_1 = \langle \mathcal{O}(-H-E), \mathcal{O}(-H), \mathcal{O}, \mathcal{O}(C) \rangle$, $\widetilde{\mathcal{A}}_2 = \langle \mathcal{O} \rangle$, and $\widetilde{\mathcal{A}}_3 = \langle \mathcal{O}(C) \rangle$. We have $\mathcal{O}_E(-1) \in \widetilde{\mathcal{A}}_1$.

Now let $\mathcal{A}_i = \pi_*\widetilde{\mathcal{A}}_i$ so

$$D^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$$

where $\mathcal{A}_1 = \langle R \rangle$ for

$$R = \pi_*(\mathcal{O}(-H-E)) \cong \pi_*\mathcal{O}(-H)$$

and $\mathcal{A}_2 = \langle \mathcal{O} \rangle$ and $\mathcal{A}_3 = \langle \mathcal{O}(d) \rangle$.

Finally, let's look at the Brauer obstruction. If $\mathcal{T} = \langle D^b(K_1\text{-mod}), ... \rangle$ we have $K_0(\mathcal{T}) = \bigoplus K_0(D^b(K_i\text{-mod}))$, which is torsion free. But $K_0(D^b(X))_{\mathrm{tors}} = \mathrm{Br}(X) = H^2(X, \mathbf{G}_m)_{\mathrm{tor}}$.