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Lecture Eight (Hechen Hu, Generation of Derived Categories, 3/26/25)

6.1 Preliminaries

Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}$ an object. Denote by $\langle E \rangle_1$ the strictly full (i.e. closed under isomorphism classes) subcategory containing objects isomorphic to direct summands of finite direct sums $\bigoplus_i E[n_i]$ of shifts. For $n > 1$, let $\langle E \rangle_n$ denote the full subcategory consisting of direct summands of objects X fitting into the triangle

$$A \to X \to B \to A[1]$$

with $A \in \langle E \rangle_1$ and $B \in \langle E \rangle_{n-1}$. Each $\langle E \rangle_n$ is closed under taking direct summands and shifts but not necessarily cones, hence they are not triangulated subcategories. However, the category $\langle E \rangle \coloneqq \bigcup_n \langle E \rangle_n$ is a triangulated subcategory. Call a subcategory *thick* if it is closed under taking direct summands.

Proposition (Lemma 0ATG)**.** $\langle E \rangle$ is the smallest strictly full and thick triangulated subcategory of $\mathcal{D}$.

Definition (Definition 09SJ)**.**

1. E is a *classical generator* if $\langle E \rangle = \mathcal{D}$;
2. E is a *strong generator* if $\langle E \rangle_n = \mathcal{D}$ for some n;
3. E is a *weak generator* (or *generator*) if for any nonzero object K there is some n such that $\mathrm{Hom}_{\mathcal{D}}(E, K[n]) \neq 0$.

This can be generalized to a collection of objects $\mathcal{E}$. For example, weak generation means that $\mathcal{E}^{\perp} = 0$.

Example. If E_i is an collection of exceptional objects forming a semiorthogonal decomposition of $\mathcal{D}$, then $\bigoplus_i E_i$ is a classical generator.

Remark. In the definition of a generator we can also shift in E, i.e. consider $\mathrm{Hom}_{\mathcal{D}}(E[n], K)$.

Remark. A classical generator is always a generator. If the category has a strong generator, then a classical generator is also strong.

6.2 Generation by Perfect Objects

For a scheme, let $D(\mathcal{O}_X)$ be the derived category of complexes of $\mathcal{O}_X$-modules and $D_{\text{QCoh}}(\mathcal{O}_X)$ the subcategory consisting of complexes with quasi-coherent cohomology sheaves. Recall that a complex (of $\mathcal{O}_X$-modules) is *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles. From now on, assume that X is quasi-compact and quasi-separated. The main theorem we wish to prove is $D_{\text{QCoh}}(\mathcal{O}_X)$ is generated by a single perfect object.

Remark. $D_{\text{QCoh}}(\mathcal{O}_X) \cong D(\text{QCoh}(\mathcal{O}_X))$ when X is separated.

6.2.1 The Affine Case

Let $X = \operatorname{Spec} A$ be an affine scheme.

Proposition (Lemma 06Z0)**.** All functors in the triangle

$$\begin{array}{ccccc} & & D_{\text{QCoh}}(\mathcal{O}_X) & & \\ & \overset{\text{inclusion}}{\nearrow} & & \overset{R\Gamma(X,-)}{\searrow} & \\ D(\text{QCoh}(\mathcal{O}_X)) & & \xleftarrow[\sim]{\qquad\qquad} & & D(A) \end{array}$$

are equivalences of triangulated categories. Moreover, for all $E \in D_{\text{QCoh}}(\mathcal{O}_X)$ one has that $H^0(X; E) = H^0(X; H^0(E))$.

Sketch of a Proof. The functors $R\Gamma(X, -)$ and $\sim$ are adjoints (they are identified as $R\pi_*$, $L\pi^*$ with $\pi\colon (X, \mathcal{O}_X) \to (*, A)$). Thus we have the unit-counit maps

$$\begin{aligned} \widetilde{R\Gamma(X, E)} \to E, &\qquad E \in D_{\text{QCoh}}(\mathcal{O}_X) \\ M^\bullet \to R\Gamma(X, \widetilde{M^\bullet}), &\qquad M^\bullet \in D(A) \end{aligned}$$

which can be proven to be quasi-isomorphisms. □

Thus $\mathcal{O}_X$ is the perfect object generating $D_{\text{QCoh}}(\mathcal{O}_X)$.

6.2.2 The General Case

Note the following fact: if the cohomology sheaves of a complex E is supported in a closed subset $T \subset X$, then $E|_{X-T}$ is exact so it is zero in the derived category. By abuse of notation we will write $E|_{X-T} = 0$ and say that E is *supported* in E.

Lemma (Lemma 08DF)**.** *Temporarily we will work over general ringed spaces $(X, \mathcal{O}_X)$. Let $j\colon U \to X$ be an open subspace and $T \subset X$ a closed subset containing in U. For objects (in $D(\mathcal{O}_U)$ or $D(\mathcal{O}_X)$) supported on T, one has the isomorphisms $(-) \cong Rj_*(-|_U)$ and $j_!(-) \cong Rj_*(-)$, $j_!$ the direct image with proper support (in the open immersion case this is the same as extension by zero).*

Proof Sketch. Let $V = X - T$ and look at the sheaves over the open cover U, V. □

Lemma (Section 08CL)**.** *Again we work over general ringed spaces.*

1. *If $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a morphism of ringed spaces, the Lf^* preserves perfectness.*

2. *If any two of the elements in a distinguished triangle in $D(\mathcal{O}_X)$ is perfect, so is the third.*

3. *Perfectness is preserved under $\overset{L}{\otimes}$ and $\oplus$;*

4. *Let $j\colon U \to X$ be an open subspace. If $E \in D(\mathcal{O}_U)$ is a perfect object supported on a closed subset $T \subset U$ such that $j(T)$ is also closed in X, then Rj_*E is also perfect.*

Proof of the Last Statement. Since perfectness is local, suffices to prove that $R_{j_*}E|_U$ and $R_{j_*}E|_V$, $V = X - j(T)$, are perfect. Now observe that $R_{j_*}E|_U = E$ and $R_{j_*}E|_V = 0$. □

Lemma (Lemma 05QT)**.** *Consider a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$. If g has a right inverse $s\colon Z \to Y$, then $f \oplus s\colon X \oplus Z \to Y$ is an isomorphism.*

Proof. One can prove that in a distinguished triangle the composition of two consecutive maps are zero. The existence of s says that $Z \to X[1]$ is the zero map so there is a split short exact sequence

$$0 \to \operatorname{Hom}(-,X) \to \operatorname{Hom}(-,Y) \to \operatorname{Hom}(-,Z) \to 0$$

and we can conclude by Yoneda. □

Now going back to schemes.

Lemma (Lemma 08EG)**.** *Let $X = \operatorname{Spec} A$ be an affine scheme and U a quasi-compact open subscheme. For every perfect complex $E \in D(\mathcal{O}_U)$ there is an integer r and a finite locally free sheaf $\mathcal{F}$ such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object in $D(\mathcal{O}_X)$.*

Proof. By Lemma 08EE there is a bounded above complex $\mathcal{F}^\bullet$ of finite free A-modules such that $\mathcal{F}^\bullet|_U$ is isomorphic to E in the derived category. Because U is quasi-compact and E is locally of finite tor dimension, in fact E has finite tor dimension. Say that the tor groups are zero for any $i \notin [a,b]$ and pick $r < a$. Set

$$\mathcal{K} := \operatorname{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \operatorname{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r)$$

and $\mathcal{F} = \mathcal{K}|_U$. By our choice of r the sheaf $\mathcal{F}$ is flat and of finite presentation, thus is finite locally free. Then $\mathcal{F} \to \mathcal{F}^r|_U \to \mathcal{F}^{r+1}|_U \to \cdots$ is a bounded complex of vector bundles isomorphic to E. If we let $P = (\mathcal{F}^r \to \mathcal{F}^{r+1})$, then by truncation there is a distinguished triangle

$$\mathcal{F}[-r-2] \to P|_U \to E \to \mathcal{F}[-r-1]$$

If we choose $r \ll 0$, then $E \to \mathcal{F}[-r-1]$ is necessarily zero so one has a splitting $P|_U \cong \mathcal{F}[-r-2] \oplus E$ (representable functors are cohomological, see Lemma 05QT). □

The preceding lemma also holds when X is quasi-compact and quasi-separated, however we will not give a proof, see Lemma 09IQ.

Lemma (Lemma 09IR)**.** *Let $j\colon U \to X$ be an inclusion of open subscheme into $X = \operatorname{Spec} A$ such that $U = D_{f_1} \cup \cdots \cup D_{f_r}$ for $f_i \in A$. Let $K \in D(\mathcal{O}_X)$ correspond to the Koszul complex. For $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. $E = Rj_*(E|_U)$;

2. $\operatorname{Hom}(K[n], E) = 0$ *for all* $n \in \mathbb{Z}$.

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. By adjunction

$$\operatorname{Hom}(K[n], Rj_*(E|_U)) \cong \operatorname{Hom}(K|_U[n], E) = 0$$

for all $n \in \mathbb{Z}$ because $K|_U = 0$. It suffices to prove the statement for N, i.e. assume that $E|_U = 0$. Let e_i be natural numbers. There are distinguished triangles

$$K^\bullet(f_1^{e_1}, \cdots, f_i^{e_i'}, \cdots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \cdots, f_i^{e_i'+e_i''}, \cdots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \cdots, f_i^{e_i''}, \cdots, f_r^{e_r}) \to \cdots$$

so if we let $K_e := K(f_1^e, \cdots, f_r^e)$, $e \geqslant 0$, then $\operatorname{Hom}(K[n], E) = 0$ implies that $\operatorname{Hom}(K_e[n], E) = 0$ for all e. By Lemma 08E3, in this setting ($H^i(E)|_U = 0$) given any $s \in H^0(X; E)$ there is $e \geqslant 0$ and a morphism $K_e \to E$ such that s is in the image of $H^0(X; K_e) \to H^0(X; E)$. Combining these two statements, it follows that $R\Gamma(X, E)$ is exact so $E = 0$. □

Remark. The last statement is easier to see in the case E is a complex with only one term in degree 0 and $U = D_f$ is a principal open subset. In this case $K_e = (A \xrightarrow{\cdot f^e} A)$. The condition $H^0(E)|_U = 0$ says that E is supported in $X - U$. The statement is saying that given any global section $s \in \Gamma(X; E)$ there is some e such that there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \Gamma(X, E) \\ {\scriptstyle \cdot f^e}\uparrow & & \uparrow \\ A & \longrightarrow & 0 \end{array}$$

with the top row given by sending 1 to a unit times s. The commutativity asserts that $f^e s = 0$, which is classically known.

Remark. The preceeding lemma shows that K is a perfect generator of the kernel of the restriction map $D_{\text{QCoh}}(\mathcal{O}_X) \to D_{\text{QCoh}}(\mathcal{O}_U)$.

Theorem 6.2.1 (Theorem 09IS)**.** *There exists a perfect object P of $D(\mathcal{O}_X)$ such that the following are equivalent for all $E \in D_{QCoh}(\mathcal{O}_X)$:*

1. $E = 0$;

2. $\operatorname{Hom}(P[n], E) = 0$ *for all* $n \in \mathbb{Z}$.

Thus $D_{QCoh}(\mathcal{O}_X)$ is generated by a single perfect object.

Proof. The affine case is already know. By induction it suffices to do the case $X = U \cup V$, U quasi-compact and $V = \operatorname{Spec} A$ affine, such that the theorem holds for U. Let $P \in D_{\text{QCoh}}(U)$ be a perfect generator. By the lemma (that we never proved) there is a perfect object $Q \in D(\mathcal{O}_X)$ with P a direct summand of $Q|_U$. Let $Z = X - U$ so that Z is a closed subset of V with quasi-compact complement $V - Z$. Then we can choose $f_1, \cdots, f_r \in A$ such that $Z = V(f_1, \cdots, f_r)$. Let $K \in D(\mathcal{O}_X)$ be the perfect object corresponding to the Koszul complex of f_i over A. Since K is supported on $Z \subset V$ closed, $K' := Rj_{V,*}K$, $j_V \colon V \to X$, is also perfect. We claim that $Q \oplus K'$ is a generator for $D_{\text{QCoh}}(\mathcal{O}_X)$.

Let $E \in D_{\text{QCoh}}(\mathcal{O}_X)$ such that $\text{Hom}((Q \oplus K')[n], E) = 0$ for all $n \in \mathbb{Z}$. If we can show that $E = Rj_{U,*}E|_U$, then we are done because then

$$\text{Hom}(Q[n], E) \cong \text{Hom}(Q|_U[n], E|_U)$$

and $\text{Hom}(Q|_U[n], E|_U)$ contains $\text{Hom}(P[n], E|_U)$ as a direct summand.

The proof that $E = Rj_{U,*}E|_U$ uses a Mayer-Vietoris type argument. By another lemma $Rj_{V,*}K \cong j_{V,!}K$ so one has an adjunction

$$\text{Hom}(K'[n], E) \cong \text{Hom}(K, E|_V) = 0$$

Then the previous lemma says that $E|_V \cong Rj_{U\cap V,*}E|_{U\cap V}$. For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have the Mayer-Vietoris sequence

$$0 \to \mathcal{F} \to j_{U,*}\mathcal{F}|_U \oplus j_{V,*}\mathcal{F}|_V \to j_{U\cap V,*}\mathcal{F}|_{U\cap V} \to 0$$

which gives a distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

c.f. Lemma 08GW. The isomorphism $E|_V \cong Rj_{U\cap V,*}E|_{U\cap V}$ gives a section of $Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V}$, which by a lemma gives the desired isomorphism

$$E \to Rj_{U,*}E|_U$$

□

Example. Consider the case of $X = \mathbb{P}^n_A$. We claim that $P = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-n)$ is a generator. Twisting the Koszul complex for x_i in $A[x_i]$ by $\mathcal{O}_X(n+a)$ gives an exact complex

$$0 \to \mathcal{O}_X(a) \to \cdots \to \mathcal{O}_X(a+i)^{\binom{n+1}{i}} \to \cdots \to \mathcal{O}_X(a+n+1) \to 0$$

c.f. Lemma 0BQS. The case $a = -n-1$ says that $\mathcal{O}_X(-n-1) \in \langle P \rangle$ and by induction $\mathcal{O}_X(-m) \in \langle P \rangle$ for all $m \geqslant 0$. By adjunction

$$\begin{aligned}
\text{Hom}(\mathcal{O}_X(-m), E[p]) &= \text{Ext}^{-p}(\mathcal{O}_X(-m), E) \\
&\cong \text{Ext}^{-p}(\mathcal{O}_X, R\text{Hom}(\mathcal{O}_X(-m), E)) \\
&\cong H^{-p}(X; R\text{Hom}(\mathcal{O}_X(-m), E)) \\
&\cong H^{-p}(X; E \overset{L}{\otimes} \mathcal{O}_X(m)) \\
&\cong H^{-p}(X; E \overset{L}{\otimes} \mathcal{O}_X(1)^{\otimes m})
\end{aligned}$$

so the last cohomology group is zero for all $p \in \mathbb{Z}$ and all $m \geqslant 0$. However, since $\mathcal{O}_X(1)$ is ample, this forces $E = 0$, c.f. Lemma 0BQR.

Example (Remark 0BQT)**.** If $f\colon X \to Y$ is a map between quasi-compact and quasi-separated schemes, then by adjunction we have the following equivalences:

1. Rf_* is injective, i.e. $Rf_*K = 0$ if and only if $K = 0$.
2. Lf^* takes generators to generators.

These two conditions hold when f is a composition of affine morphisms or open immersions. In particular,

1. If X is quasi-affine, then $\mathcal{O}_X$ is a generator;
2. If $X \subset \mathbb{P}^n_A$ is a quasi-compact locally closed subscheme, then $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-n)$ is a generator.