SEMI-ORTHOGONAL DECOMPOSITIONS SEMINAR

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1. Summary

In this seminar we study work on the derived categories of coherent sheaves on schemes, particularly focusing on methods related to semi-orthogonal decompositions. After building up the basic theory of derived categories, we will examine various cases of semi-orthogonal decompositions arising in geometry. Finally, we will see some examples of how these techniques are used in modern algebraic geometry research, with the specifics flexible depending on participant interest!

2. Introduction

The study of derived categories is an exciting topic. Historically, they were introduced as an homological tool that allowed us to study derived functors on algebraic varieties, which we quickly recall.

Recall (cf. [Har77, Ch. III]) that given a coherent sheaf F on a scheme X, we define its *cohomology* $H^i(X, F)$ by taking an injective resolution, applying the global sections functor, and taking cohomology of the resulting complex. Taking injective resolutions is a bit tricky: they are not necessarily isomorphic, but *quasi-isomorphic*. The quasi-isomorphism is not unique, but unique up to *homotopy*.

Following these ideas, Jean-Louis Verdier introduced the *derived category* $\mathrm{D}(\mathcal{A})$ as a place to run these computations. Informally, its objects are complexes of objects in an abelian category $\mathcal{A}$, and morphisms are defined up to homotopy, and formally inverting quasi-isomorphisms. As such, it helps to make various statements about derived functors more precise.

Many years later, people started to realize that the derived category of an algebraic variety contains a lot of geometric data of the variety itself. For example, Bondal and Orlov proved in [BO01] that $\mathrm{D}^b(\mathrm{Coh}(X))$ can recover X itself, provided that K_X is ample or anti-ample. Furthermore, the study of these derived categories has proven fruitful to many questions of interest to classical algebraic geometry, such as conjectured criteria for rationality in [Kuz10a] and connections with the minimal model program in [Bri02].

Studying these categories is tricky, as they tend to be complicated. In this seminar we will focus on *semi-orthogonal decompositions*, one of the most successful techniques to analyze triangulated categories. Informally, these play the role of realizing a category as an "extension" of two (or more) pieces. As in other cases, we can look at the smaller pieces to gather information about the original category.

Date: January 2025.

3. Plan

We will have weekly meetings each Wednesday, 1:00–2:30pm. The first two lectures will be focused on introducing derived categories and semi-orthogonal decompositions.

After that, we plan to discuss various examples on semi-orthogonal decompositions (projective bundles, blow-ups, Grassmannians, singular varieties, Kuznetsov components), and so on. At the same time, we will cover various technical tools that help manipulating these objects, such as mutations, Serre functors, various criteria for fully faithfulness, Hochschild cohomology, and so on.

At last, we will discuss various applications of semi-orthogonal decompositions to study various questions in algebraic geometry. Topics here include (categorical) resolutions of singularities, stability conditions, derived equivalences, and so on.

The distribution below is our suggestion. Please contact either of the organizers if you are interested in giving any of the talks (those with “TBA”), or if you want to suggest an additional topic. We welcome your ideas!

(1) **January 29$^{\text{th}}$**: A fast introduction to derived categories (Nicolás Vilches)
Construct the derived category of an abelian category, and derived functors. Include a bit of the technicalities (e.g. triangulated axioms) and explicit computations. References for this are [Huy06, Chapters 1–3] and [Har66].

(2) **February 5$^{\text{th}}$**: Introduction to semi-orthogonal decompositions (Amal Mattoo)
Define admissible subcategories, and introduce SOD. Explain the equivalence with fully faithful functors admitting an adjoint. Provide basic examples: $D^b(X) = \langle \ker Rf_*, Lf^* D^b(Y)\rangle$ if $f\colon X \to Y$ is smooth, exceptional objects. See [Huy06, §1.4] for example.

(3) **February 12$^{\text{th}}$**: SOD’s from geometry (Wenqi Li)
Discuss SOD’s for projective bundles and for blow-ups. Discuss explicit examples, e.g. the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, and compare it with the blow-up of $\mathbb{P}^2$ at two points. See [Beĭ83, Orl92] for the original presentations, and [Huy06, §11.1–2].

(4) **February 19$^{\text{th}}$**: More examples on SOD’s (TBA)
Some options include the following. First, discuss Kapranov’s SOD for the Grassmannian, following [Kap84]. Second, look at derived categories of quartic hypersurfaces (from [Kap86]) or intersection of quartic hypersurfaces (in [BO95, §2]). Third, introduce twisted derived categories (e.g. [Cal00]) and mention the semi-orthogonal decomposition of Brauer–Severi varieties from [Ber09]. (Of course, no need to do all of them!)

(5) **February 26$^{\text{th}}$**: Kuznetsov components (TBA)
Introduce Kuznetsov components for Fano varieties, following [Kuz10a] or [Huy23, §7.1]. Prove basic properties, mention cool theorems (e.g. categorical Torelli for cubic threefolds [Huy23, 7.2.4]), and mention some of the conjectures (e.g. that they should detect rationality of cubic fourfolds [Kuz10a]). Time permitting, give the description of some examples, e.g. [Huy23, §7.3.2].

(6) **March 5$^{\text{th}}$**: Mutations and Serre functors (TBA)
Introduce mutations following [BK89] or [Kuz10a, §2.1]. Highlight the use of Serre functors to produce adjoints (cf. [Huy06, §1.3]. Give examples

(e.g. go back to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, and try to relate the two SOD's; or [Kuz21] for other fun examples).

(7) **March 12th**: Surfaces with rational singularities (Vidhu Adhihetty)
Start by defining rational singularities and relating its derived category with a resolution (e.g. [KKS22, §2.1]). Explain why we need to work on the (left) unbounded derived category. Give basic descriptions of the kernel of Rf_* (in [KKS22, Lemma 2.1]). Mention how to use this, together with an SOD on the resolution, to get an SOD on the singular variety.

(8) **March 26th**: Computational results on SOD's (Hechen Hu)
Give computations of invariants on SOD's and relate them to the original category. Some options here include discussing how to construct generators of derived categories (cf. [BvdB03]), Hochshild cohomology (cf. [Kuz09], Serre functors on Kuznetsov components ([KP21]). Alternatively, discuss more technical tools, e.g. [Kuz10b] to produce SOD's under base change.

(9) **April 2nd**: Derived equivalences via SOD's (Yoonjoo Kim)
Show how SOD's are used to tackle derived equivalences problems. First, discuss the proof of the derived equivalence on the standard flop in [BO95, §3]. Second, continue the discussion in [BO95, §3] for three-dimensional flops. Third, discuss [Bri02] (especially the construction of a perverse t-structure and the perverse Hilbert space). (As usual, you can pick your favorites here!)

(10) **April 9th**: Categorical resolutions (Rafah Hajjar Muñoz)
Introduce categorical resolutions of singularities, and describe how to construct them. References are [KL15, KS23].

(11) **April 16th** Stability conditions and SOD's (Sofia Wood)
Introduce stability conditions, following e.g. [Bri07]. Give examples of stability conditions on Kuznetsov components, say via [BLMS23]. Alternatively (or also?) discuss [CP10].

(12) **April 23th**: TBD

(13) **April 30th**: TBD

As we mentioned above, participants are welcome to share and present other papers of interest. We have included some extra topics on the references.

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