33 lectures on quasimaps and elliptic stable envelopes

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Contents

Lecture 1. An overview from physics

An introductory discussion of: indices of supersymmetric operators, their relation to topological K-theory, K-theoretic Field Theories, moduli of vacua/Gibbs states, 3-dimensional mirror symmetry.

1.1

We begin with an overview of the contents of this course.

The overarching problem of interest is a generalization of an old (classical) problem in representation theory and harmonic analysis. Suppose a group G acts on a Riemannian manifold (M, g) by isometries, and let $\varphi \colon M \to \mathbb{R}$ be a G-invariant function. Then one can ask:

what are the eigenspaces of the Laplacian $-\Delta + \varphi$ as G-modules?

In mathematical physics, this corresponds to the propagation of a quantum particle on M. Knowing the spectrum of this Laplacian tells us about not just the time-evolution operator, but the time-evolution operator *in the presence* of the G-action, e.g. quantities such as

$$\mathrm{tr}\left(\mathrm{evolution}(T) \cdot g\right), \quad g \in \mathsf{G}.$$

There are very well-known classical cases of this problem: homogeneous spaces $M = \mathsf{G}/H$; $M = \mathbb{R}^n$ with the action of $\mathsf{G} = \mathrm{O}(n)$ by rotation and $\varphi = \varphi(r)$ is radial (which is just the study of spherical harmonics). But in general, this problem is difficult.

1.2

A great simplification to this problem occurs in the presence of supersymmetry. Supersymmetry means the following constraints.

- The Hilbert space
$$\mathcal{H} = \mathcal{H}_{\mathrm{even}} \oplus \mathcal{H}_{\mathrm{odd}}$$
is $\mathbb{Z}/2$-graded. Geometrically, $\mathcal{H}$ should be viewed as the space of sections of an "even" bundle $V_{\bar{0}}$ and an "odd" bundle $V_{\bar{1}}$ on M, namely
$$\mathcal{H}_{\mathrm{even}} = H^0(M, V_{\bar{0}}), \quad \mathcal{H}_{\mathrm{odd}} = H^0(M, V_{\bar{1}}).$$

- The Laplacian factors as a square

$$-\Delta + \varphi = (D + D^*)^2$$

  where D swaps $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$, and $D^2 = (D^*)^2 = 0$. Here D^* is the adjoint.

An example to keep in mind is when $V_{\bar{i}}$ are bundles $\Omega^{\text{even/odd}}M$ of differential forms on M, and D is the de Rham differential.

1.3

In this supersymmetric setting, let $\mathcal{H} = \bigoplus_\lambda \mathcal{H}^\lambda$ be its decomposition as a G-module into its G-eigenspaces. For $\lambda \neq 0$, an easy argument shows that

$$\cdots \xrightarrow{D} \mathcal{H}^\lambda_{\text{even}} \xrightarrow{D} \mathcal{H}^\lambda_{\text{odd}} \xrightarrow{D} \cdots \tag{1}$$

is an exact sequence of G-modules. Rephrasing, in the representation ring $R(\mathsf{G})$,

$$\mathcal{H}^\lambda_{\text{even}} \ominus \mathcal{H}^\lambda_{\text{odd}} = 0 \in R(\mathsf{G}).$$

On the other hand, when $\lambda = 0$, the space $\mathcal{H}^0$ of ground states creates a very interesting element

$$\text{Index} \coloneqq \mathcal{H}^0_{\text{even}} \ominus \mathcal{H}^0_{\text{odd}} \in R(\mathsf{G}).$$

Formally, Index is the same as $\mathcal{H}_{\text{even}} - \mathcal{H}_{\text{odd}}$, using the entire Hilbert space instead of just ground states. Equivalently,

$$\text{str}_{\mathcal{H}}(\text{evolution} \cdot g) = \text{tr}_{\text{Index}}(g).$$

Here str is the supertrace, and we used that time-evolution on ground states is just the identity operator. From this presentation, it is clear that Index is a topological invariant, as it is the index of an actual elliptic differential operator.

1.4

To compute the index, we can use the Atiyah–Singer formula. This is not the place to discuss the formula in detail, and instead we will give only a sense of its structure. It is written in terms of G-equivariant sheaves on M, in the language of equivariant K-theory. Equivariant K-theory is a functor

$$\begin{pmatrix} \text{topological space} \\ M \text{ with } G\text{-action} \end{pmatrix} \to \begin{pmatrix} \text{abelian groups} \\ K^i_{\mathsf{G}}(M) \text{ with product} \end{pmatrix}.$$

It is an example of a cohomology theory — a way to transform geometry into algebra. This is an essential tool for us, since the whole subject of geometric representation theory is about the interplay of geometry and algebra via such functors. We will discuss cohomology theories in more detail later.

Example. If M is compact and Hausdorff, then

$$K^0_{\mathsf{G}}(M) = \{\mathsf{G}\text{-equivariant vector bundles on } M\}.$$

One can think of this as representations of G, but indexed by points of M. More formally, a vector bundle V is G-equivariant if there is a G-action on its total space such that the G-action on M induces G-linear maps

$$V\big|_m \xrightarrow{m \mapsto g\cdot m} V\big|_{g\cdot m}$$

of the fibers of V. In particular, if M is just a point,

$$K^0_{\mathsf{G}}(\mathrm{pt}) = R(\mathsf{G})$$

is just the representation ring. In general, $K^0_{\mathsf{G}}(M)$ is a semi-ring with operations given by $\oplus$ and $\otimes$, and we make it into a ring by formally adding $\ominus$.

The other K-groups are in fact periodic, and involve suspensions of M.

1.5

The actual content of the Atiyah–Singer theorem is as follows. Recall that over M we had two vector bundles V_{even} and V_{odd} with an operator D going between them. Since the index is a virtual representation of the group G, it yields a map from this data landing in $K_{\mathsf{G}}(\mathrm{pt})$. The theorem is that there is an intermediate step consisting of an index *sheaf*

$$\mathcal{I}ndex \in K_{\mathsf{G}}(M)$$

such that its *pushforward* from $K_{\mathsf{G}}(M)$ to $K_{\mathsf{G}}(\mathrm{pt})$ computes exactly the index. In principle, this pushforward can be computed in cohomology via Riemann–Roch, which is a great theoretical tool. But in practice, pushforwards in K-theory are usually computed in other, easier ways.

1.6

The really important thing to emphasize today is the passage

$$(V_{\text{even}}, V_{\text{odd}}, D) \rightsquigarrow \mathcal{I}ndex \in K_{\mathsf{G}}(M).$$

The perspective is that pushforwards are like integrals: while at first the mechanics of how to integrate certain functions is very important, later we care more about what goes under the integral sign. Naively,

$$\mathcal{I}ndex \approx V_{\text{even}} \ominus V_{\text{odd}},$$

but this is not right because there is a differential operator D between them. In particular, D is *not* linear over functions. To encode the derivatives in D, we must consider $\pi\colon T^*M \to M$. The pullbacks $\pi^* V_{\text{even}}$ and $\pi^* V_{\text{odd}}$ have a linear map between them which is the *symbol* of D, and the operator D being elliptic means that $\mathrm{symbol}(D)$ is an isomorphism away from the zero section $M \subset T^*M$. Hence

$$\pi^* V_{\text{even}} \ominus \pi^* V_{\text{odd}} \in K_{\mathsf{G}}(T^*M)$$

is supported only on M, and by excision (which exists in equivariant K-theory) is therefore an element $\mathcal{I}ndex \in K_{\mathsf{G}}(M)$, as desired. This is a "local index" of sorts.

1.7

A natural geometric question is whether $\mathcal{I}ndex$ comes from an even smaller subset $M' \subset M$. For example,

$$\operatorname{tr}(g \cdot \text{evolution}(T))$$

involves only diagonal elements of g, and therefore only the fixed locus $M^g \subset M$. This sort of restriction is really important if M is infinite-dimensional, which is usually the case in quantum field theories (in contrast to quantum mechanics).

A QFT in $(d+1)$-dimensions, meaning a d-dimensional space plus one time dimension, has M which is roughly

$$M = \mathsf{Maps}(\text{space} \to X) \tag{2}$$

for some X. (For now, view X as the configuration space at a point.) For example, in $(2+1)$-dimensions, we are mapping Riemann surfaces into X.

A general QFT and such M is difficult to work with, but the index localizes to a much smaller $M' \subset M$ if there is *extended* supersymmetry, i.e. when there are other operators like D. The number of these operators is usually denoted $\mathcal{N}$. One can imagine that extra such operators produce a bigger complex than just (1), which cancels out even more stuff in $\mathcal{H}$. Usually, M' is the space of solutions to some PDEs defined by these additional supersymmetry operators, and, very importantly,

$$M' = \bigsqcup(\text{finite-dimensional spaces}).$$

This is more tractable.

From now on we replace M with M' and never look at the original M again.

1.8

The specific M of interest to us is the subset of (2) where there is a fixed complex structure on both the source and target, and maps are required to be *holomorphic*. Namely let

$$M = \mathsf{Holo}\left((C, p_1, \ldots, p_n) \to X\right)$$

be holomorphic functions from a marked compact Riemann surface to an algebraic variety X. One can construct with algebraic geometry some element in $K_{\mathsf{G}}(M)$ that stands in for the local index of D; this is some complicated construction we omit for now. To compute the actual index, it is actually better not to immediately do the pushforward all the way to a point. Instead, there is a forgetful map

$$M \to \{(C, p_1, \ldots, p_n)\} \times X^n$$

which forgets the holomorphic map f and only remembers the values $f(p_1), \ldots, f(p_n)$, and we first do an intermediate pushforward along this map instead, which yields an element of

$$K_{\mathsf{G}}\left(\mathsf{Moduli}(C, p_1, \ldots, p_n) \times X^n\right).$$

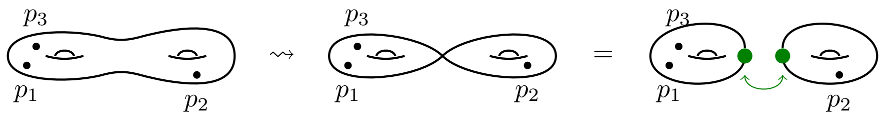


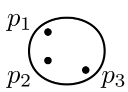
Figure 1: A possible degeneration of $(C, p_1, \ldots, p_n)$ in a CohFT.

Here the group action is by $\mathsf{G} \coloneqq \mathrm{Aut}(C, p_1, \ldots, p_n) \times \mathrm{Aut}(X)$. There is a very nice language to talk about equivariant K-classes on this space, because actually they form a compatible collection when one varies over all genera and number of marked points. Such a structure is called a *cohomological field theory* (CohFT) [KM94].

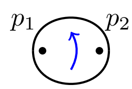
The basic argument in CohFTs is to *degenerate* C in some controllable fashion into multiple components linked by nodes, and then there is some mechanism which glues the K-classes on these components into a K-class on C. This requires passing from M to a slightly larger moduli space

$$\overline{M} \supset M$$

of possibly-singular C; all our constructions extend to $\overline{M}$ as well. By such degeneration arguments, the entire CohFT is determined by very few basic tensors, e.g.



$\in K_{\mathrm{Aut}(X)}(X^3),$



$\in K_{\mathbb{C}_q^\times \times \mathrm{Aut}(X)}(X^2)$

$q \in \mathbb{C}^\times = \mathrm{Aut}(\mathbb{P}^1, 0, \infty)$.

Note that in the 2-pointed case, there is the non-trivial automorphism group $\mathbb{C}^\times = \mathrm{Aut}(\mathbb{P}^1, 0, \infty)$, whose coordinate we denote q. This sort of reconstruction is unsurprising in physics: if it is known how a field theory behaves on flat space, then one should be able to put it onto any geometry. One of our main goals will be to identify these basic tensors in terms of some geometric representation theory of certain quantum loop groups acting on the equivariant K-theory $K_{\mathrm{eq}}(X)$ of X.

1.9

Example. Here is a silly (but not so silly) example. If $X = \{0, 1\}$ is a two-point space, then $K(X) = \mathbb{Z} \oplus \mathbb{Z}$ just records the dimensions of the vector spaces at the two points. Then

$$K(X \times X) = \mathsf{Mat}(2 \times 2, \mathbb{Z})$$

acts naturally on $K(X)$, essentially by matrix multiplication. Although this example sounds silly, actually X really is the local configuration space of e.g. $\mathfrak{sl}_2$-vertex models. These are physical systems involving configurations on a lattice, with some rules on how configurations at various vertices interact. Then, quite precisely, $K(X) \otimes_{\mathbb{Z}} \mathbb{C}$ is a module for the quantum group $U_\hbar(\mathfrak{gl}_2)$ which governs the vertex model. There is a standard evaluation map

$$U_\hbar(\mathfrak{gl}_2) \leftarrow U_\hbar(\widehat{\mathfrak{gl}}_2)$$

and this is the quantum loop group.

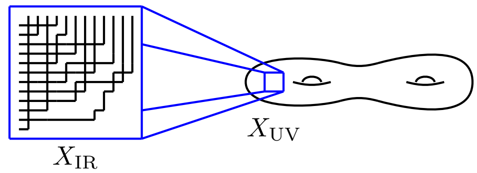


Figure 2: A configuration in the six-vertex model, consisting of lattice paths.

One may wonder, and this is an important point: how can we possibly have a holomorphic map into $X = \{0,1\}$? The answer is that this $X = X_{\mathrm{IR}}$ is the space of states defined *microscopically*, whereas for the index computation one can assume that C is very large because the index is independent of metric. Then the lattice is well-modelled by $\mathbb{R}^2$. Vacuum states in $\mathbb{R}^2$ can clearly exist for infinite time, and so they are equivalent to states in $\mathbb{R}^2 \times \mathbb{R}$ where the second $\mathbb{R}$ is time. These are known as *Gibbs states*. In the six-vertex model, for example, Gibbs states are parametrized by the average density of horizontal and vertical paths, i.e. by

$$X_{\mathrm{UV}} \coloneqq [0,1]^2,$$

which has a natural complex structure coming from an interesting function called surface tension (which for the six-vertex model we only know slightly explicitly). In the end, the holomorphic maps are to X_{UV}, not to X_{IR}.

1.10

Many different theories will not only have the same index, but also the same moduli space of vacua X_{UV} by virtue of appearing the same from large distances. However, we are actually interested in situations where different theories have the same index for highly *non-trivial* reasons, as follows.

Our $X = X_{\mathrm{IR}}$ will have more supersymmetry than minimal. We said that the basic example of supersymmetry is the de Rham operator in real geometry. For more supersymmetry, we use the Dolbeault operator in complex geometry. To get even more supersymmetry, we use hyperkähler manifolds, where there are *four* differential operators corresponding to $1, i, j, k$. The algebraic analogue of hyperkähler manifolds is algebraic symplectic varieties. Notably, the group G is allowed to scale the symplectic form ω_X, and the weight $\hbar \in K_{\mathsf{G}}(\mathrm{pt})$ of ω_X will be the deformation parameter for the quantum groups. All constructions in G-equivariant K-theory live over this base ring $K_{\mathsf{G}}(\mathrm{pt})$.

For certain such X, there is an amazing phenomenon called 3d mirror symmetry. If X is viewed as a "Lie algebra of the XXI century" then this is like Langlands duality. Namely there are pairs

$$X \leftrightarrow X^\vee$$

and there is a relationship between

$$\mathcal{I}ndex \in K_{\mathrm{eq}}\left(\mathsf{Moduli}(C, p_i) \times X^n\right)[[z]]$$

for X and $X^\vee$. Here z is a variable that remembers the degree

$$\deg(f) := f_*([C]) \subset H_2(X, \mathbb{Z}),$$

and z^d is a character of the torus Z dual to $H_2(X, \mathbb{Z})$. In 3d mirror symmetry, there is a change of variables

$$\mathsf{Z} \leftrightarrow \mathsf{A}^\vee, \quad \mathsf{A} \leftrightarrow \mathsf{Z}^\vee$$

where A is the maximal torus in the group $\mathrm{Aut}(X, \omega)$ of symplectic automorphisms, and likewise for $\mathsf{A}^\vee$ and $\mathrm{Aut}(X^\vee, \omega^\vee)$. Poles of $\mathcal{I}ndex$ in A and Z are called *equivariant* and *Kähler roots*. Usual Langlands is recovered for $X = T^*\mathsf{G}/B$ and a maximal torus $\mathsf{A} \subset \mathsf{G}$, and these are really the roots (and co-roots) in usual Lie theory. Of course, $\mathcal{I}ndex$ contains more information than just where the roots are.

A long term, optimistic goal for this course is to prove 3d mirror symmetry in this form, whenever it is defined.

Lecture 2. An introduction to vertices and quantum groups

q-hypergeometric functions and Macdonald polynomials as examples of vertex functions, quantum groups and their categories of modules, R-matrices, reconstruction of a quantum group from R-matrices.

2.1

As discussed last time, we are concerned with a moduli space of holomorphic maps

$$f\colon (C, p_1, \ldots, p_n) \to X$$

where the source is supposed to be space (in spacetime) and the target X is a component of the moduli of vacua. However, our basic building blocks (in flat spacetime) involve only very specific C, namely

$$C = \mathbb{C} = \mathbb{R}^2,$$

with some assumptions on what happens "at infinity".

In algebro-geometric language, this means we actually consider $C = \mathbb{P}^1$, and impose the boundary condition that f takes on a specified value $f(\infty) \in X$ at the point $\{\infty\} = \mathbb{P}^1 \setminus \mathbb{C}$. In fact, not only should f take on this value, but it should take on the value in a non-singular way, with $f'(\infty)$ finite. This sort of setup appears often, e.g. Nekrasov partition functions (in 4d gauge theory) or Givental's J-function. The locus of such functions f is an open subset of the space of maps we were considering.

2.2

We want to integrate the local index sheaf $\mathcal{I}ndex$ on this open set. In K-theory, integration is a pushforward, which can be thought of as taking sections (and higher cohomologies) of a holomorphic vector bundle. But in general, the vector space of sections is infinite-dimensional.

This is remedied by working equivariantly, with respect to the two group actions present in the setup: $\mathrm{Aut}(\mathbb{P}^1,\infty)$ (automorphisms preserving ∞) and $\mathrm{Aut}(X)$. Hence this vector space of sections is a module for $\mathrm{Aut}(\mathbb{P}^1,\infty)\times\mathrm{Aut}(X)$, and its *character* is a well-defined rational function for a given degree $d=\deg f$.

Generating functions are used everywhere in mathematical physics, and also in this course. To sum these characters over all degrees d, we introduce a new variable z and obtain generating series of the form

$$\sum_d \mathbf{z}^d\cdot(\text{rational function}).$$

The rational functions are in terms of *equivariant variables*, namely weights $q\in\mathrm{Aut}(\mathbb{P}^1,\infty)$ and in $\mathrm{Aut}(X)$, and recall that we can think of $\mathbf{z}$ as a character of the torus $H^2(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}^\times$. If we denote by q the weight of $\mathrm{Aut}(\mathbb{P}^1,\infty)$, such series can be thought of as a generalization of q-hypergeometric functions like the very classical

$$\Phi\left[\begin{array}{c|c} a,b & \\ c & \mathbf{z},q\end{array}\right]:=\sum_d \mathbf{z}^d\frac{(a)_d(b)_d}{(q)_d(c)_d},$$

where $(x)_d=(1-x)(1-qx)\cdots(1-q^{d-1}x)=(x)_\infty/(q^dx)_\infty$. Such expressions occur very naturally in equivariant K-theory.

2.3

What unites all of our series is that they are solutions of regular q-difference equations in the variables $\mathbf{z}$, and also of regular q-difference equations in those $\mathrm{Aut}(X)$-variables which are regular in $\mathrm{Aut}(X,\omega)\subset\mathrm{Aut}(X)$. (This is not obvious at all, and requires proof.) 3d mirror symmetry $X\leftrightarrow X^\vee$, as discussed last lecture, exchanges these two types of q-difference equations. Note that while for compact geometries 3d mirror symmetry is an *equality* of indices, for non-compact geometries the indices are different and only their q-difference equations are preserved (and swapped).

Example. Consider the Macdonald polynomial $P_\lambda(\mathbf{z};q,t)$ for $\mathrm{GL}(n)$. It is the "terminating" specialization

$$(a_1,\ldots,a_n)=(q^{\lambda_1}t^{n-1},\ldots,q^{\lambda_n})$$

of a certain hypergeometric function in $\mathbf{a}$; here "terminating" means that variables x appearing in q-factorials are specialized so that $(x)_d=0$ for all $d\geq d_0$.

There is a remarkable *label/argument* symmetry which can be viewed as a consequence of 3d mirror symmetry: the q-difference equations in $\mathbf{a}$ are the same as those in $\mathbf{z}$, and therefore in the terminating case the resulting solutions are *equal*. This is an example of the general theory for the self-mirror $X=T^*\,\mathrm{GL}(n)/B=X^\vee$. In more generality, away from type A, q-difference operators for Macdonald polynomials come from Cherednik algebras, and to exchange the roots $\mathbf{a}$ with the co-roots $\mathbf{z}$ one must pass to the Langlands dual.

Macdonald polynomials are ubiquitous, and there are many ways to think about their q-difference equations, but the perspective we take is that they are a special case of the so-called *dynamical equations* associated to $U_\hbar(\mathfrak{g}_{KM})$, studied by Etingof and Varchenko [EV02] and others. We will need to generalize $U_\hbar(\mathfrak{g}_{KM})$.

2.4

What is a quantum group? We will take an approach which does *not* involve generators and relations, which is not a productive way to think about our quantum groups for the simple reason that they will be infinitely generated. Writing generators and relations for an algebra is akin to writing coordinates and equations for an algebraic variety; we will try to work like an algebraic geometer, with equations lurking in the background but not necessarily made explicit.

To motivate the discussion, let's begin with an example. Let G be a finite group, $\Bbbk$ be a field (or, later, maybe a ring), and consider the group algebra $\Bbbk G$ consisting of formal linear combinations $\sum_{g \in G} c_g g$. This algebra captures in a sense everything about the representation theory of G, e.g.

$$\mathbb{C}G = \bigoplus_{\text{irreps } M} \operatorname{End}_{\mathbb{C}}(M).$$

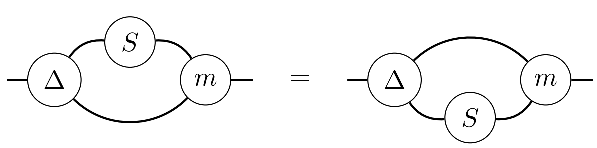
On the other hand, it forgets a lot of information about G because many different groups can have the same group algebra. To reconstruct G, observe that $\Bbbk G$ has two structures that a normal algebra does not. The category of G-modules has

1. a tensor product $(M_1, M_2) \to M_1 \otimes M_2$, with unit $\mathbb{1}$, and
2. duals $M^\vee := \operatorname{Hom}(M, \Bbbk)$,

and G must act on them in some way. Namely, an element $g \in G$ acts on a tensor product by $g \otimes g$, on the unit by some scalar in $\Bbbk$, and on duals by $(g^{-1})^T$. These respectively define maps

$$\begin{aligned} \Delta\colon \Bbbk G &\to \Bbbk G \otimes \Bbbk G & g &\mapsto g \otimes g \\ \epsilon\colon \Bbbk G &\to \Bbbk & & \\ S\colon \Bbbk G &\to \Bbbk G & g &\mapsto g^{-1} \end{aligned}$$

called the *coproduct*, *counit*, and *antipode*. (Note that we omit the transpose in $(g^{-1})^T$, so the antipode is an algebra *anti*-automorphism.) These maps satisfy all sorts of relations. For example, that $g^{-1}g = gg^{-1} = 1$ means (in graphical shorthand)



$$m((S \otimes \mathrm{id})\Delta(x)) \quad = \quad m((\mathrm{id} \otimes S)\Delta(x)) \quad = \quad \text{unit} \circ \text{counit}.$$

By definition, these relations are axioms of a *Hopf algebra*. Notably, in a Hopf algebra, the set of axioms is self-dual in the sense that it is invariant under reversing all arrows, and so given a Hopf algebra one can form its *dual* Hopf algebra. For example, $\Bbbk G$ is dual to the Hopf algebra $\Bbbk[G]$ of functions on G, where the product is multiplication of functions and coproduct is

$$\Delta f(g_1, g_2) = f(g_1 g_2).$$

2.5

More generally, if G is a Lie group, there are many ways to create Hopf algebras from G:

- the group algebra $\Bbbk G$, just like for finite groups, or more generally signed measures on G (which become delta masses when G is finite) or distributions on G, a subset of which is the universal enveloping algebra $U(\mathfrak{g})$;
- any covariant functor of G, e.g. homology $H_*(G, A)$ with coefficients in a ring A.

Their dual Hopf algebras are:

- the ring of functions $\Bbbk[G]$, in the appropriate category (smooth, algebraic, etc.);
- the cohomology $H^*(G, A)$ (which was the original case investigated by Hopf).

2.6

A *quantum group* is a deformation of any of these in the world of Hopf algebras, but generally with the following feature. Note that the Hopf algebra $U(\mathfrak{g})$ is not commutative but has a coproduct $\Delta\xi = \xi \otimes 1 + 1 \otimes \xi$ which is *co*commutative. (This coproduct is the infinitesimal version of $\Delta(g) = g \otimes g$.) On the dual side, the Hopf algebra $\Bbbk[G]$ is commutative but not cocommutative. Quantum groups generally give up on *both* commutativity and cocommutativity. In particular,

$$\Delta \neq \Delta^{\text{op}} \coloneqq (12)\Delta$$

where (12) is the permutation of tensor factors. However, Δ and Δ^{op} will be related by a cocommutation relation, like how in $U(\mathfrak{g})$ the commutator $[-,-]$ (taking values in $\mathfrak{g}$) relates the multiplication m with $m \circ (12)$.

2.7

Instead of exploring the general theory of quantum groups, we will specialize to a particular class of examples. Concretely, we want deformations

$$U(\widehat{\mathfrak{g}}) \rightsquigarrow U_\hbar(\widehat{\mathfrak{g}})$$

where the underlying Lie algebra is

$$\widehat{\mathfrak{g}} = \mathsf{Maps}(\mathbb{G} \to \mathfrak{g})$$

for some 1-dimensional algebraic group $\mathbb{G}$ (additive, multiplicative, or an elliptic curve). For example, if $t \in \mathbb{G}$ is a coordinate, then as modules

$$\begin{aligned} \widehat{\mathfrak{g}} &= \mathfrak{g}[t], \quad \mathbb{G} = \mathbb{G}_a \\ \widehat{\mathfrak{g}} &= \mathfrak{g}[t^{\pm 1}], \quad \mathbb{G} = \mathbb{G}_m, \end{aligned}$$

and commutators are taken coefficient-by-coefficient. The category of modules we want to study deforms the category of *evaluation representations* of $U(\widehat{\mathfrak{g}})$. These representations arise from pulling back highest weight representations of $\mathfrak{g}$ along the evaluation maps

$$\widehat{\mathfrak{g}} \xrightarrow{\text{evaluate at } a} \mathfrak{g}, \quad a \in \mathbb{G}.$$

If M is a highest weight module for $\mathfrak{g}$, let $M(a)$ denote its evaluation representation (at a). Our category of modules is then spanned by elements $\otimes_i M_i(a_i)$.

2.8

Prior to deformation, in $\bigotimes_i M_i(a_i)$, it is clear that different points a_i don't talk to each other at all, since polynomials may take any set of values at any set of given points. In particular, the tensor product is irreducible if all the M_i are irreducible and the points $a_i \neq a_j$ are distinct, and any map between such tensor products is either zero or invertible. Upon deformation,

$$M_1(a_1) \otimes M_2(a_2) \quad \text{and} \quad M_1(a_1) \otimes^{\text{op}} M_2(a_2)$$

are still irreducible modules for generic a_1 and a_2, but what will happen is that there is an intertwiner

$$R_{M_1,M_2}(a_1,a_2)\colon M_1(a_1) \otimes M_2(a_2) \to M_1(a_1) \otimes^{\text{op}} M_2(a_2)$$

called the *R-matrix*, which is an isomorphism only for a_1, a_2 generic (meaning, away from finitely many points). In general it is a rational function in a_1 and a_2 with non-zero determinant. Actually, since prior to deformation there is a *loop rotation* automorphism of $\mathbb{G}$ itself acting on $\widehat{\mathfrak{g}}$, if we wish for the deformation to preserve the loop rotation then this matrix R must depend only on $u \coloneqq a_1/a_2$.

2.9

We are not claiming yet that such a structure always exists. Instead, our strategy will be to construct the R-matrix R first, and then the rest, including $\mathfrak{g}$ itself, will follow. The construction will be done geometrically, and then we will run some kind of reconstruction process to obtain the quantum group. Our R will satisfy the following relations.

1. (Unitarity) Setting $R^\vee(u) \coloneqq (12)R(u)$, the composition

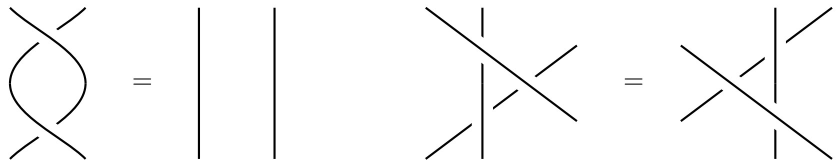
$$M_1 \otimes M_2 \xrightarrow{R^\vee(u)} M_2 \otimes M_1 \xrightarrow{R^\vee(u^{-1})} M_1 \otimes M_2$$

   is the identity. This is not the case for all quantum groups, but ours will have this property.

2. (Yang–Baxter equation) The two different ways to change $M_1 \otimes M_2 \otimes M_3$ to $M_3 \otimes M_2 \otimes M_1$ are equal:

$$R^{(12)}(a_1/a_2)R^{(13)}(a_1/a_3)R^{(23)}(a_2/a_3) = R^{(23)}(a_2/a_3)R^{(13)}(a_1/a_3)R^{(12)}(a_1/a_2). \quad (3)$$

Pictorially, viewing $R^\vee$ as a sort of braiding operator, these two properties are often depicted as



Note that the presence of the *spectral parameter* u in the R-matrix makes the theory vastly more general than for the classical case without spectral parameter, corresponding to taking $R(0)$ or $R(\infty)$. In particular, the unitarity property $R^{(21)}(u^{-1})R^{(12)}(u) = 1$ is lost in such specializations.

Remark. The *spectral* YBE (3) actually predates quantum groups, which originated from the study of vertex models in 2d statistical physics. There, an R-matrix $R^{(12)}(a_1/a_2)$ literally encodes some interaction of states living in vector spaces $M_1(a_1)$ and $M_2(a_2)$, where the a_i are parameters called *rapidities*. From the point of view of statistical mechanics, we think of the YBE as an invariant of the model with respect to the local arrangement of interactions, which is a combinatorial analogue of things like covariance with respect to local changes of coordinates.

2.10

The data $\{R_{M_i,M_j}(a_i/a_j)\}$ for a collection of modules $\{M_i\}$ is what will be constructed geometrically. Given this data, the remarkable fact is that there is a reconstruction procedure which automatically makes all tensor products $\bigotimes M_{k_i}(a_i)$ into modules for a certain quantum group, such that the R-matrices are precisely the desired cocommutation relations. Two good references for these sorts of ideas are [Res89] and [ES98]. We will explain the reconstruction procedure in the next lecture.

Lecture 3. Constructing quantum groups from R-matrices

More on R-matrices, comultiplication in Yangians, relations in a quantum group, R-matrix as an interface, geometric construction of the R-matrix for $Y(\mathfrak{sl}_2)$, geometric meaning of the Yang–Baxter equation.

3.1

The goal today is to continue the discussion of R-matrices and quantum groups, and in particular the chicken-and-egg relationship between the two: if one has a full-fledged quantum group satisfying some co-commutation relation, then there will be an R-matrix, and vice versa. We will go with the historical route, where the R-matrix comes first. It will be easier for us to construct the R-matrix, and then deduce the quantum group action from it.

3.2

The starting point is a collection of vector spaces $M_i(a)$, where a is the *spectral variable*. It could be a complex number, or more generally we can think of $M_i(a)$ as a free module over $\Bbbk[a]$ or $\Bbbk[a^{\pm 1}]$. Here $\Bbbk$ is some ground ring like $\mathbb{Z}[\hbar^{\pm 1}]$. In reality, for us,

$$M_i(a) = \text{Cohomology}_{\text{equivariant}}(X)$$

will be the equivariant cohomologies (e.g. ordinary cohomology, K-theory, elliptic cohomology) of some space X, and the variable

$$a \in \mathbb{G}_m \subset \text{Aut}(X)$$

is an equivariant parameter. Acting on these $M_i(a)$ will be an R-matrix

$$R\colon M_1(a_1) \otimes M_2(a_2) \to M_1(a_1) \otimes M_2(a_2)$$

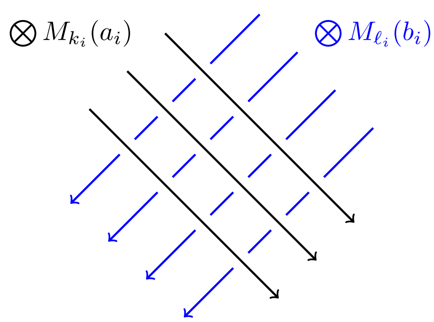
satisfying the Yang–Baxter equation.

Example. The simplest example is the quantum group $U_\hbar(\widehat{\mathfrak{gl}}_2)$, which is produced from $X_k = T^* \operatorname{Gr}(k, \mathbb{C}^n)$. The next simplest is $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$, which is produced from $X_k = \text{Hilb}(\mathbb{C}^2, k)$, which is the space of ideals in $\mathbb{C}[x_1, x_2]$ of codimension k. We write $\mathbb{C}$ here but really the ground field can be anything.

3.3

From the data of these R-matrices, we would like to make all the $M_i(a)$ into modules for a quantum group. This takes three steps.

1. Extend the operators $R_{M_1,M_2}(a_1/a_2)$ to arbitrary tensor products of the form $\bigotimes M_{k_i}(a_i)$ by, pictorially, the composition

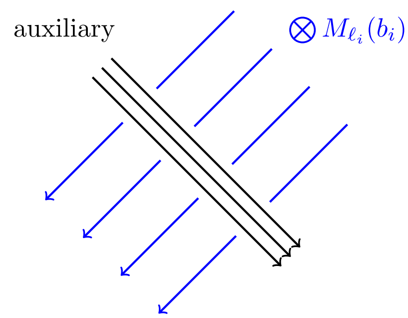
   

   . (4)

   One easily verifies that this composition still satisfies the YBE, since each "strand" $M_{k_i}(a_i)$ of the composite $\bigotimes M_{k_i}(a_i)$ in the diagram can individually be moved.

2. Extend the operators to dual modules. (The antipode, if it exists, is actually unique; this is like saying that for a group, the inverse is uniquely determined. So it is not exactly a new structure.)

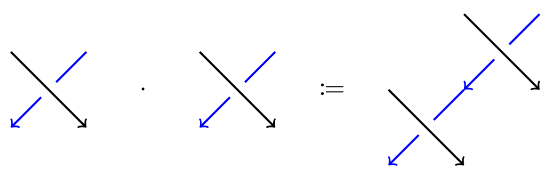
3. Define operators on the second factor $\bigotimes M_{\ell_i}(b_i)$ of (4), by taking matrix elements of the operator



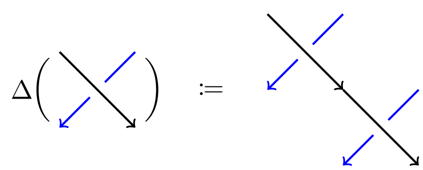
in the auxiliary space. Each matrix element is an $\mathrm{End}(\bigotimes M_{\ell_i}(b_i))$-valued function of spectral parameters b_i. This is precisely the sort of thing we want in our quantum loop group, which is then defined to be the algebra generated by these matrix elements. We can take coefficients (or any other linear functionals) of these operators in b_i, or we also can leave them alone, as is, as generating functions in b_i.

3.4

Clearly the resulting operators form an algebra: multiplication is composition, which can be drawn as



But completely analogously, there is a comultiplication given by


(5)

3.5

The comultiplication needs a bit of discussion if $\dim M_i = \infty$. If all $\dim M_i < \infty$, then all matrices are of finite size and this composition is well-defined, but otherwise in principle it is an infinite linear combination. To make sense of this, recall that our modules M are cohomologies of a space X which splits as $X = \bigsqcup_{k\geq 0} X_k$. So there is a grading on M which is bounded below. R-matrices will be constructed from some simple geometric operation which preserves this grading. Hence, taking graded matrix elements, the comultiplication

$$\Delta\colon U_\hbar(\widehat{\mathfrak{g}}) \to U_\hbar(\widehat{\mathfrak{g}}) \widehat{\otimes} U_\hbar(\widehat{\mathfrak{g}})$$

is in principle an infinite sum and requires the completion $\widehat{\otimes}$, but each term in the sum acts locally nilpotently. Put differently, all but finitely many terms act by zero on a vector of any given degree.

3.6

Depending on whether we work in ordinary cohomology, K-theory, or elliptic cohomology, we will get R-matrices which are functions of the spectral variable $u \in \mathbb{G}_a$, $\mathbb{G}_m$, or an elliptic curve, respectively. For an explicit example, suppose we use the additive group $\mathbb{G}_a$. The resulting quantum groups are called *Yangians*, and for them the R-matrices have a good expansion around $u = \infty$:

$$R(u) = 1 - \frac{r}{u} + O\left(\frac{1}{u^2}\right) \quad u \to \infty. \tag{6}$$

The linear term r is the *classical* r-matrix, and is a canonical element

$$r = \sum e_\alpha \otimes e^\alpha \in \widehat{S^2\mathfrak{g}} \tag{7}$$

where $\mathfrak{g}$ is a Lie algebra with a non-degenerate invariant bilinear form. It can be decomposed into finite-dimensional root subspaces $\mathfrak{g}_\alpha$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha.$$

In principle there can be infinitely many roots, which is how we get infinite sums in (7) or the coproduct.

3.7

Example. As an example of coproduct computation, consider Δe_α. By definition, this means to take the $1/u$ coefficient in the composition (5). This coefficient is additive, and therefore

$$\Delta e_\alpha = e_\alpha \otimes 1 + 1 \otimes e_\alpha.$$

Since the R-matrix intertwines Δ and Δ^{op},

$$[R, \Delta e_\alpha] = 0.$$

This means R is $\mathfrak{g}$-invariant.

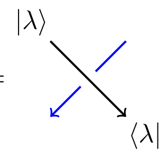
3.7.1

Example. To see an infinite sum appearing in the coproduct, let $|\lambda\rangle \in M$ be a lowest weight vector, i.e.

$$\mathfrak{g}_\alpha |\lambda\rangle = 0, \quad \alpha < 0$$
$$h |\lambda\rangle = \lambda(h) |\lambda\rangle, \quad h \in \mathfrak{h}.$$

Using the expansion (6), compute the matrix coefficient

$$R^{|\lambda\rangle,\cdot}_{\langle\lambda|,\cdot} =$$

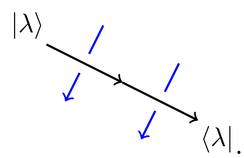


$$= 1 - \frac{h_\lambda}{u} + \frac{h_\lambda^{(1)}}{u^2} + \cdots$$

where $h_\lambda := \sum_\alpha \lambda(h_\alpha) h_\alpha$. The next term $h_\lambda^{(1)} \in U_\hbar(\mathfrak{g})$ can be interpreted as the $\hbar$-deformation of $t \cdot h_\lambda \in U(\mathfrak{g}[t])$.

(To spoil the upcoming story a little, these terms h_λ and $h_\lambda^{(1)}$ have the following geometric meaning. The grading, coming from $X = \bigsqcup X_k$, can be viewed as given by the rank of some universal bundle on X. Then h_λ acts by exactly the rank of this universal bundle, and its older siblings $h_\lambda^{(1)}$ and higher-order terms act by c_1 and higher Chern classes of this universal bundle.)

Let's compute $\Delta(h_\lambda^{(1)})$. By definition, this is the $1/u^2$ coefficient in the composition



There are three ways to get a $1/u^2$ term: either $1/u^2$ comes purely from the first, or from the second factor, or both factors contribute $1/u$. Hence

$$\Delta(h_\lambda^{(1)}) = h_\lambda^{(1)} \otimes 1 + 1 \otimes h_\lambda^{(1)} + \cdots$$

where the $\cdots$ denotes the latter case. But we know the $1/u$ term of the R-matrix very well. Using it, we see that

$$\cdots = \sum_{\alpha,\beta} \langle\lambda| e_\beta e_\alpha |\lambda\rangle.$$

By weight considerations, and using that $[e_{-\alpha}, e_\alpha] = -h_\alpha$,

$$\langle\lambda| e_\beta e_\alpha |\lambda\rangle = \begin{cases} 0 & \text{unless } \alpha+\beta = 0,\ \alpha > 0 \\ -(\lambda,\alpha) & \alpha+\beta=0,\ \alpha > 0. \end{cases}$$

We conclude that

$$\Delta(h_\lambda^{(1)}) = h_\lambda^{(1)} \otimes 1 + 1 \otimes h_\lambda^{(1)} - \sum_\alpha (\lambda,\alpha) r_{-\alpha}, \quad r_{-\alpha} = e_{-\alpha} \otimes e_\alpha \in \mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_\alpha.$$

As discussed earlier, we see that in principle this sum over α could be infinite, but each piece acts locally nilpotently.

3.8

Remark. Usually, the condition that $[R, \Delta g] = 0$ (which was somewhat trivial to derive) and

$$R\Delta h_\lambda^{(1)} = \Delta^{\mathrm{op}} h_\lambda^{(1)} R \tag{8}$$

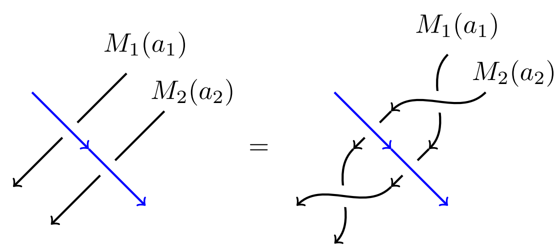
(which we just derived) determine R uniquely up to a scalar multiple. Hence if we are defining R geometrically, it suffices to understand the Lie algebra involved along with how c_1 of universal bundles comultiply. An example of this is $\mathrm{Hilb}(\mathbb{C}^2)$. The Lie algebra is $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$, generated by operators α_n and commutation relations $[\alpha_n, \alpha_m] = n\delta_{n+m}c$ for a central element c. The classical r-matrix is

$$r = (\text{diagonal part}) + \sum_n \alpha_n \otimes \alpha_{-n}.$$

Bilinear expressions in α_n should remind us of Sugawara-like formulas for the Virasoro algebra. In fact $\alpha_n^{(1)}$ in this case are indeed Virasoro operators, and then (8) immediately implies the R-matrix is the *Liouville reflection operator*.

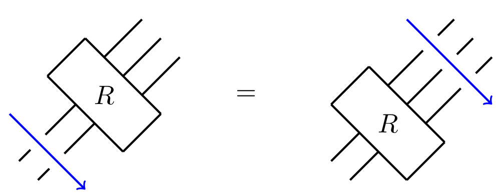
3.9

It is obvious that the R-matrices do give the braiding of the modules we considered, since by Yang–Baxter

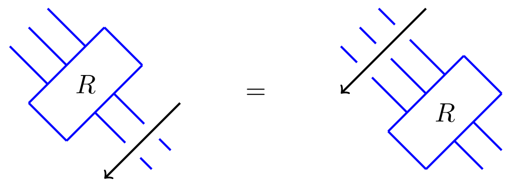


Flipping the roles of physical vs auxiliary spaces, we see that Yang–Baxter is both the commutation *and* co-commutation relation.

More generally, the braiding is an example of a morphism in the category we constructed. Namely, we have constructed a tensor category (modulo some details about poles): objects are $\bigotimes M_{k_i}(a_i)$ and morphisms are maps that commute with R-matrices in the sense that



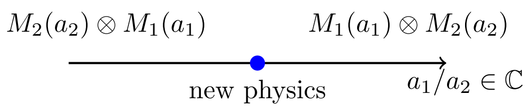
By flipping physical vs auxiliary spaces, we see therefore that morphisms also give relations between matrix elements, in $U_\hbar(\widehat{\mathfrak{g}})$:



Such a perspective fits well with older perspectives on reductive groups $G \subset \mathrm{GL}(V)$, e.g. H. Weyl's idea that thinking about G is the same as thinking about all G-invariant operators $\mathrm{Hom}_G(V^{\otimes k}, V^{\otimes \ell})$.

3.10

How do we actually construct R-matrices? We will begin this discussion now, and return to it in more detail later. R-matrices are examples of morphisms in this tensor category. A morphism which intertwines $K_{\mathrm{eq}}(X)$ and $K_{\mathrm{eq}}(X')$, where both X and X' are moduli of vacua, can be understood physically as an interface in some variation of parameters of a QFT. At some critical point, we will transition from X to X'. At this critical point, there are *extra* degrees of freedom coming from both X and X', and in general the physical theory is richer there (e.g. more massless particles). This is relevant for R-matrices because essentially we will construct them via behavior at a special point



where the axis is $a_1/a_2 \in \mathbb{C}$. At this special value of a_1/a_2 we will construct a larger module, which for temporary lack of a better notation we call $M_1(a_1) \circledast M_2(a_2)$, such that there are embeddings

$$M_1(a_1) \otimes M_2(a_2) \hookrightarrow M_1(a_1) \circledast M_2(a_2) \hookleftarrow M_2(a_2) \circledast M_1(a_1)$$

and the cokernels of both embeddings are *torsion*, e.g. annihilated by something like $(1 - a_1/a_2\hbar^{\cdots})^N$. In other words, there is some *enriched* tensor product $\circledast$ which is symmetric and better in some sense, and the embeddings are isomorphisms up to some prescribed poles.

3.11

Having a map of modules with torsion cokernel is a very typical scenario in equivariant cohomology. A kindergarten illustration is to take $X = \mathrm{pt}$ with a trivial action of $\mathbb{C}^\times \ni a$. Then

$$K_{\mathrm{eq}}(X) = \mathbb{Z}[a^{\pm}]$$

is the representation ring of $\mathbb{C}^\times$. Take $Y = \mathbb{C}$ with the defining action of $\mathbb{C}^\times$. Then there is a projection $Y \to X$, and pullback gives an isomorphism

$$K_{\mathrm{eq}}(Y) = K_{\mathrm{eq}}(X)$$

since any $\mathbb{C}^\times$-equivariant vector bundle on Y is specified by how it looks at 0. Now consider a different map, namely the *pushforward* $K_{\text{eq}}(X) \to K_{\text{eq}}(Y)$. The image is generated by the structure sheaf $\mathcal{O}_0$, which is the quotient

$$0 \to x\mathbb{C}[x] \to \mathbb{C}[x] \to \mathcal{O}_0 \to 0$$

where x is the coordinate on Y, with weight a^{-1}. Hence in equivariant K-theory,

$$\mathcal{O}_0 = (1 - a^{-1})\mathcal{O}_{\mathbb{C}},$$

and

$$\operatorname{coker}\left(K_{\text{eq}}(X) \to K_{\text{eq}}(Y)\right) = \mathbb{Z}[a^{\pm 1}]/\langle(1 - a^{-1})\rangle$$

is torsion. This is a very typical feature of pushforwards.

3.12

Example. We would like to study the action of $U_\hbar(\widehat{\mathfrak{gl}}_2)$ on $K_{\text{eq}}(X_n)$, where $X_n = \bigsqcup_k T^* \operatorname{Gr}(k, n)$. For example,

$$X_1 = \text{pt} \sqcup \text{pt}.$$

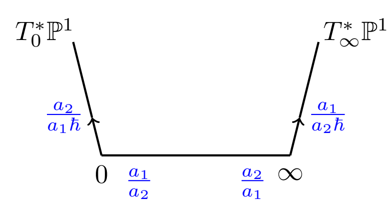
On $\operatorname{Gr}(k, n)$, act by $(a_1, \ldots, a_n) \in \operatorname{GL}(n)$ along with a variable $\hbar$ which scales cotangent fibers (by its inverse). So

$$K_{\text{eq}}(X_1) = \mathbb{Z}[\hbar^\pm, a_1^\pm]^{\otimes 2}$$

which we abbreviate as "$\mathbb{C}^2(a_1)$". Hence we need two different maps to a bigger module

$$\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \hookrightarrow K_{\text{eq}}(X_2) \hookrightarrow \mathbb{C}^2(a_2) \otimes \mathbb{C}^2(a_1)$$

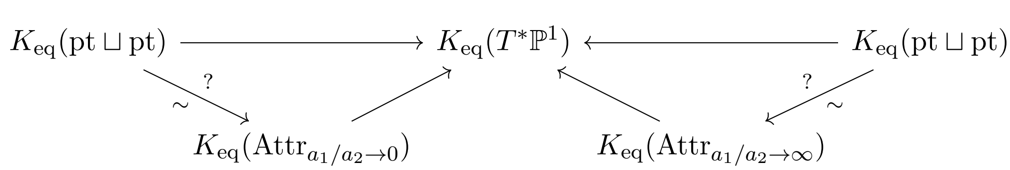
which turns out to be $K_{\text{eq}}(X_2)$. Since $X_2 = \text{pt} \sqcup T^*\mathbb{P}^1 \sqcup \text{pt}$, we should focus on $K_{\text{eq}}(T^*\mathbb{P}^1)$. Draw $T^*\mathbb{P}^1$ with the following weights:



In the 4-dimensional $\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2)$, the component $T^*\mathbb{P}^1$ is the piece of "middle" weight with respect to the Cartan $\mathfrak{h}$, i.e. we can normalize things so that there is one vector of weight -2, two of weight 0, and one of weight 2. Hence we need two maps

$$K_{\text{eq}}(\text{pt} \sqcup \text{pt}) \to K_{\text{eq}}(T^*\mathbb{P}^1) \leftarrow K_{\text{eq}}(\text{pt} \sqcup \text{pt}).$$

To produce these maps geometrically, we factor them into two pieces:



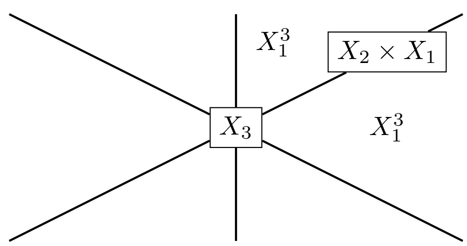
where the *attracting manifolds* Attr have to do with the attracting/repelling behavior of points in $T^*\mathbb{P}^1$ with respect to the action of a_1/a_2 (Figure 3). The pushforwards $\text{Attr} \to T^*\mathbb{P}^1$ supply the torsion cokernels we want, and it suffices to supply the isomorphisms marked with ?; in particular, it is not enough to know that there is some abstract isomorphism, since we need the explicit isomorphism to construct the R-matrix explicitly.

(a) Points whose limits exist as $a_1/a_2 \to 0$

(b) Points whose limits exist as $a_1/a_2 \to \infty$

Figure 3: Attracting manifolds for different a_1/a_2.

These explicit isomorphisms will be different in different cohomology theories, and in particular they are *not* unique. Non-uniqueness is good, because it accounts for the elliptic quantum group having lots of dynamical parameters. From the proposed construction, it is also clear exactly where the R-matrix will have poles, e.g. at $a_2/a_1\hbar$ in Figure 3a and at $a_1/a_2\hbar$ in Figure 3b. Finally, the explicit isomorphisms must satisfy the following analogue of the Yang–Baxter equation. Consider $X_3 = \bigsqcup_k T^* \operatorname{Gr}(k, 3)$, where now there are various chambers



for the parameters $a = (a_1, a_2, a_3)$, and for every type of chamber we have indicated the a-fixed locus $(X_3)^a$. The Yang–Baxter analogue is that various compositions of our explicit isomorphisms in this diagram are compatible, e.g.

$$\begin{array}{ccc} & K_{\text{eq}}(X_2 \times X_1) & \\ \swarrow\sim & & \nwarrow\sim \\ K_{\text{eq}}(X_3) & \xleftarrow{\sim} & K_{\text{eq}}(X_1 \times X_1 \times X_1) \end{array}$$

commutes. This is some actual geometric problem.

Lecture 4. Equivariance and equivariant K-theory

Moduli of vacua in QFTs with extended supersymmetry, their response to variation of external parameters, general idea of "stable envelopes", equivariant cohomology theories, introduction *to equivariant K-theory, equivariant K-theory of the projective space.*

4.1

Let X be a moduli of vacua in some very supersymmetric quantum field theory in $2+1$ dimensions. We didn't discuss this in general, but we did discuss some examples: $T^* \operatorname{Gr}(k,n)$ and $\operatorname{Hilb}(\mathbb{C}^2, k)$. There are two general remarks to make about such X.

- Due to the extended supersymmetry, X wants to be a hyperkähler manifold, or a holomorphic symplectic variety. This is clear for both our examples.
- To be a vacuum state among all possible states usually means to minimize some function, e.g. in statistical mechanics Gibbs states minimize the functional called free energy. So X wants to be the *critical locus* of some functions on

$$(\text{configurations of fields})/\text{gauge}.$$

What happens if we perturb this function? For a real manifold and a de Rham operator, there is a way to twist by an arbitrary function like in Morse theory. On a complex manifold, the function has to be holomorphic, and then we get Picard–Lefschetz theory. On a hyperkähler manifold, we'd like to have a "quaternionic holomorphic" function.

4.2

One important scenario of this perturbation is the following. Consider a complex field $\varphi = x+iy$ and the potential $m|\varphi|^2/2$. If the symplectic form is $\omega \coloneqq dx \wedge dy$, then one way to interpret this potential is as the *moment map* which generates rotations with angular velocity m. More canonically, if we think of m as an element of $\mathfrak{u}(1) = \operatorname{Lie} U(1)$ acting by rotation on the plane, we should view

$$m\frac{|\varphi|^2}{2} = \langle \text{moment map}, m\rangle.$$

4.3

Suppose now that U is a compact group that acts by automorphisms of

$$(X, \underbrace{\omega_1}_{\omega_{\mathbb{R}}}, \underbrace{\omega_2, \omega_3}_{\omega_{\mathbb{C}}}).$$

Let $\mathfrak{u} \coloneqq \operatorname{Lie} \mathsf{U}$. Then the total moment map is

$$\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})\colon X \to \mathfrak{u}^* \otimes \mathbb{R}^3.$$

We would like to work equivariantly with respect to an action that does not necessarily preserve the hyperkähler structure, but rather rotates the "sphere" of symplectic forms $\omega_{\mathbb{R}}, \omega_2, \omega_3$ around the $\omega_{\mathbb{R}}$ axis. This extra rotation will be denoted $\mathbb{C}^\times_\hbar \supset U(1)_\hbar$, and in fact this $\hbar$ is the deformation parameter of the quantum group.

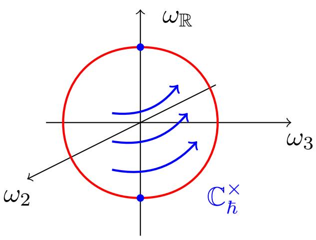


Figure 4: The "sphere" of symplectic forms $\omega_{\mathbb{R}}$, ω_2 and ω_3 with the $\mathbb{C}_\hbar^\times$ rotation.

To preserve this equivariance, μ must paired with a $\mathbb{C}_\hbar^\times$-invariant element m, giving

$$\langle \mu_{\mathbb{R}}, m \rangle = \text{function}, \quad m \in \mathfrak{u}.$$

Now there are two different options, which are related by 3d mirror symmetry (we will see why later):

1. view this U as an external field, i.e. some global symmetry which induces a correction to whatever potential we had before;
2. view this U as a gauge symmetry.

4.3.1 Option 1

If U is an external field, then the new critical locus is taken *without* varying the new variable m, and therefore (think about $m|\varphi|^2/2$)

$$\operatorname{crit}_X \langle \mu_{\mathbb{R}}, m \rangle = X^m$$

is the fixed locus in X of m. If $\mathsf{A} \subset \mathsf{U}$ is the maximal torus, then changes in this fixed locus X^m as m varies defines a wall-and-chamber arrangement in $\mathrm{Lie}(\mathsf{A})$. Generically the fixed locus is X^{A}, but e.g. at the origin the fixed locus is $\operatorname{crit}_X 0 = X$ itself.

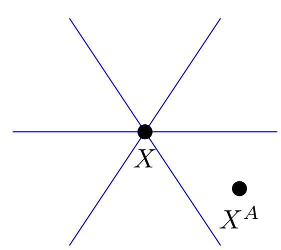


Figure 5: Chambers in $\mathrm{Lie}(\mathsf{A})$ defined by the fixed locus X^m for $m \in \mathrm{Lie}(\mathsf{A})$.

Example. Consider the simple case $\mathsf{U} = U(1)$, and the space $\mathfrak{u}(1)$ of all possible m. The difference between $m > 0$ and $m < 0$ is that the force induced by the potential is attracting vs repelling (and at $m = 0$ there is no force).

While from the perspective of the compact group $U(1)$ nothing really changes from $m > 0$ to $m < 0$, its complexification $\mathrm{GL}(1)$ knows that the "attracting"/"repelling" directions from the origin have changed.

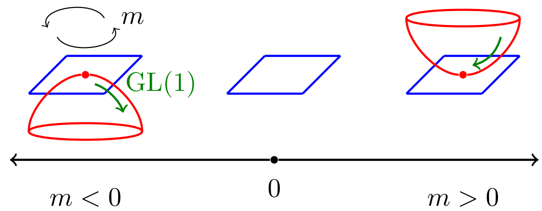


Figure 6: The potential $m|\varphi|^2/2$ as $m \in \operatorname{Lie} U(1)$ varies. The green arrow indicates the action of $\mathrm{GL}(1) \coloneqq U(1) \otimes \mathbb{C}$.

4.3.2

In general, different chambers in A therefore correspond to different attracting/repelling directions for A. Since A preserves $\omega_{\mathbb{C}}$, these attracting/repelling directions come in pairs, and, moving from chamber to chamber, some attracting ones become repelling and vice versa, but the total number of attracting or repelling directions remains the same.

Recall that as an external field goes through a critical value, there tends to be more physics at the critical value and less away from it. For us, this critical value is at the origin, so we would like to have an "interface" between X and X^{A}. Ideally, the moduli spaces X and X^{A} are smooth hyperkähler manifolds, and holomorphic Lagrangians naturally form correspondences between them. So we want an interface which is a holomorphic Lagrangian submanifold, which looks like the attracting manifold of $X^{\mathsf{A}} \subset X$. (If the potential is turned on, every point is driven by a force either away or toward a fixed point in X^{A}.) However, the attracting manifold is only locally closed, and its closure is very singular and unsuitable as an interface. The *stable envelopes* we will define later are good correspondences that come as close as possible to such Lagrangians.

4.3.3 Option 2

If U is viewed as a gauge symmetry, then the field m is allowed to vary, and in particular both x and $m \in \mathfrak{u}^* \otimes \mathbb{R}^3$ are allowed to vary in the computation $\operatorname{crit}_X \langle \mu(x), m \rangle$. So not only does m fix x, i.e. $m \cdot x = 0$, but we have more equations for the critical locus:

$$m \cdot x = 0, \quad \mu_{\mathbb{R}}(x) = \mu_{\mathbb{C}}(x) = 0. \tag{9}$$

It also means we take the quotient by U. This whole procedure is called *hyperkähler* or *holomorphic symplectic reduction*:

$$X \mapsto X /\!/\!/\!/ \mathsf{U}.$$

The four slashes in the notation $/\!/\!/\!/$ remind us that the equations (9), along with the quotient by U, cut down the dimension by *four* times $\dim_{\mathbb{R}} \mathsf{U}$. More generally, we can make $\mu_{\mathbb{R}}(x) = \theta \in (\mathfrak{u}^*)^{\mathfrak{u}}$ and set

$$X /\!/\!/\!/_\theta \mathsf{U} \coloneqq \left\{ \begin{array}{c} m \cdot x = 0 \\ \mu_{\mathbb{R}}(x) = \theta,\ \mu_{\mathbb{C}}(x) = 0 \end{array} \right\} / U.$$

This is an extra degree of freedom and corresponds to a choice of GIT stability. For example, if $\mathfrak{u} = \mathfrak{gl}_n$ then $(\mathfrak{u}^*)^{\mathfrak{u}}$ is the 1-dimensional center.

4.3.4

In analogy with option 1, we get a θ-line instead of an m-line, and there is a wall-and-chamber arrangement in the space of all possible θ, which is now an external field. If θ is generic and θ_0 is not, then

$$X^{\theta\text{-stable}} \hookrightarrow X^{\theta_0\text{-stable}}.$$

Analogous to what we had for m, we would like a good interface between these θ. We will see that this interface is not really different from stable envelopes, and is in fact their 3d mirror dual.

4.4

Let X be a non-singular algebraic variety, perhaps holomorphic symplectic and some other assumptions. Suppose a torus A acts on X and decompose $\mathrm{Lie}(\mathsf{A})$ into chambers $\mathfrak{C}$ as in Figure 5. The general problem solved by (K-theoretic) stable envelopes is to find the nicest possible map

$$K_{\mathrm{eq}}(X^{A}) \xrightarrow{\sim} K_{\mathrm{eq}}\left(\mathrm{Attr}_{\mathfrak{C}}(X^{A})\right) \tag{10}$$

where the attracting manifold for the chamber $\mathfrak{C}$ is

$$\mathrm{Attr}_{\mathfrak{C}}(X^{\mathsf{A}}) \coloneqq \left\{ (f,x) \in X^{\mathsf{A}} \times X : \begin{array}{c} \lim_{u\to 0} \sigma(u)\cdot x = f \\ \text{for some } \sigma\colon \mathbb{C}^\times \to \mathsf{A},\, d\sigma \in \mathfrak{C} \end{array} \right\}.$$

Equivariance is with respect to the centralizer of A in $\mathrm{Aut}(X)$, i.e. we keep as much equivariance as possible. In particular, the $\mathbb{C}^\times_\hbar$ scaling $\omega_{\mathbb{C}}$ with weight $\hbar$ is included. Of course, K-theory can be replaced with ordinary cohomology, elliptic cohomology, $D^b\mathsf{Coh}$, etc.

In the setting of relevance where we want to get quantum groups, e.g. $T^*\,\mathrm{Gr}(k,n)$ where there is a natural action of $\mathrm{GL}(\mathbb{C}^n) \supset \mathsf{A}$, all the fixed loci are products of varieties X' of the *same kind*. Therefore there is a decomposition

$$K_{\mathrm{eq}}(X^{\mathsf{A}}) \cong \bigotimes_{X'} K_{\mathrm{eq}}(X')$$

and this is what gives R-matrices and makes the modules into a *braided* tensor category, as discussed last time. In general, away from Nakajima quiver varieties, there is some similar structure but it does not give quantum groups.

4.5

We now begin a discussion of equivariant cohomology theories, to understand the source and target of (10). These theories are: equivariant cohomology $H^*_{\mathrm{eq}}(X)$, equivariant K-theory $K_{\mathrm{eq}}(X)$, and equivariant elliptic cohomology $\mathrm{Ell}_{\mathrm{eq}}(X)$. In principle there is a mother of all these theories, called equivariant cobordism, which will be discussed next lecture. We are mainly interested in elliptic cohomology: our interfaces live on circles along the underlying Riemann surface, and taking indices means time is made periodic — so manifestly there is an elliptic curve (or at least a torus) on which our interface is a quantum field theory. However, it is a little difficult to explain what $\mathrm{Ell}_{\mathrm{eq}}(X)$, so instead we will begin with the more concrete $K_{\mathrm{eq}}(X)$. Hopefully our discussion of K-theory and cobordism together prepare us to tackle elliptic cohomology.

4.6

K-theory has many incarnations, the easiest of which is probably *topological* K-theory. For Nakajima quiver varieties in general, algebraic K-theory and topological K-theory are actually the same, so it doesn't really matter which one we think of. Let G be a compact group acting on a topological space X, which for convenience we assume is compact. Then an element of $K_{\mathsf{G}}(X)$ is a G-equivariant vector bundle, meaning a vector bundle $\mathcal{E}$ on X with a linear action

$$\mathcal{E}\big|_x \to \mathcal{E}\big|_{g\cdot x}$$

of G lifting the G-action on X. This is exactly like the representation theory of G, but instead of a single vector space there is a family of them labeled by points $x \in X$.

4.7

There are natural operations on $K_{\mathsf{G}}(X)$. Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$, one can form

$$\mathcal{E} \oplus \mathcal{F} \quad \text{and} \quad \mathcal{E} \otimes \mathcal{F}.$$

These operations make the set of G-equivariant vector bundles into a *semiring*, and then $K_{\mathsf{G}}(X)$ is obtained by adding in formal differences $\ominus$. This is the Grothendieck K-group construction. In representation theory this is very familiar: a *virtual* representation is the (formal) difference of two representations, and its character is the difference of the two characters. Hence

$$K_{\mathsf{G}}(\mathrm{pt}) = (\text{representation ring of } \mathsf{G}).$$

In general, the representation ring is $\mathbb{Z}[\mathsf{G}]^{\mathsf{G}}$ where G acts by conjugation. But we are mostly interested in abelian groups G, for which $K_{\mathsf{G}}(\mathrm{pt}) = \mathbb{Z}[\mathsf{G}]$.

4.8

K-theory is a *contravariant* theory, meaning that if there is a map of equivariant spaces

$$\begin{array}{ccc} \mathsf{G}_1 & \longrightarrow & \mathsf{G}_2 \\ \circlearrowright & & \circlearrowright \\ X & \longrightarrow & Y \end{array}$$

then there is a pullback of vector bundles $f^*\colon K_{\mathsf{G}_2}(X_2) \to K_{\mathsf{G}_1}(X_1)$. Since every space maps to a point, $K_{\mathsf{G}}(X)$ is in general an algebra over $K_{\mathsf{G}}(\mathrm{pt})$. When we begin discussing elliptic cohomology, there will be no more rings and instead we will work with

$$\operatorname{Spec} K_{\mathsf{G}}(X) \to \operatorname{Spec} K_{\mathsf{G}}(\mathrm{pt}) = \mathsf{G}.$$

4.9

Moving away from topology, there are two *algebraic* versions of K-theory. One can take algebraic G-equivariant vector bundles, but in algebraic geometry there is the more flexible object of G-equivariant *coherent sheaves*. These are sheaves $\mathcal{E}$ which have finite presentation, i.e. can be written

$$\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E}$$

in terms of finitely many generators $\mathcal{F}_1$ and finitely many relations $\mathcal{F}_2$. (Both $\mathcal{F}_i$ must be locally free, and for Noetherian schemes the finiteness of the latter is automatic.) The group G then actually acts on the $\mathcal{F}_i$, but for a more intrinsic description of G-equivariant sheaves see [CG97, Chapter 5], [Mer05].

Definition. Let

$$K_{\mathsf{G}}(X) \coloneqq K(\mathsf{Coh}_G(X))$$
$$K_{\mathsf{G}}^{\mathsf{perf}}(X) \coloneqq K(\mathsf{Vect}_G(X))$$

denote the Grothendieck K-group of G-equivariant coherent sheaves and of G-equivariant vector bundles respectively. A short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ gives a relation

$$[\mathcal{E}_2] = [\mathcal{E}_1] \oplus [\mathcal{E}_3].$$

A basic fact in algebraic geometry is that if X is non-singular, then

$$K_{\mathsf{G}}(X) = K_{\mathsf{G}}^{\mathsf{perf}}(X)$$

because every coherent sheaf has a finite locally free resolution.

4.10

K-theory is supposed to be a functor, so now we discuss its functorial properties. Given a map $f\colon X \to Y$ of algebraic varieties, factor it as a composition

$$X \xrightarrow{\iota} \Gamma_f \xrightarrow{\pi} Y$$

where $\Gamma_f \coloneqq \{(x, f(x))\} \subset X \times Y$ is the graph of f. Then it is easy to apply ι to sheaves, namely

$$\iota_*\mathcal{E} = (\text{extension by zero outside } \Gamma_f).$$

But we should *not* define $\pi_*[\mathcal{G}]$ to be $[\pi_*\mathcal{G}]$, since for example the pushforward of an exact sequence ceases to be exact. The correct definition includes all higher cohomology groups:

$$\pi_*[\mathcal{G}] \coloneqq \sum_i (-1)^i [R^i\pi_*\mathcal{G}]. \tag{11}$$

By long exact sequences in cohomology, this is a well-defined map between K-theories if each term $R^i\pi_*\mathcal{G}$ is a coherent sheaf. This is implied if f is proper. Note that additional restrictions on f are necessary in order to land inside $K_{\mathsf{G}}^{\mathsf{perf}}(X) \subset K_{\mathsf{G}}(X)$.

Actually, in equivariant theory, the properness condition on f may be relaxed. One can have a module which is infinite-dimensional but nonetheless has finite multiplicities for the group. The simplest example is to take the map $f\colon \mathbb{A}^1 \to \mathrm{pt}$. Letting $x \in \mathbb{A}^1$ denote the coordinate function,

$$f_*\mathcal{O}_{\mathbb{A}^1} = \mathbb{C}[x].$$

If we have an action which scales the $\mathbb{A}^1$ with weight t, then x has weight t^{-1} and

$$\text{character of } \mathbb{C}[x] = 1 + t^{-1} + t^{-2} + \cdots \tag{12}$$

which is a well-defined expression. It suffices to assume, for f, that it can be contracted to a proper subset. Here "contraction" implies a choice of attracting/repelling directions, and that choice is reflected by the direction of expansion of infinite series like (12). For $\mathbb{A}^1$, acting by t contracts everything to the point 0.

4.11

There is also a pullback in K-theory. If π is a projection, then $\pi^*[\mathcal{E}]$ is just the usual pullback of $\mathcal{E}$. However, $\iota^*[\mathcal{G}]$ is not just $[\iota^*\mathcal{G}]$ since $\iota^*\mathcal{G} = \mathcal{G} \otimes \mathcal{O}_\Gamma$ and tensoring with $\mathcal{O}_\Gamma$ is not exact. Like in (11) we must include all higher Tors:

$$\iota^*[\mathcal{G}] \coloneqq \sum_i (-1)^i \operatorname{Tor}^i(\mathcal{G}, \mathcal{O}_\Gamma).$$

To compute this, we can resolve *either* $\mathcal{G}$ or $\mathcal{O}_\Gamma$ by locally free sheaves. Some assumptions either on the map f or the sheaf $\mathcal{G}$ are necessary in order for the sum to be finite and therefore well-defined. For example, it suffices for f to be locally free (in which case the resolution is just one term), or more generally for f to be flat or of finite Tor dimension. Equivariantly these restrictions may again be relaxed, e.g. if $X \coloneqq \{xy = 0\} \subset \mathbb{A}^2$ then $\mathcal{G} = \mathcal{O}_0$ only has infinite resolutions, but the character like in (12) is well-defined.

4.12

Example. We want to compute $K_{\mathrm{GL}(V)}(\mathbb{P}(V))$. The first two steps of this computation are very general. Firstly, recall that

$$\mathbb{P}(V) = (V \setminus \{0\}) / \operatorname{GL}(1).$$

Equivariant theories are actually *simpler* than non-equivariant ones, for the reason that many important spaces can be exhibited as quotients. To access their cohomology, we can use the principle in K-theory that if G acts freely on X, then there is an isomorphism

$$K(X/\mathsf{G}) \underset{\text{invariants}}{\overset{\text{pullback}}{\rightleftarrows}} K_{\mathsf{G}}(X) .$$

(These maps are in fact equivalences of categories, so even the higher K-groups are the same.) Hence

$$K_{\mathrm{GL}(V)}(\mathbb{P}(V)) = K_{\mathrm{GL}(V)\times \mathrm{GL}(1)}(V \setminus \{0\})$$

and this new GL(1) is the center of GL(V). Abbreviate $\mathsf{G} := \mathrm{GL}(V) \times \mathrm{GL}(1)$.

Cohomology theories in general are supposed to take certain sequences of spaces to long exact sequences of cohomology groups. The incarnation here is that if $Y \hookrightarrow X$ is a closed G-invariant embedding, then there is an exact sequence

$$K_{\mathsf{G}}(Y) \xrightarrow{\text{pushforward}} K_{\mathsf{G}}(X) \xrightarrow{\text{restriction}} K_{\mathsf{G}}(X \setminus Y) \to 0.$$

Continuing to the left, there are then higher K-groups $K^1_{\mathsf{G}}(X \setminus Y)$ and so on. Applying this to $\mathbb{P}(V)$,

$$K_{\mathsf{G}}(\mathrm{pt}) \to K_{\mathsf{G}}(V) \to K_{\mathsf{G}}(V \setminus \{0\}) \to 0.$$

So it remains to compute the first two terms and the cokernel of the map.

A sheaf on a vector space V which is equivariant with respect to e.g. the scaling action can, firstly, be equivariantly resolved by free sheaves, and secondly these free sheaves are fully determined by their fibers over $0 \in V$. In other words, more canonically, the pullback

$$K_{\mathrm{eq}}(\mathrm{pt}) \to K_{\mathrm{eq}}(V)$$

is an isomorphism. Hence, putting everything together,

$$K_{\mathrm{GL}(V)}(\mathbb{P}(V)) = \operatorname{coker}\left(K_{\mathsf{G}}(\mathrm{pt}) \to K_{\mathsf{G}}(\mathrm{pt})\right)$$

where the map is pushforward along the inclusion $\{0\} \hookrightarrow V$. The remaining beautiful step is to compute this pushforward using the beautiful, classical *Koszul resolution*. We will do this in Lecture 6.

Lecture 5. Equivariant cobordism and rigidity

Lecture by Igor Krichever.

5.1

The main focus of today will be elliptic cohomology, whose little sisters, K-theory and ordinary cohomology, we have already seen. It is perhaps easier to understand them all as certain evaluations of the mother of the whole family: unitary cobordism.

Definition. An *extraordinary cohomology theory* is a functor

$$h^*\colon \begin{pmatrix} \text{pairs } (X, A) \text{ of} \\ \text{topological spaces} \end{pmatrix} \to (\text{an abelian category}).$$

To be on the safe side, we assume the topological spaces are CW complexes. The functor h^* must satisfy all the Eilenberg–Steenrod axioms (homotopy, long exact sequence, excision, disjoint union, dimension) except for the dimension axiom requiring $h^n(\mathrm{pt}) = 0$ for $n > 0$.

The dimension axiom holds in ordinary cohomology but not in K-theory, for instance.

5.2

We will talk about multiplicative cohomology theories and more. These are "nice" theories with a notion of Chern classes for complex vector bundles. For them to exist, we require that all complex vector bundles $\xi \to X$ are h-orientable; for us, this just means there is a Thom isomorphism

$$h^*(X) \cong h^*(B(\xi), S(\xi))$$

where $B(\xi)$ (resp. $S(\xi)$) is the bundle over X of unit disks (resp. unit spheres) in ξ. The isomorphism is defined by multiplication by the *Thom class* $t(\xi) \in h^{2n}(B, S)$. Such an isomorphism is more difficult but can also be established in more general cohomology theories like K-theory and cobordism.

The existence of the Thom class plays an essential role in all localization theorems. Let $\iota\colon X \to B$ be the inclusion of the zero section. Then $\iota^* t(\xi) = e(\xi)$ is the Euler class of the vector bundle, and as a result there is the so-called *Gysin exact sequence*

$$\cdots \to h^{*-2n}(X) \xrightarrow{\cdot e(\xi)} h^*(X) \to h^*(S) \xrightarrow{\delta} h^{*-2n+1}(X) \to \cdots .$$

It is used to derive the localization formula, but also can be used to easily compute

$$h^*(\mathbb{CP}^\infty) \cong h^*(\mathrm{pt})[[u]]$$

where $u \coloneqq e(\mathcal{O}_{\mathbb{CP}^\infty}(1))$ is the Euler class of the universal line bundle.

5.3

The *Brown representation theorem* says that any cohomology theory is uniquely defined by its *spectrum*, which is a sequence $M_n \to M_{n+1} \to \cdots$ of spaces with maps between them such that there is a homotopy equivalenced $S^1 \wedge M_n \to M_{n+1}$ of the suspension with the next space in the list. Then

$$h^*(X, A) = \varinjlim_{k\to\infty} [S^k \wedge X/A, M_{n+k}]$$

and the corresponding homology theory is

$$h_*(X, A) = \varinjlim_{k\to\infty} [S^{k+n}, X/A \wedge M_k].$$

However sometimes it is highly non-trivial to construct the spectrum, e.g. for elliptic cohomology. From a pedestrian point of view, it is better to treat this theorem as an existence result.

5.4

We will discuss cohomology theories in much more down-to-earth terms. Before we do so, we require some facts from the theory of formal commutative 1-dimensional groups. A formal group over a ring K can be treated as a series

$$f(u, v) \in K[[u, v]]$$

called the *formal group law*, which must satisfy certain properties:

1. (identity) $f(u,0) = u$;
2. (commutativity) $f(u,v) = f(v,u)$;
3. (associativity) $f(f(u,v),w) = f(u,f(v,w))$.

Properties 1 and 2 imply that

$$f(u,v) = u + v + \sum_{i+j>1} \alpha_{ij} u^i v^j, \quad \alpha_{ij} \in K,$$

and associativity imposes many relations on the coefficients α_{ij}. Given a cohomology theory h^*, one checks that the series

$$f_h(u,v) \coloneqq c_1(\mathcal{O}(1) \boxtimes \mathcal{O}(1)) \in h^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = h^*(\mathrm{pt})[[u,v]]$$

is a formal group law. We will see that this formal group law almost (and in some cases, completely) defines the cohomology theory.

Example. 1. The additive formal group law is $f(u,v) = u + v$. This is how Chern classes behave in ordinary cohomology theory: $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$.

2. The multiplicative formal group law is $f(u,v) = u + v + uv$. This is how Chern classes behave in K-theory.

3. A less obvious one is the *elliptic* formal group law

$$f(u,v) = \frac{u\sqrt{P(v)} + v\sqrt{P(u)}}{1 - \epsilon u^2 v^2}$$

where $P(t) \coloneqq 1 - 2\delta t^2 + \epsilon t^4$. The variables δ, ϵ belong to the underlying ring $K \coloneqq \mathbb{Z}[1/2][\delta, \epsilon]$.

5.5

A result of Lazard [Laz55] says there exists a universal formal group $F(u,v)$ over a special ring $\widehat{K}$, called the *Lazard ring*, such that any other formal group $f(u,v)$ over a ring K can be written as

$$f(u,v) = \widetilde{h}(F(u,v))$$

for some homomorphism $\widetilde{h}\colon \widehat{K} \to K$. In other words, formal groups are classified by the choice of homomorphism $\widetilde{h}$.

The Lazard ring $\widehat{K}$ is obtained as follows. Over $\mathbb{Q}$, all formal groups are isomorphic. Namely, they can all be written in the form

$$f(u,v) = g^{-1}(g(u) + g(v))$$

for some $g(u) = u + \sum_{i>1} \lambda_i u^i$ called the *logarithm* of f. If instead we plug in

$$g(u) \coloneqq \sum_{n=0}^{\infty} \frac{x_n}{n+1} u^{n+1}$$

for independent formal variables x_n, then the resulting $f(u,v)$ has coefficients which are rational polynomials of the x_n. Then $\widehat{K}$ is the smallest ring containing these coefficients.

The magical fact, due to Quillen [Qui69] and also present in works of Miščenko, Novikov, and Buchstaber, is that the formal group of unitary cobordism is exactly Lazard's universal formal group. In fact the starting point was Miščenko's work [Nov67] showing that for unitary cobordism

$$g(u) = \sum_{i=0}^{\infty} \frac{[\mathbb{CP}^n]}{n+1} u^{n+1}, \tag{13}$$

but there was a need to check that there were no additional relations among the coefficients. This was done by Quillen.

5.6

Let U^* denote unitary cobordism, which we will define shortly. As a consequence of Lazard's theorem, any cohomology theory h^* has an associated homomorphism $\widetilde{h}\colon U^*(\mathrm{pt}) \to h^*(\mathrm{pt})$, and

$$h^*(X,Y) = U^*(X,Y) \otimes_{\widetilde{h}} h^*(\mathrm{pt}) \tag{14}$$

as 2-graded theories (meaning grading by odd and even degrees). Such a tensor product preserves all the Eilenberg–Steenrod axioms except possibly the exactness of long exact sequences, and we say $\widetilde{h}$ is *Landweber flat* if it indeed also preserves exactness. We are interested in performing this procedure in reverse, to construct a cohomology theory from (14) given a homomorphism from $U^*(\mathrm{pt})$ into a ring.

5.7

To define the cobordism theory U^*, it is easier to begin with unitary *bordism* theory. Two n-dimensional manifolds M^n and $\widetilde{M}^n$ are *bordant*, written $M^n \sim \widetilde{M}^n$, if there exists an $(n+1)$-dimensional manifold W with

$$\partial W^{n+1} = M^n \cup (-\widetilde{M}^n). \tag{15}$$

Here the minus sign means to take the opposite orientation. Then, given a pair (X,A) of topological spaces, the *oriented bordism* group is

$$\Omega(X,A) \coloneqq \{\text{maps } (M,\partial M) \to (X,A)\}/(\sim \text{ bordism}),$$

where $(M,\partial M)$ is any manifold with boundary.

There are many flavors of bordism theory, given by putting additional structure on W^{n+1}, which we have already implicitly done above by taking orientation into account. For example, in the unoriented version the minus sign in (15) is irrelevant and is omitted. For the *unitary bordism* group $U_*(X,A)$, the stable tangent bundles of bordisms W^{n+1} are required to carry a complex structure.

Remark. The spectrum $\mathrm{MU}(n)$ of unitary cobordism theory is very simple: they are Thom spaces of universal $U(n)$-bundles over $PU(n)$.

5.8

There is a canonical isomorphism $U_{2n}(\mathrm{pt}) \cong U^{2n}(\mathrm{pt})$ of coefficient rings of bordism and cobordism, and so we now begin the study of homomorphisms $U_*(\mathrm{pt}) \to \mathbb{Q}$. The ring structure on $U_*(\mathrm{pt})$ is given by direct product of manifolds.

Lemma. *Every homomorphism $\widehat{h}\colon U_*(\mathrm{pt}) \to \mathbb{Q}$ arises as*

$$\widehat{h}(X) = \left\langle \prod_{i=1}^{n} \frac{x_i}{h(x_i)}, X \right\rangle$$

for some power series $h(x) = x + \sum \lambda_i x^i$, where in the rhs we treat the k-th elementary symmetric polynomial of the x_i as the k-th Chern class.

Definition. The series $h(x)$ is a *Hirzebruch multiplicative genus.*

Alternatively and perhaps more concretely, the formal group associated to a homomorphism $\widehat{h}\colon U_*(\mathrm{pt}) \to \mathbb{Q}$ has logarithm given by $g_h(u) \coloneqq \sum_{n=0}^{\infty} \frac{1}{n+1}\widehat{h}([\mathbb{CP}^n])u^{n+1}$, cf. (13), and the Hirzebruch genus is its functional inverse

$$h(x) = g_h^{-1}(x).$$

This is an observation due to Novikov. All classical genera can be obtained in this way, e.g. Todd genus $\mathrm{td}[\mathbb{CP}^n] \coloneqq 1$, signature $\mathrm{sign}[\mathbb{CP}^{2n}] \coloneqq 0$ and $\mathrm{sign}[\mathbb{CP}^{2n+1}] \coloneqq 1$, and A-genus for spin manifolds.

5.9

In many cases, we don't know how to define a cohomology theory, e.g. the definition of elliptic cohomology is very complicated. So, as a good approximation, instead of a cohomology theory we just consider a pair

$$(\text{cohomology theory}) \approx U^* + (\text{Hirzebruch genus}).$$

There is a good notion of generalized elliptic genus, which is a particular Hirzebruch genus, capturing a 4-parameter family of formal groups. All classical genera, and elliptic genus, are particular cases.

Definition. Let

$$\Phi(x, z|\omega_1, \omega_2) \coloneqq \frac{\sigma(z-x)}{\sigma(x)\sigma(z)} e^{\zeta(z)x}$$

where ζ and $\sigma \coloneqq \zeta'$ are Weierstrass functions for the elliptic curve E with periods $2\omega_1$ and $2\omega_2$. It is periodic in z with

$$\Phi(x, z + 2\omega_\alpha) = \Phi(x, z)$$

but is not an elliptic function of z because it has an essential singularity at $z = 0$. In the variable x,

$$\Phi(x + 2\omega_\alpha, z) = \Phi(x, z)e^{2\zeta(z)\omega_\alpha - 2\eta_\alpha z},$$

so it is a section of a certain line bundle on E, and near $x = 0$ $\Phi(x, z) = x^{-1} + O(1)$. The Hirzebruch genus of interest is

$$\widehat{\varphi}(x, z, k|\omega_1, \omega_2) \coloneqq \frac{e^{kx}}{\Phi(x, z|\omega_1, \omega_2)}$$

By definition, any such series defines a genus, and the result is the *generalized elliptic genus*.

5.10

In fact the function Φ was known from a different perspective, even before the introduction of elliptic genera. It is the simplest case of a Baker–Akhiezer function, and satisfies the Lamé equation

$$\left(\frac{d^2}{dx^2} - 2\wp(x)\right)\Phi(x, z) = \wp(x)\Phi(x, z)$$

where $\wp(x)$ is the Weierstrass elliptic function. It also satisfies a remarkable "addition-type" relation (for all z, which we omit in the notation)

$$\Phi(x + y)[\wp(y) - \wp(x)] = \Phi'(x)\Phi(y) - \Phi(x)\Phi'(y)$$

underlying its use in many settings, e.g. elliptic Calogero–Moser systems. Finally,

$$\Phi(x, z)\Phi(-x, z) = \wp(z) - \wp(x).$$

It is an easy exercise to check that upon specializing $z = \omega_\alpha$ to a half-period, $\Phi(x, \omega_\alpha)$ is the elliptic formal group. Classical genera correspond to degenerations of this elliptic curve to nodal rational curves, i.e. to multiplicative and additive formal groups, which parallels the degenerations of the elliptic Calogero–Moser system to the trigonometric and rational cases.

The genus Φ is the most general "nice" genus. To explain what is so nice about it, we return to the original goal of defining equivariant cobordism.

5.11

For a space X with action by a group G, equivariant *bordism* theory is

$$U_*^{\mathsf{G}}(X, A) \coloneqq \{\mathsf{G}\text{-equivariant maps } (M, \partial M) \to (X, A)\}/(\sim \text{ unitary bordism}).$$

Equivariant *cobordism* $U_{\mathsf{G}}^*(X, A)$ is harder to define. With Poincaré duality in mind, cobordism is the obvious dual to bordism. However, we will try to study $U_{\mathsf{G}}^*(X, A)$ without an explicit definition. For instance, an obvious requirement is that if X is a free G-manifold

$$U_{\mathsf{G}}^*(X) = U^*(X/\mathsf{G}).$$

More generally, let X be a space with G-action. Then there is the *Borel construction*

$$X_{\mathsf{G}} \coloneqq (X \times E\mathsf{G})/\mathsf{G},$$

which is a bundle over the classifying space $B\mathsf{G}$ with fiber X. The space $E\mathsf{G}$ is the universal contractible G-bundle on BG. One can try to imitate ordinary cohomology and consider $U^*(X_{\mathsf{G}})$ as the G-equivariant version of $U^*(X)$, but this is not what we are looking for; already in K-theory, $K_{\mathsf{G}}(X) \neq K(X_{\mathsf{G}})$, see e.g. [].

5.12

Characteristic classes are elements $\chi \in U^*(B\mathsf{U})$ of the cohomology of the classifying space. If ξ is a G-vector bundle on X then the Borel construction gives a bundle

$$\xi_{\mathsf{G}} \coloneqq (\xi \times E\mathsf{G})/\mathsf{G}$$

on X_{G}, and the G-equivariant version of χ is defined to be

$$\chi^{\mathsf{G}}(\xi) \coloneqq p_! \chi(\xi_{\mathsf{G}}) \in U^*(B\mathsf{G})$$

where $p\colon BU \to B\mathsf{G}$ is the projection.

Let $\mathsf{H} \subset \mathsf{G}$ be a normal subgroup. Then a very general localization theorem says

$$f_*(\chi(X, \xi_{\mathsf{G}})) = \sum_S f_{S,*}\left(\frac{\chi(S, \xi_{\mathsf{G}}|_S)}{e(\mathcal{N}_{S/X}|_S)}\right)$$

where S ranges over connected components of the fixed locus of the H action. This formula is well-known in cohomology, K-theory, etc. in the case where $\mathsf{H} = \mathsf{G}$, but as we will see later it is helpful to write it for any H.

5.13

Any cohomology theory defines a homomorphism $\widehat{h}\colon U^*(\mathrm{pt}) \to \mathbb{Q}$, but this can actually be extended to functors between cohomology theories, e.g. $\widehat{\widehat{h}}\colon U^*(X) \to K^*(X) \otimes \mathbb{Q}$. The equivariant version

$$h^{\mathsf{G}}(X)\colon \widehat{\widehat{h}}(p_!(1)) \in K_{\mathsf{G}}(\mathrm{pt}) \otimes \mathbb{Q}$$

is called the *index genus*. While the usual Hirzebruch genus is valued in $\mathbb{Q}$, this is valued in G-representations. For example, while all classical genera are indices $\dim\operatorname{coker} - \dim\ker$ of certain elliptic operators, the index genus amounts to just taking $\operatorname{coker} - \ker$ instead.

If G is connected, then representations of G in cohomologies $H^*(X, \mathbb{Z})$ with integer coefficients must be trivial. Atiyah and Hirzebruch showed that this implies $h^{\mathsf{G}}(X)$ is constant, namely

$$\operatorname{im} h^{\mathsf{G}} \subset \mathbb{Q} \subset K_{\mathsf{G}}(\mathrm{pt}) \otimes \mathbb{Q},$$

known as a *rigidity property*. They used rigidity to prove many remarkable results about genera, e.g. that for S^1-manifolds $\operatorname{sign}(X) = \sum_S \operatorname{sign}(S)$ where S ranges over S^1-fixed components. Such a statement is clear for things like Euler characteristic, but is non-trivial and surprising for the signature. In fact the whole development of elliptic cohomology stemmed from attempts to prove Witten's conjecture on the rigidity of elliptic genus for S^1-manifolds; elliptic genus was originally realized as an index of a Dirac operator on loop spaces, which is difficult to make rigorous mathematically.

5.14

We conclude with the rigidity property of generalized elliptic genus, which provides an explanation for why elliptic cohomology, K-theory, etc. are so special: they arise from a general genus with this rigidity property.

Theorem ([Kri90]). *Generalized elliptic genus is rigid for* SU*-manifolds.*

Lecture 6. Comparing K-theory to elliptic cohomology in an example

Koszul resolutions, localization, first steps in elliptic cohomology, Chern classes, equivariant K-theory and equivariant elliptic cohomology of $\mathrm{Hilb}(\mathbb{C}^2)$*, Thom spaces.*

6.1

We return to the computation of $K_{\mathrm{GL}(V)}(\mathbb{P}(V))$. Recall what we have done so far. The first step is to express

$$\mathbb{P}(V) = (V \setminus \{0\})/\,\mathrm{GL}(1)$$

where $\mathrm{GL}(1)$ acts by scaling V. There is a general principle that $K_{\mathsf{G}}(Y) = K(Y/\mathsf{G})$ for the free action of a group G on a space Y, and therefore

$$K_{\mathrm{GL}(V)}(\mathbb{P}(V)) = K_{\mathrm{GL}(V)\times \mathrm{GL}(1)}(V \setminus \{0\}). \tag{16}$$

In fact it is a nice exercise to deduce from this general principle that $K_{\mathsf{G}}(\mathsf{G}/H \times X) = K_H(X)$. Continuing, the excision long exact sequence associated to the embedding $\iota\colon \{0\} \hookrightarrow V$ is

$$\cdots \to K^1_{\mathsf{G}}(V \setminus \{0\}) \to K_{\mathsf{G}}(\{0\}) \xrightarrow{\iota_*} K_{\mathsf{G}}(V) \xrightarrow{\text{restriction}} K_{\mathsf{G}}(V \setminus \{0\}) \to 0,$$

and we want the cokernel of ι_*.

If $\pi\colon V \to \{0\}$ is the projection, then $K_{\mathsf{G}}(\{0\}) \xrightarrow{\pi^*} K_{\mathsf{G}}(V)$ is an isomorphism. This is because any module over a polynomial ring has a finite locally free resolution of length $\dim V$ by Hilbert's syzygy theorem, but a locally free G-equivariant module is actually free by using the G-action to move around.

Hence we write ι_* as $R(\mathsf{G}) \xrightarrow{\iota_*} R(\mathsf{G})$, and all that remains is to compute the equivariant resolution of the image $\iota_*\mathcal{O}_0$ of the generator. This is the subject of Koszul resolutions.

6.2

Let $\mathsf{T} \subset \mathsf{G}$ be the maximal torus. Any G-equivariant resolution is in particular a T-equivariant resolution. Say $\dim V = 2$, and let $R \coloneqq \mathbb{C}[x, y]$. The typical T-equivariant module is a quotient R/I for a T-equivariant ideal I, which we equivalently write as a partition. Then, as shown in Figure 7:

- R/I is generated by a single element ;

- the kernel of $R \cdot$ ■ $\to R/I$ is generated by elements ■;
- the kernel of $\bigoplus R \cdot$ ■ $\to R \cdot$ ■ is the free module generated by elements ■.

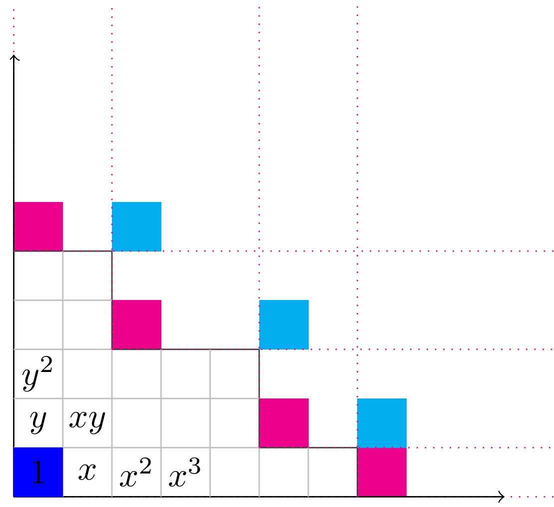


Figure 7: A monomial ideal I, and generators in the Koszul resolution of R/I

In summary, there is a free resolution

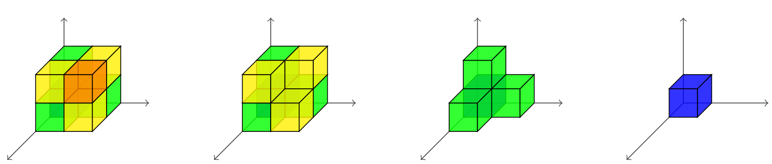
$$0 \to \bigoplus R \cdot ■ \to \bigoplus R \cdot ■ \to R \cdot ■ \to R/I \to 0.$$

of length 2, in accordance with Hilbert's syzygy theorem.

6.3

As another example, consider $\dim V = 3$ and the analogous Koszul resolution of $\mathcal{O}_0 = R/\langle x_1, x_2, x_3 \rangle$, where we visualize $\mathbb{C}[x_1, x_2, x_3]$-modules by drawing 3-dimensional boxes:

$$0 \longrightarrow Rx_1x_2x_3 \longrightarrow \bigoplus_{i<j} Rx_ix_j \longrightarrow \bigoplus_i Rx_i \longrightarrow R \longrightarrow \mathcal{O}_0 \longrightarrow 0.$$



This is a T-equivariant resolution as written, but it is clear how to write it in a G-equivariant fashion:

$$0 \to \wedge^3 V^* \otimes \mathcal{O}_V \xrightarrow{d} \wedge^2 V^* \otimes \mathcal{O}_V \xrightarrow{d} V^* \otimes \mathcal{O}_V \xrightarrow{d} \mathcal{O}_V \xrightarrow{d} \mathcal{O}_0 \to 0.$$

Here it is important to remember that the symbols x_i are *functions* on V and therefore live in V^*. The differential d may be written in a manifestly $\mathrm{GL}(V)$-equivariant way as

$$d = \sum_i \frac{d}{dx_i} \otimes x_i,$$

where d/dx_i is meant in the odd sense: commute x_i to the front if it exists, and then $d/dx_i = 1$ if it exists and 0 otherwise. One can check that $d^2 = 0$ using that $(d/dx_i)^2 = 0$ and d/dx_i and d/dx_j anti-commute. Finally, the extra equivariance $s \in \mathrm{GL}(1)$ from acts by s^{-k} on $\wedge^k V^*$. In the identification (16), it is a good exercise to verify $s \mapsto \mathcal{O}(1)$.

6.4

It follows that, in general,

$$K_{\mathrm{GL}(V)}(\mathbb{P}(V)) = K_{\mathrm{GL}(V)}(\mathrm{pt})[s^{\pm 1}] \Big/ \Big\langle \sum (-s)^i \wedge^i V = 0 \Big\rangle.$$

This can equivalently be written in coordinates as follows. Any conjugation-invariant function on $\mathrm{GL}(V)$ is uniquely determined by its restriction to the maximal torus T, whose coordinates we denote by $a_1, \ldots, a_n$, and therefore $K_{\mathrm{GL}(V)}(\mathrm{pt})$ consists of symmetric polynomials in the a_i. As for the relation, a basis of $\wedge^i V$ is given by i-element subsets of $\{1, \ldots, n\}$, e.g.

$$\wedge^3 V = a_1 a_2 a_3 + a_1 a_2 a_4 + \cdots = e_3(a_1, a_2, a_3)$$

is the third elementary symmetric polynomial, and the generating function of elementary symmetric polynomials is $\prod(1 - s a_i)$. Hence

$$K_{\mathrm{GL}(V)}(\mathbb{P}(V)) = (\text{sym polys in } a_i)\,[s^{\pm 1}] \Big/ \Big\langle \prod (1 - s a_i) = 0 \Big\rangle \tag{17}$$

$$= \bigcup_i \{s = a_i^{-1}\}. \tag{18}$$

It is instructive, especially later for elliptic cohomology, to visualize (18) as the union of tori $\{s = a_i^{-1}\}$ over the base $\operatorname{Spec} K_{\mathrm{GL}(V)}(\mathrm{pt})$. This is shown in Figure 8 for $\mathbb{P}(\mathbb{C}^2)$, where for clarity we use $\log s \in \mathbb{C}$ instead of $s \in \mathbb{C}/(\text{lattice})$ as the "vertical" coordinate.

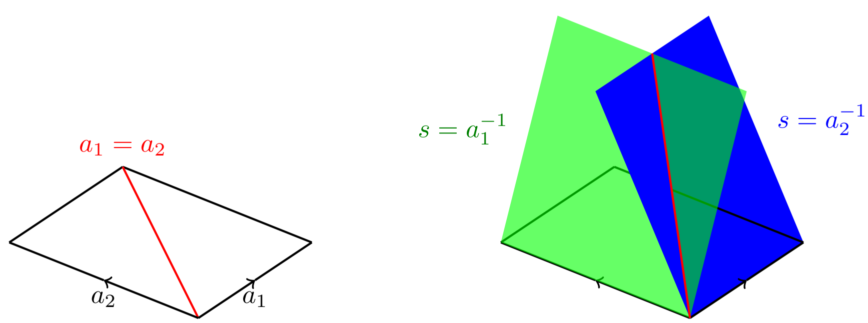


Figure 8: Visualization of $\operatorname{Spec} K_{\mathrm{GL}(2)}(\mathbb{P}(\mathbb{C}^2)) \to \operatorname{Spec} K_{\mathrm{GL}(2)}(\mathrm{pt})$

6.5

There are some lessons to draw here that are also relevant for elliptic cohomology. Warning: we will mildly abuse notation and let G denote both the compact group G_c and the split reductive group G_{alg}. The relation between the two is that

$$\mathsf{G}_{\text{alg}} = \operatorname{Spec} \bigoplus_{\substack{\text{irreps } V \\ \text{of } \mathsf{G}_c}} \operatorname{End}(V),$$

while by Peter–Weyl the ring of functions on G_c is the *completion* of this direct sum. Which of these two is meant by G should be clear from context. Let $\mathsf{G}_{ss} \subset \mathsf{G}_c$ denote the subgroup of semisimple elements.

6.6 Lesson 1

There is a stratification of the base $\operatorname{Spec} K_{\mathsf{G}}(\text{pt}) = \mathsf{G}_{ss}/\mathsf{G} = \mathsf{T}/W$ by different types of fixed loci. To be more precise, there is a diagram

$$\begin{array}{ccc} \operatorname{Spec} K_{\mathsf{G}}(X) & \longleftarrow & \operatorname{Spec} K(X^g) \\ \downarrow & & \downarrow \\ \operatorname{Spec} K_{\mathsf{G}}(\text{pt}) & \hookleftarrow & \{g\} \end{array}$$

and the fixed loci X^g are different (namely, larger) for non-generic g. Figure 9 shows this for $\mathbb{P}^2$, where $a_1 = 1$ for simplicity:

- the generic fiber $(1-s)(1-a_2 s)(1-a_3 s) = 0$ consists of three points, corresponding to $K((\mathbb{P}^2)^t) = K(\text{pt} \sqcup \text{pt} \sqcup \text{pt})$ for generic $t \in \mathsf{T}$;
- the hyperplane $\{a_2 = a_3\}$ has fiber $(1-s)(1-a_2 s)^2 = 0$ corresponding to the fixed locus of $\operatorname{diag}(1, a_2, a_2)$, namely $K(\text{pt} \sqcup \mathbb{P}^1)$;
- the origin $\{a_1 = a_2 = a_3\}$ has fiber $(1-s)^3 = 0$ corresponding to $K(\mathbb{P}^2)$.

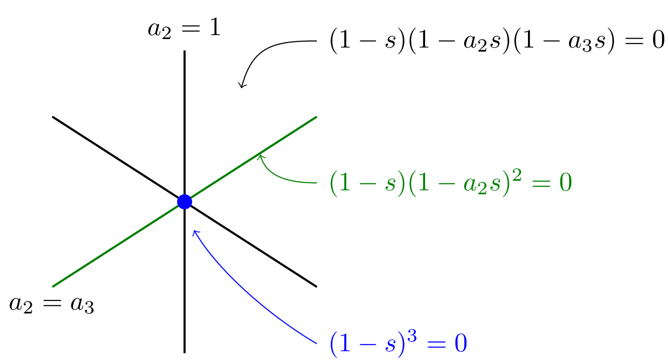


Figure 9: Stratification and fibers of $\operatorname{Spec} K_{GL(3)}(\mathbb{P}(\mathbb{C}^3)) \to \operatorname{Spec} K_{GL(3)}(\text{pt})$

One can view localization, roughly, as the statement that the fiber over a generic $t \in \mathsf{T}/W$ is $K(X^t) = K(X^{\mathsf{T}})$, noting that a generic point generates the whole torus T. The more standard statement of localization arises by changing coefficients to a field, thereby capturing only the generic point.

6.7

The analogue in elliptic cohomology is as follows. View the base $\operatorname{Spec} K_{\mathsf{G}}(\mathrm{pt})$ as G/G, the space of G-local systems on a circle S^1 defined by their monodromy up to conjugation. Then

$$\mathrm{Ell}_{\mathsf{G}_c}(\mathrm{pt}) = (\mathsf{G}_c\text{-local systems on an elliptic curve } E).$$

This is purely topological, but Narasimhan–Seshadri (for genus 1) tells us it is isomorphic to the space of sums of stable G-bundles on E of degree zero, which is exactly the dual abelian variety $E^\vee := \mathrm{Pic}^0(E)$. For various reasons it is better here to replace E by $E^\vee$, which doesn't change anything up to (non-canonical) isomorphism since $E = \mathrm{Pic}^0(E^\vee)$. But the only stable bundles on E are those of rank 1. Hence, for $\mathsf{G} = \mathrm{GL}(n)$ for simplicity,

$$\begin{aligned}\mathrm{Ell}_{\mathsf{G}_c}(\mathrm{pt}) &= \{\text{sums of line bundles on } E^\vee \text{ of degree } 0\}/W \\ &= E^{\mathrm{rank}\,\mathsf{G}}/W \\ &= E \otimes_{\mathbb{Z}} \mathrm{cochar}(\mathsf{T})/W,\end{aligned}$$

and we take the last line as the definition of $\mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})$. Note that unlike in cohomology or K-theory, this is no longer an *affine* scheme. Also, from the construction it is clear that for a subgroup $\mathsf{H} \subset \mathsf{G}$ there is a canonical inclusion

$$\mathrm{Ell}_{\mathsf{H}}(\mathrm{pt}) \subset \mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})$$

The stratification we obtain on $\mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})$ will correspond to those subgroups H for which the fixed loci are bigger than generic. For example, for our running example of $\mathbb{P}(V)$, the analogue of (18) is

$$\begin{array}{ccccccc}\mathrm{Ell}_{\mathrm{GL}(V)}(\mathbb{P}(V)) & = & \bigcup_i \{s = a_i^{-1}\} & \subset & E^n/S(n) & \times & E \\ & & & & \cup\!\!\!\!\!\!\; & & \cup \\ & & & & (a_1, \dots, a_n) & & s\end{array}$$

where $n := \dim V$.

More generally, elliptic cohomology $\mathrm{Ell}_{\mathsf{G}}(X)$ is *covariant* with respect to $(\mathsf{G}_1, X_1) \to (\mathsf{G}_2, X_2)$. This is in contrast to the contravariance of $H_{\mathsf{G}}(X)$ and $K_{\mathsf{G}}(X)$, and is because we took Spec. When $X_1 = X_2 = \mathrm{pt}$, we are back to the case of just group homomorphisms. A good example to keep in mind comes from

$$1 \to \mu_3 \to \mathrm{GL}(1) \xrightarrow{z^3} \mathrm{GL}(1) \to 1,$$

which, using that $\mathrm{Ell}_{U(1)}(\mathrm{pt}) = E$, induces

$$0 \to E[3] \to E \xrightarrow{3} E \to 0.$$

(The convention for abelian varieties is to write the group law additively.) Writing $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, we see the kernel $E[3]$ consists of nine lattice points in $\mathbb{Z} + \mathbb{Z}\tau$. They are interpreted as pairs of commuting elements in μ_3, or, equivalently, μ_3-bundles over E.

6.8 Lesson 2

Let G act on X and let $\mathcal{P} \to X$ be a principal H-bundle. A topologist would say such bundles are classified by maps $X \to BH$; more intrinsically, we say they are classified by $X \to [\mathrm{pt}/H]$. Pullback therefore gives a map

$$\operatorname{Spec} K(X) \to \operatorname{Spec} K([\mathrm{pt}/H]) = \operatorname{Spec} K_H(\mathrm{pt})$$

and this is the *Chern class*. Concretely, we will need this for rank-n (complex) vector bundles, which give

$$\operatorname{Spec} K(X) \to K_{\mathrm{GL}(n)}(\mathrm{pt}) = (\text{sym polys in } x_1, \ldots, x_n)$$

where the *Chern roots* $x_1, \ldots, x_n$ of V are coordinates for the maximal torus $\mathsf{T} \subset \mathrm{GL}(n)$. For example, the symmetric polynomial $e_k(x_1, \ldots, x_n)$ corresponds to $\wedge^k V$. Note that everything can be made equivariant with respect to G, e.g. the G-equivariant Chern class is $\operatorname{Spec} K_{\mathsf{G}}(X) \to \operatorname{Spec} K_{\mathsf{G}\times\mathrm{GL}(n)}(\mathrm{pt})$.

The story is literally the same for elliptic cohomology, only now the target of the Chern class map is $\mathrm{Ell}_{\mathrm{GL}(n)}(\mathrm{pt}) = S^n E \coloneqq E^n/S(n)$.

6.9

A collection of vector bundles $V_1, V_2, \ldots$ generate $K_{\mathsf{G}}(X)$ as a λ-ring (or with Adams operations), meaning to include $\wedge^k V_i$ for all k, if the Chern class map

$$\operatorname{Spec} K_{\mathsf{G}}(X) \hookrightarrow K_{\mathsf{G}}(\mathrm{pt}) \times \prod S^{\operatorname{rank} V_i} \mathsf{T}$$

is an embedding. For example, $\mathcal{O}(1)$ generates $K(\mathbb{P}^n)$. There is a standard argument, known as the resolution of the diagonal, for determining whether some collection of vector bundle $\{V_i\}$ generates $K_{\mathsf{G}}(X)$. Let $\Delta \subset X \times X$ be the diagonal and suppose there is a resolution

$$\mathcal{O}_\Delta = \sum \mathcal{F}_j \boxtimes \mathcal{G}_k \in K_{\mathsf{G}}(X \times X) \tag{19}$$

such that all the $\mathcal{F}_j$ and $\mathcal{G}_k$ belong to the algebra generated by the V_i. When X is compact, convolution with $\mathcal{O}_\Delta$ is the identity operator on $K_{\mathsf{G}}(X)$, and applying this operator to $\mathcal{F} \in K_{\mathsf{G}}(X)$ gives

$$\mathcal{F} = \sum \chi(p_2^* \mathcal{F} \otimes \mathcal{G}_k) \mathcal{F}_j \in K_{\mathsf{G}}(X).$$

Hence $\{V_i\}$ generates all of $K_{\mathsf{G}}(X)$. More strongly, the resolution (19) already provides such a generating set: take either the $\{\mathcal{F}_j\}$ or the $\{\mathcal{G}_k\}$.

All the varieties we will consider have a natural resolution of the diagonal.

6.10

Example. On $\mathbb{P}^n$ there is the tautological sequence

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^{n+1} \to \mathcal{Q} \to 0.$$

Let $\mathcal{S} \coloneqq \mathcal{O}(-1)$ for brevity, and on $\mathbb{P}^n \times \mathbb{P}^n$ write $\mathcal{S}_i$ or $\mathcal{Q}_i$ to mean the respective bundle on the i-th factor. Consider the composition

$$s\colon \mathcal{S}_1 \to \mathbb{C}^{n+1} \to \mathcal{Q}_2,$$

which is an element of $\mathrm{Hom}(\mathcal{S}_1, \mathcal{Q}_2) = \mathcal{O}(1) \boxtimes \mathcal{Q}$. The diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ is exactly the locus $\{s = 0\}$. The resolution of the diagonal $\mathcal{O}_\Delta$ is therefore the Koszul complex of s, which evidently involves only powers of $\mathcal{O}(1)$.

6.11

Example. On $\mathrm{Hilb}(\mathbb{C}^2, n)$ there is a similar tautological sequence

$$0 \to \mathcal{I} \to \mathbb{C}[x_1, x_2] \to \mathbb{C}[x_1, x_2]/I \to 0$$

and the same argument shows that the quotient bundle generates the equivariant K-theory. While it is difficult to write what $K_{\mathsf{T}}(\mathrm{Hilb}(\mathbb{C}^2, n))$ is in equations, it is straightforward to describe it explicitly, using the quotient bundle, in the embedding

$$\operatorname{Spec} K_{\mathsf{T}}(\mathrm{Hilb}(\mathbb{C}^2, n)) \subset \mathsf{T} \times \operatorname{Spec} K_{\mathrm{GL}(n)}(\mathrm{pt}).$$

Let $s_1, \ldots, s_n$ be the coordinates of $K_{\mathrm{GL}(n)}(\mathrm{pt})$, up to the permutation action of $S(n)$ (cf. the $\mathbb{P}^n$ case where there was only one such coordinate s), and let t_1, t_2 be coordinates of T acting on $\mathbb{C}^2$ by scaling the axes. Then, over a generic point $\{s_i\}$, the fiber of $\operatorname{Spec} K_{\mathsf{T}}(\mathrm{Hilb}(\mathbb{C}^2, n))$ is a union over T-fixed points, which are partitions as in Figure 7, and for each partition the values of $\{s_i\}$ are the T-weights of the boxes of the partition, i.e. the monomials in $\mathbb{C}[x_1, x_2]/I$. Remember that $x_1, x_2 \in (\mathbb{C}^2)^*$ and have weights t_1^{-1} and t_2^{-1}. Over non-generic $\{s_i\}$, where fixed loci are bigger than usual, the components corresponding to different partitions interact in some complicated way.

The elliptic cohomology $\mathrm{Ell}_{\mathsf{T}}(\mathrm{Hilb}(\mathbb{C}^2, n))$ picture is exactly the same, but now the components are elliptic curves instead of tori.

6.12 Lesson 3

Let $\mathcal{E} \to X$ be a vector bundle, let $\mathbb{P}(\mathcal{E})$ be its projectivization, and consider $K_{\mathsf{G}}(\mathbb{P}(\mathcal{E}))$. When $\mathcal{E}$ is a trivial bundle over a point, this is our earlier computation for projective space, but the exact same argument shows that

$$K_{\mathsf{G}}(\mathbb{P}(\mathcal{E})) = K_{\mathsf{G}}(X)[s^{\pm 1}] \Big/ \Big\langle \sum (-s)^i \wedge^i \mathcal{E} = 0 \Big\rangle.$$

For instance, take $\mathcal{E}$ and compactify it fiber-wise by adding a projective space at infinity to get the *projective closure* $\mathbb{P}_X(\mathbb{C} \oplus \mathcal{E})$ of $\mathcal{E}$. Its K-theory is shown in Figure 10: the $s = 1$ component comes from the trivial factor $\mathbb{C}$, and the other components come from the Chern roots ϵ_i of $\mathcal{E}$.

The *Thom space* of $\mathcal{E}$ is the "twisted suspension"

$$\mathrm{Thom}(\mathcal{E}) \coloneqq \mathbb{P}_X(\mathbb{C} \oplus \mathcal{E})/\mathbb{P}_X(\mathcal{E}),$$

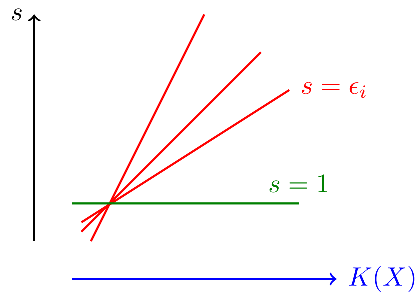


Figure 10: Spec K_{G} of $\mathbb{P}_X(\mathbb{C} \oplus \mathcal{E})$

and its equivariant K-theory therefore consists of functions on Spec $K_{\mathsf{G}}(\mathbb{P}_X(\mathbb{C}\oplus\mathcal{E}))$ that vanish on the $\{s = \epsilon_i\}$ components, i.e. the ideal generated by $(1 - \epsilon_i)$ in $K_{\mathsf{G}}(X)$. As a sheaf over $K_{\mathsf{G}}(X)$, this K-theory is isomorphic to $K_{\mathsf{G}}(X)$ and this is the Thom isomorphism.

6.13

However, something different happens in elliptic cohomology, where modding by the ideal generated by the $(1 - \epsilon_i)$ is *not* isomorphic to the structure sheaf over $\mathrm{Ell}_{\mathsf{G}}(X)$. This is the first place where the two theories differ.

Lecture 7. The Thom sheaf and Thom isomorphism

Modification of the Thom isomorphism in elliptic cohomology, Theta bundles and theta functions, Bott periodicity, pushforwards in elliptic cohomology, change of groups in equivariant elliptic cohomology.

7.1

We continue with the discussion of various aspects of K-theory, and compare and contrast them with elliptic cohomology. To begin, let's set up notation. Let

$$E = \mathrm{Pic}_0(E^\vee)$$

where the dictionary between $E^\vee$ and E is

$$E^\vee = \begin{cases} \text{smooth elliptic curve} \\ \text{nodal curve} \\ \text{cuspidal curve} \end{cases} \quad \rightsquigarrow \quad E = \begin{cases} \text{smooth elliptic curve} \\ \mathbb{G}_m \\ \mathbb{G}_a \end{cases}.$$

It is possible to take $E^\vee$ to be nodal with multiple components, in which case line bundles on $E^\vee$ are line bundles on the normalization along with the data of how they glue, namely multiple copies of $\mathbb{G}_m$. Note that, as written, everything is defined over families.

Let G be a compact (usually connected) topological group. We would like an *elliptic cohomology* functor

$$\mathrm{Ell}\colon (\mathsf{G}, X) \to \text{(super)schemes}$$

covariant in (G, X). The super-ness of the target is due to super-commutativity in cohomology between odd and even parts; it is not so important since we mostly work only on spaces with even cohomology. If there were an elliptic cohomology *ring*, the target (super)scheme should be viewed as its Spec. More precisely, due to covariance, we should write

$$\mathrm{Ell}\colon (\mathsf{G}, X) \to \text{(super)schemes over } \mathrm{Ell}_{\mathsf{G}}(\mathrm{pt}),$$

where, as discussed earlier, $\mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})$ is the space of G-bundles on $E^\vee$. (This picture gets more complicated if G is not connected.)

We will be able to understand a lot of elliptic cohomology by analogy with K-theory, using the fact that

$$E = \mathbb{G}_m/q^{\mathbb{Z}} \quad |q| < 1,$$

due to Jacobi over $\mathbb{C}$ or to Tate over $\mathbb{Q}_p$. Another useful picture to keep in mind is that, roughly,

$$\mathrm{Ell}(X) \approx K(\mathsf{Loops}(X)).$$

This came up earlier in our discussion of interfaces on Riemann surfaces, where the loops live, and is natural from the perspective of supersymmetric QFTs in dimension $2+1$.

7.2

So far we've discussed the equivariant K-theory of one specific type of space, for which the answer is the same in topological or algebraic K-theory: for a vector bundle $V \to X$,

$$K_{\mathsf{G}}(\mathbb{P}(V)) = K_{\mathsf{G}}(X)[s^{\pm 1}]/\Big\langle \prod (1 - v_i s) = 0 \Big\rangle$$

for any $\mathsf{G} \subset \mathrm{GL}(V)$, where v_i are the Chern roots of V (i.e. the restriction of V to the maximal torus of G).

The Thom space of a vector bundle V can be viewed as the bundle of unit balls in V with the boundary spheres collapsed to a point, or, perhaps a better picture, as the normal bundle to the embedding $V \to \mathrm{tot}(V)$ with the "infinity" of the vector bundle collapsed to a point. Algebraically we can take the projective closure $\mathbb{P}_X(V \oplus \mathcal{O}_X)$, which contains the projective spaces $\mathbb{P}_X(V)$ at infinity, and then define

$$\mathrm{Thom}(V) \coloneqq \mathbb{P}_X(V \oplus \mathcal{O}_X)/\mathbb{P}_X(V).$$

This can be viewed as a twisted suspension of X, where the twist happens both in the bundle and the G-action.

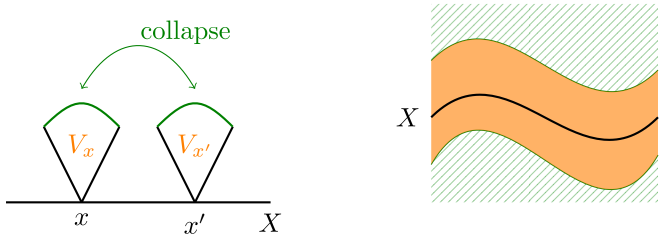


Figure 11: Two views of the Thom space of a vector bundle $V \to X$. Green regions are collapsed to a point.

7.3

As discussed last time, the K-theory of Thom(V) is the ideal $\prod_{i=1}^{r}(1-v_i)$ in the K-theory of X. In elliptic cohomology, recall that these Chern roots v_i are coordinates on the target of the Chern class map

$$\mathrm{Ell}_{eq}(X) \xrightarrow{c} \mathrm{Ell}_{\mathrm{GL}(r)}(\mathrm{pt}) = S^r E \ni (v_1, \ldots, v_r).$$

There is a natural divisor $S^{r-1}E \subset S^r E$ consisting of points $(v_1, \ldots, v_{r-1}, 0)$, up to permutation. From the perspective of $\mathrm{Ell}_{\mathrm{GL}(r)}(\mathrm{pt})$ as a moduli of sums of degree-0 line bundles on $E^\vee$, this is the codimension-1 locus where the bundles have a section, known as the *theta divisor* D_Θ. Hence, in elliptic cohomology, the Thom *sheaf* is

$$\mathrm{Thom}(V) = c^*\Big(\text{ideal of } \prod(1-v_i)\Big) = c^*(\mathcal{O}(-D_\Theta)).$$

Definition. Denote the *Thom sheaf* by $\Theta(-V)$. More generally, note that

$$\Theta(V_1 \oplus V_2) = \Theta(V_1) \otimes \Theta(V_2),$$

and therefore we can consider a homomorphism $\Theta\colon K(X) \to \mathrm{Pic}(\mathrm{Ell}(X))$.

The Thom isomorphism is therefore *not* an isomorphism in elliptic cohomology. Perhaps this serves as an indication that the Thom isomorphism in ordinary cohomology or K-theory, where it really is an isomorphism, is deeper and more non-trivial than one may initially suspect. While the Thom sheaf on Spec of cohomology or K-theory is the trivial line bundle, elliptic cohomology is a non-affine scheme where there are many *non-trivial* line bundles, one of which is the elliptic Thom sheaf.

7.4

A short digression on theta functions is appropriate here. The ordinary ϑ function comes from the embedding $S^0E \to S^1E = E$, which is

$$D_\Theta = \{0\} \subset E.$$

There is a unique section of $\mathcal{O}(D_\Theta)$ by Riemann–Roch, which is exactly ϑ. In the presentation $E = \mathbb{G}_m/q^{\mathbb{Z}}$, the origin is $\{x = 1\} \subset \mathbb{G}_m$, and the appropriate function on $\mathbb{G}_m$ is

$$\vartheta(x) \coloneqq (x^{1/2} - x^{-1/2}) \prod_{n>0} (1 - q^n x)(1 - q^n/x). \tag{20}$$

One can quickly verify that it is a section of the correct line bundle by computing

$$\vartheta(qx) = -q^{1/2} x^{-1} \vartheta(x).$$

Note that in (20) we have allowed a square root of x, which lets us normalize $\vartheta(x)$ so that it is odd under the involution $x \mapsto 1/x$ (corresponding to the involution $p \mapsto -p$ on E), namely

$$\vartheta(x^{-1}) = -\vartheta(x).$$

From (20) it is clear that ϑ has a unique zero at $x = 1$ (in the fundamental domain), and converges if $|q| < 1$.

7.5

Returning to the Thom isomorphism, in K-theory the Thom sheaf really is trivial and therefore

$$K_{\text{eq}}(\text{Thom}(V)) \cong K_{\text{eq}}(X)$$

by multiplication by $\prod(1 - v_i)$. For the trivial bundle $\mathbb{C} \to X$,

$$\text{Thom}(\mathbb{C} \to X) = \Sigma^2 X \tag{21}$$

is the two-fold suspension of X. (The one-fold suspension is shown in Figure 12.) Then the

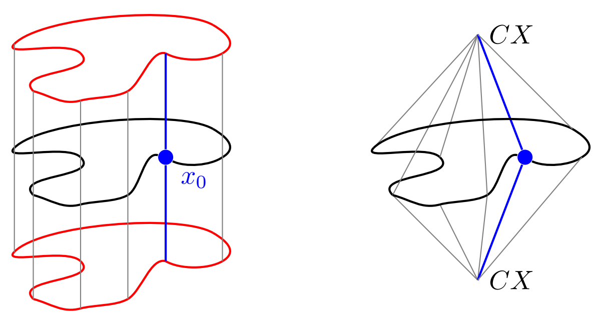


Figure 12: The suspension $\Sigma^1 X \coloneqq D^1 \times X / S^0 \times X \cup D^1 \times \{x_0\}$

Thom isomorphism is

$$K(\Sigma^2 X) \cong K(X) \tag{22}$$

which is known as *Bott periodicity*.

7.6

In general,

$$(\mathbb{C}^r\text{-bundles over } X) = [X, \mathrm{Gr}(r,\infty)],$$

because every bundle $V \to X$ is a quotient of the trivial bundle $\mathbb{C}^N$ of sufficiently large rank, by picking sufficiently many sections of V. Therefore V is an r-dimensional subspace of $\mathbb{C}^N$, and, as we don't know what N is, it suffices to send $N \to \infty$. The space

$$\mathrm{Gr}(r,\infty) \coloneqq \mathsf{Mat}_{\text{full rank}}(r \times \infty)/\operatorname{GL}(r)$$

is therefore denoted $BU(r)$, the classifying space of rank-r bundles. In K-theory, we study bundles up to stable equivalence, i.e. up to trivial summands. So

$$K(X) = [X, \mathbb{Z} \times \mathrm{BU}(\infty)] \tag{23}$$

where $\mathbb{Z}$ keeps track of the rank of the bundles. Note that $\mathbb{Z} \times \mathrm{BU}(\infty)$ is only a group *up to homotopy* with respect to $\oplus$ of bundles.

A bundle on ΣX is the data of a clutching function on X, since it can be trivialized on each cone leaving only the gluing data. So

$$K(\Sigma^1 X) = [X, U(\infty)].$$

On the other hand, from (23),

$$\begin{aligned} K(\Sigma^1 X) &= [\Sigma^1 X, \mathbb{Z} \times \mathrm{BU}(\infty)] \\ &= [X, \Omega^1(\mathbb{Z} \times \mathrm{BU}(\infty))] \end{aligned}$$

where $\Omega^1(-)$ denotes the space of loops. Hence there is a homotopy equivalence

$$U(\infty) \simeq \Omega^1(\mathbb{Z} \times \mathrm{BU}(\infty)).$$

Repeating with $\Sigma^2 X$ and using (22),

$$\Omega^1 U(\infty) \simeq \mathbb{Z} \times \mathrm{BU}(\infty).$$

So there is an infinite chain of spaces, each of which is loops of the previous one, and one way to state Bott periodicity is that this chain is 2-periodic. In particular, the homotopy groups of $U(\infty)$ and $\mathbb{Z} \times \mathrm{BU}(\infty)$ are also 2-periodic and only differ by a shift by one.

In the stable homotopy category, suspension Σ is like a shift [1] for complexes, and

$$K^{-i}(X) = K(\Sigma^i X).$$

7.7

What is Bott periodicity in elliptic cohomology? From the Thom isomorphism,

$$\mathcal{O}^{-2}_{\mathrm{Ell}(X)} = \mathcal{O}_{\mathrm{Ell}(\Sigma^2 X)} = c^*\mathcal{O}(-[0])$$

where $[0]$ is the divisor of $\{0\} \subset E$. Recall from (21) that $\Sigma^2 X$ is the Thom space of the *trivial* line bundle, so here $c\colon E \to E$ is the zero map. Of course then any pullback c^* results in a trivial line bundle on E, but it is important to write them uniformly in terms of the modulus q because on the moduli of elliptic curves they still form an interesting line bundle. The fiber of the trivial bundle is

$$\mathcal{O}(-[0])\big|_0 = \mathfrak{m}_0/\mathfrak{m}_0^2 = T_0^* E$$

where $\mathfrak{m}_0$ is the maximal ideal of $0 \in E$. The first equality is because $\mathcal{O}(-[0])$ consists of functions vanishing at 0, and the second equality is by definition. In the language of enumerative geometry, this is the class ψ_1 on the moduli of elliptic curves. Hence

$$\mathcal{O}^{i-2}_{\mathrm{Ell}(X)} = \mathcal{O}^{i}_{\mathrm{Ell}(X)} \otimes \psi_1.$$

7.8

We use the Thom sheaf to formulate a pushforward in elliptic cohomology. While pullbacks are defined for any continuous G-equivariant map, pushforwards are only defined for *complex-oriented* maps $f\colon X \to Y$. In algebraic geometry, pushforwards are always defined but it is convenient to factor them into an inclusion followed by a projection; similarly, complex-oriented maps are those which factor as

$$f\colon X \xrightarrow{\text{inclusion}} V_Y \xrightarrow{\text{complex vector bundle}} Y$$

such that the normal bundle of $X \hookrightarrow V_Y$ is also a complex vector bundle N. In this setting, there are two Thom isomorphisms

$$\Theta_X(-N) \cong \mathcal{O}_{\mathrm{Ell}(\mathrm{Thom}(N))}$$
$$\mathcal{O}_{\mathrm{Ell}(\mathrm{Thom}(V))} \cong \Theta_Y(-V)$$

In fact there is a map

$$\Theta_X(-N) \to \Theta_Y(-V) \tag{24}$$

between them, induced by the *collapse map* $\mathrm{Thom}(V) \to \mathrm{Thom}(N)$ where any points not in a neighborhood of X in $\mathrm{Thom}(V)$ are all identified together. There is also a tautological map

$$\Theta_X(f^*V) \to \Theta_Y(V) \tag{25}$$

over $\mathrm{Ell}_{\mathrm{GL}(\mathrm{rank}\,V)}(\mathrm{pt})$. Pushforward in elliptic cohomology is the twist of (24) by (25),

$$f_*\colon \Theta_X(-N_f) \to \mathcal{O}_{\mathrm{Ell}(Y)}, \tag{26}$$

where $N_f \coloneqq N - f^*V$ is the *normal bundle* to the map f. The usual definition of N_f in algebraic geometry is in terms of the graph of f, which for our f is equivalent to our definition.

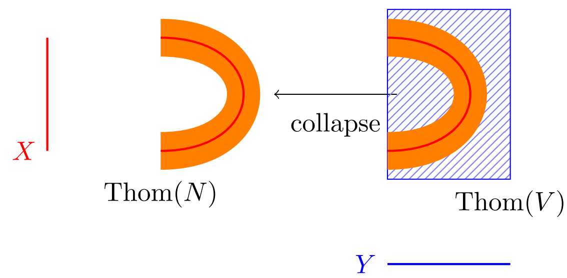


Figure 13: The Pontrjagin–Thom collapse map

Remark. Observe that in (26) the contributions from inclusion and projection enter with opposite signs. If one regards pushforwards as integration $\int$, then heuristically we obtain that $\int = 1/(\text{inclusion})$, and, letting x_i be the Chern roots of the normal bundle,

$$\begin{aligned}(\text{inclusion}) &\sim \prod \vartheta(x_i)\\ (\text{integration}) &\sim \frac{1}{\prod \vartheta(x_i)}.\end{aligned}$$

This is more or less the same as in K-theory, only every x_i is replaced by $\vartheta(x_i)$.

7.9

We conclude with yet another perspective on the computation of $K(\mathbb{P}(V))$. In K-theory, $K_{\mathsf{G}}(\mathsf{G}/\mathsf{H} \times X) = K_{\mathsf{H}}(X)$ simply because the data of a G-equivariant sheaf on $\mathsf{G}/\mathsf{H} \times X$ is the same as the data of an H-equivariant sheaf on X. By analogy,

$$\mathrm{Ell}_{\mathsf{G}}(\mathsf{G}/\mathsf{H} \times X) = \mathrm{Ell}_{\mathsf{H}}(X) \tag{27}$$

as well. In particular, let $\mathsf{G} \coloneqq U(n+1)$ and $\mathsf{H} \coloneqq U(n) \times U(1)$ so that $\mathbb{P}^n = \mathsf{G}/\mathsf{H}$. Let T be the maximal torus. Then

$$\begin{aligned}\mathrm{Ell}_{\mathsf{G}}(\mathbb{P}^n) = \mathrm{Ell}_{\mathsf{H}}(\mathrm{pt}) &= E \otimes_{\mathbb{Z}} \mathrm{cochar}(\mathsf{T})/W_{\mathsf{H}}\\ &= E^{n+1}/S(n) = E \times S^n E\end{aligned}$$

and the natural map $E \times S^n E \to S^{n+1} E$ is the map $\mathrm{Ell}_{\mathsf{G}}(\mathbb{P}^n) \to \mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})$. Moreover, as our computation of $\mathrm{Ell}_{\mathsf{G}}(\mathbb{P}^n)$ is equally applicable to $\mathrm{Ell}_{\mathsf{G}}(\mathbb{P}(\mathcal{E}))$ for any bundle $\mathcal{E}$, (27) gives a pullback square

$$\begin{array}{ccc}\mathrm{Ell}_{\mathsf{H}}(X) & \longrightarrow & \mathrm{Ell}_{\mathsf{G}}(X)\\ \downarrow & & \downarrow\\ \mathrm{Ell}_{\mathsf{H}}(\mathrm{pt}) & \longrightarrow & \mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})\end{array}. \tag{28}$$

This appears more obvious than it actually is, and in fact the square is not a pullback for general G and H, not even in topological K-theory. A counterexample is

$$K_{\mathrm{GL}(2)}(\mathbb{P}^1) \neq K_{\mathrm{PGL}(2)}(\mathbb{P}^1) \otimes_{K_{\mathrm{PGL}(2)}(\mathrm{pt})} K_{\mathrm{GL}(2)}(\mathrm{pt}),$$

viewing $\mathsf{G} = \mathrm{PGL}(2)$ and $\mathsf{H} = \mathrm{GL}(2)$. There is a natural map from the rhs to the lhs which is not surjective: the class $\mathcal{O}(1)$ is not in the image. The general question of which G and H make (28) a pullback square is somewhat complicated; cases where it holds include when G/H is a Grassmannian or $\mathbb{C}$ or $\mathbb{C}^*$ (which are $\mathbb{P}^1$ minus one or two points).

Lecture 8. A cellular approach to elliptic cohomology

Equivariant elliptic cohomology from the cell decomposition point of view, cofibration sequence, examples.

8.1

Previously, we approached elliptic cohomology from the perspective of Chern classes. Today we'll take a complementary approach via cell decompositions. The analogy with ordinary cohomology is that, for any coefficient ring R,

$$H^*(X, R) = H^*(\cdots \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \xrightarrow{d} \to \cdots)$$

for a chain complex of R-modules C^i which we equivalently view as sheaves on $\operatorname{Spec} R$, and there is a cup product

$$\cup\colon C^i \otimes_R C^j \to C^{i+j}$$

compatible with the differential, namely $d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^{\cdots} \alpha \cup d\beta$. This chain complex, as an element of $D^b\mathsf{Coh}(\operatorname{Spec} R)$, is more fundamental than the cohomology rings $H^*(X, R)$. Various properties of cohomology, such as the suspension isomorphism $H^{i+1}(\Sigma X) = H^i(X)$ and Mayer–Vietoris, all amount to saying that the corresponding cohomology functor

$$(\text{topological spaces } X) \to (\text{complexes of } R\text{-modules})$$

is a triangulated functor. Namely, it must preserve the structure of triangulated categories:

- a shift-by-one (to the left) functor $[1]$ given by

  $$(C^\bullet[1])^i \coloneqq C^{i+1},$$

  e.g. so that suspension is $\Sigma = [-1]$;

- triangles given by

  $$A^\bullet \xrightarrow{f} B^\bullet \to \operatorname{Cone}(f) \to A^\bullet[1]$$

  for some map f, or any triple $A^\bullet \to B^\bullet \to C^\bullet \to A[1]$ isomorphic to such a complex. The object $\operatorname{Cone}(f)$ must have $B^\bullet$ as a sub-complex and $A^\bullet[1]$ as a quotient, and therefore is constructed as

  $$\operatorname{Cone}(f) \coloneqq \left[A^\bullet[1] \oplus B^\bullet \text{ with differential } \begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix} \right].$$

  Note that the shift on $d_A[1]$ changes some signs, and the upper right block must be zero because $B^\bullet$ is a sub-complex.

8.2

For topological spaces, there is an analogous sequence called the *cofibration* or *Puppe* sequence. The only difference is that cohomology is a contravariant functor so this sequence runs the opposite way. Given $f\colon X \to Y$, it is good to first replace the target Y by something homotopy equivalent that makes f an embedding. Hence take the *mapping cylinder* of f, obtained by gluing one end of the cylinder $X \times [0,1]$ to Y via f as shown in Figure 14. Collapsing the cylinder into a cone then gives $\mathrm{Cone}(f)$. Further collapsing the other end of the cylinder gives a map $\mathrm{Cone}(f) \to \Sigma^1 X$. Putting it all together, the cofibration sequence is

$$X \hookrightarrow (\text{mapping cylinder of } f) \to \mathrm{Cone}(f) \to \Sigma^1 X \to \cdots,$$

which, as in cohomology, can be continued infinitely with $\Sigma^1 X \xrightarrow{\Sigma^1 f} \Sigma^1 Y \to \cdots$.

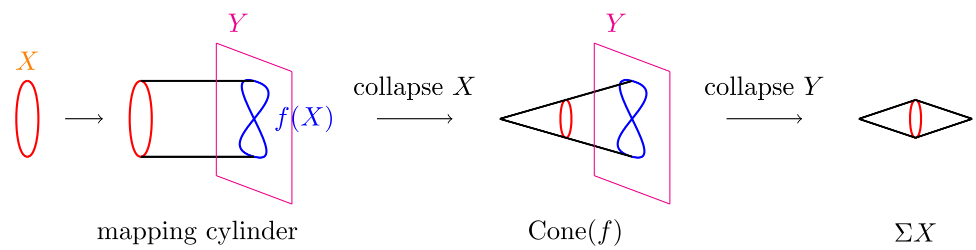


Figure 14: Cofibration sequence of a map $f\colon X \to Y$

8.3

The fundamental result is that, for any space Z, taking homotopy classes of maps $[-, W]$ out of this sequence produces an exact sequence of groups

$$\cdots \leftarrow [\cdots, W] \leftarrow [\Sigma^i X, W] \leftarrow [\Sigma^i Y, W]. \tag{29}$$

Note that $[\Sigma^i X, W]$ is a group by the same construction that makes homotopy groups into groups, see Figure 15; alternatively, by adjunction, $[\Sigma^i X, W] = [X, \Omega^i W]$ is like maps of X into the i-th homotopy group of W. Consequently $[\Sigma^i X, W]$ is abelian for $i > 1$. So, sufficiently deeply into (29), it is an exact sequence of abelian groups.

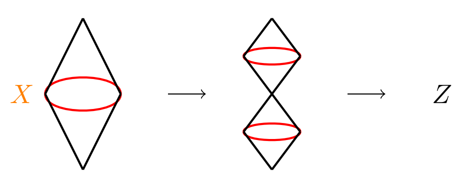


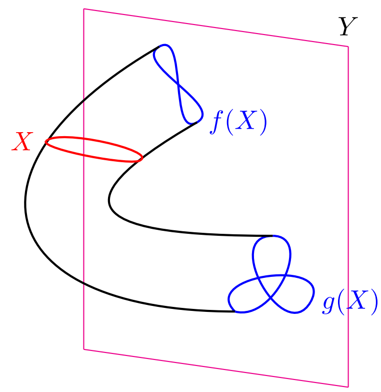
Figure 15: Composition of two elements of $[\Sigma^1 X, W]$

In cohomology theories, all long exact sequences arise from (29), and the fact that it is exact can be rephrased as the fact that the cohomology functor takes triangles to triangles.

8.4

Slightly more elaborately, homotopy colimits of spaces become homotopy limits of sheaves on $\operatorname{Spec} R$. In practice, avoiding delving too deeply into category theory, this boils down to the following construction. Suppose that, instead of a single map, we have two maps $f, g\colon X \to Y$. (The more general construction reduces to this if we think of X and Y as a disjoint union of a bunch of things which we want to glue together.) Construct a new space

$Z \quad :=$



(30)

by gluing the two ends of the cylinder $[0,1] \times X$ onto $f(X), g(X) \subset Y$ respectively. From our experience with mapping cylinders and the cofibration sequence,

$$H^*(Z) \approx \ker\left(H^*(Y) \xrightarrow{f-g} H^*(X)\right).$$

But in a triangulated category there are only cones, not kernels, so the more accurate statement is

$$H^*(Z) = \operatorname{Cone}\left(H^*(Y) \xrightarrow{f-g} H^*(X)\right)[-1].$$

This construction is used for gluing spaces from simpler pieces; as an exercise, one may check that Mayer–Vietoris is a special case of this. For example, the simplest case is to glue a single n-cell to a space W along an attachment map $f\colon S^{n-1} \to W$, in which case we take $g\colon S^{n-1} \to D^n$ to be the inclusion of the boundary, and Z is the homotopy pushout

$$\begin{array}{ccccc} & & D^n & & \\ & \nearrow & & \searrow & \\ X = S^{n-1} & & & & Z = (W \text{ with cell glued}). \\ & {\scriptstyle f}\searrow & & \nearrow & \\ & & W & & \end{array} \tag{31}$$

A space which is homotopy-equivalent to a CW complex (which we always assume it is) is obtained by successively attaching one cell at a time in this way, and its cohomology is obtained by successively taking cones as above.

8.5

In the equivariant situation, the notion of a cell becomes more interesting. Previously, the basis for topology was the *point*, and disks of different dimensions were just different incarnations of it. The equivariant replacement for a point is

$$\mathrm{pt} \rightsquigarrow (\text{orbit of a group } \mathsf{G}),$$

the “smallest units” of a G-equivariant topological space. There are many different possible orbits, all of the form G/H for some subgroup H, and so *equivariant cells* are of the form

$$D^n \times \mathsf{G}/\mathsf{H}$$

where G acts trivially on D^n. Attachment maps are the equivariant version of (31), namely

$$\begin{array}{ccccc} & & D^n \times \mathsf{G}/\mathsf{H} & & \\ & \nearrow & & \searrow & \\ X = S^{n-1} \times \mathsf{G}/\mathsf{H} & & & & Z = (W \text{ with cell glued}). \\ & \underset{f}{\searrow} & & \nearrow & \\ & & W & & \end{array}$$

By our previous non-equivariant discussion, to construct the equivariant cohomology sheaf $\mathsf{Cohom}^*_{\mathsf{G}}(-)$ it suffices to specify $\mathsf{Cohom}^*_{\mathsf{G}}(D^n \times \mathsf{G}/\mathsf{H})$. Cohomology should still be homotopy-invariant, so this is just some sheaf $\mathsf{Cohom}^*_{\mathsf{G}}(\mathsf{G}/\mathsf{H})$ on $\operatorname{Spec} \mathsf{Cohom}^*_{\mathsf{G}}(\mathrm{pt})$. We take it to be

$$\mathsf{Cohom}^*_{\mathsf{G}}(\mathsf{G}/\mathsf{H}) := \left(\begin{array}{c} \text{image of structure sheaf under} \\ \operatorname{Spec} \mathsf{Cohom}^*_{\mathsf{H}}(\mathrm{pt}) \to \operatorname{Spec} \mathsf{Cohom}^*_{\mathsf{G}}(\mathrm{pt}) \end{array} \right). \tag{32}$$

8.6

There is in fact no need for these spaces $\operatorname{Spec} \mathsf{Cohom}^*_{\mathsf{G}}(\mathrm{pt})$ to be affine schemes; for example, equivariant elliptic cohomology has no underlying “cohomology ring”. Put differently, if we think of cohomology as the cohomology of a complex of sheaves, there is no need for the sheaves to actually be modules over some ring. Stated in this way, computing cohomology just involves working with sheaves over some scheme and repeatedly taking cones.

For G-equivariant elliptic cohomology in particular, the scheme is denoted $\mathrm{Ell}_{\mathsf{G}}(\mathrm{pt})$ and is roughly the space of semistable degree-zero G-bundles on the dual elliptic curve $E^\vee$. The dual is because we would like a canonical identification

$$(\mathrm{GL}(1)\text{-bundles on } E^\vee) = \mathrm{Pic}_0(E^\vee) = E$$

with the original elliptic curve. Concretely, for connected G,

$$\mathrm{Ell}_{\mathsf{G}}(\mathrm{pt}) = E \otimes \mathrm{cochar}(\mathsf{G})/W$$

where W is the Weyl group. Even more concretely, we are mainly interested in abelian groups T, for which

$$\mathrm{Ell}_{\mathsf{T}}(\mathrm{pt}) = E \otimes \mathrm{cochar}(\mathsf{T}).$$

A map $\mathsf{T}_1 \to \mathsf{T}_2$ induces a map on cocharacter lattices, or, equivalently, a map $\mathrm{Ell}_{\mathsf{T}_1}(\mathrm{pt}) \to \mathrm{Ell}_{\mathsf{T}_2}(\mathrm{pt})$. For instance,

$$(\mathsf{T} \times \mathsf{T} \xrightarrow{\cdot} \mathsf{T}) \rightsquigarrow (\text{group law on } \mathrm{Ell}_{\mathsf{T}}(\mathrm{pt})).$$

This is directly related to the classical story, where the group law on abelian varieties comes from things like tensor product of line bundles, which has to do with operations like $\mathsf{T} \times \mathsf{T} \xrightarrow{\cdot} \mathsf{T}$ for 1-dimensional groups.

8.7

More generally, given a short exact sequence $1 \to \Gamma \to \mathsf{T} \to \mathsf{T}' \to 1$ (where the kernel Γ is often called a *quasi-torus*), there is an induced exact sequence

$$0 \to \mathrm{Ell}_{\Gamma}(\mathrm{pt}) \to \mathrm{Ell}_{\mathsf{T}}(\mathrm{pt}) \to \mathrm{Ell}_{\mathsf{T}'}(\mathrm{pt}) \to 0.$$

For example, if $\mathsf{G} = \mathbb{C}^\times$, the sequence $1 \to \mu_n \to \mathbb{C}^\times \xrightarrow{z \mapsto z^n} \mathbb{C}^\times \to 1$ induces

$$0 \to E[n] \to E \xrightarrow{\cdot n} E \to 0$$

where $E[n]$ denotes points of order n on E. Over $\mathbb{C}$, as we saw previously, $E[n]$ consists of n^2 disjoint points, but over other fields it may look very different. Over a field of characteristic $p = n$, it could be p points of fatness p or it could be 1 point of fatness p^2. In such a setting, it is clear there must be some fatness since the differential of the multiplication-by-p map is zero at $0 \in E$, which is therefore a point of fatness at least p. Such elliptic curves are called *supersingular*, and in the moduli of elliptic curves they form the locus where elliptic cohomology behaves very differently from "K-theory made q-periodic", in the sense of that $E = \mathbb{G}_m / q^{\mathbb{Z}}$.

8.8

Any abelian group is a subgroup of a torus, so we have obtained a full description of equivariant elliptic cohomology of a point. Then equivariant elliptic cohomology of (spaces homotopic to) CW complexes can be computed cell by cell.

Example. Take $U(1)$ acting on $\mathbb{CP}^1$, which decomposes into the two 0-dimensional cells 0 and ∞ and the 1-dimensional cell $U(1) \times D^1$. This is exactly the situation of (30), with

$$X = U(1), \quad Y = \{0\} \cup \{\infty\},$$

and the two maps collapse X to the two points in Y respectively.

In terms of equivariant cells, Y consists of two orbits $U(1)/U(1)$ and X is the orbit $U(1)/\{1\}$. Then elliptic cohomology of Y is $\mathcal{O}_E^{\oplus 2}$ since $E = \mathrm{Ell}_{U(1)}(\mathrm{pt})$, and by (32) elliptic

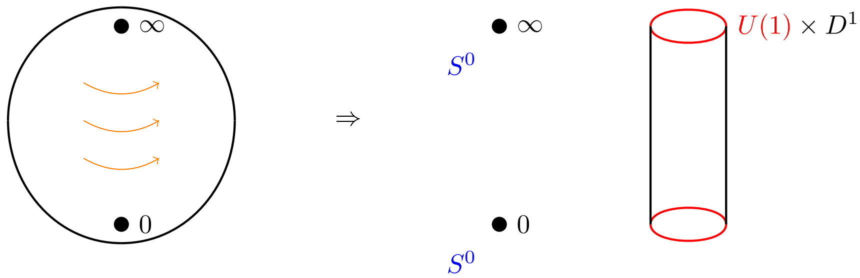


Figure 16: $U(1)$-equivariant cells of $\mathbb{CP}^1$

cohomology of X is $\mathcal{O}_0$ since $\{0\} \subset E$ is $\mathrm{Ell}_{\{1\}}(\mathrm{pt}) \hookrightarrow \mathrm{Ell}_{U(1)}(\mathrm{pt})$, and the map from one to the other is

$$\mathcal{O}_E^{\oplus 2} \ni (\varphi_1, \varphi_2) \mapsto \varphi_1(0) - \varphi_2(0) \in \mathcal{O}_0. \tag{33}$$

This is a surjection, and so its cone is the kernel

$$\{(\varphi_1, \varphi_2) \in \mathcal{O}_E : \varphi_1(0) = \varphi_2(0)\}.$$

In terms of geometry, i.e. taking spectrum, the kernel corresponds to two elliptic curves E glued together at 0. This is what we have previously computed with Chern classes. The exact sequence coming from (33) is precisely the normalization sequence

$$0 \to \mathcal{O}_{\mathrm{Ell}_{U(1)}(\mathrm{pt})} \to \mathcal{O}_{\mathrm{normalization}} \to \mathcal{O}_{\mathrm{intersection}} \to 0.$$

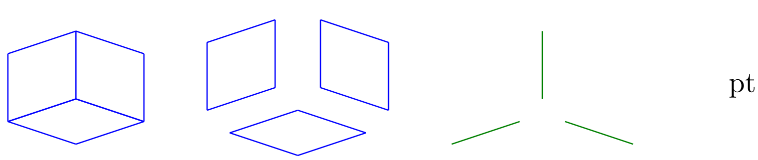
8.9

Example. The same procedure can be done for the action of $\mathsf{A} = \{\mathrm{diag}(1, a_2, a_3)\}$ on $\mathbb{CP}^2$, where now there are three kinds of orbits: a single free orbit, three orbits A/A, and three orbits of the form A/A_i where A_i are the three 1-dimensional subtori. We computed earlier that $\mathrm{Ell}_{\mathsf{A}}(\mathbb{CP}^2)$ is three copies of $E^{\oplus 2}$ glued along coordinate axes. Its normalization is

$$\mathrm{Ell}_{\mathsf{A}}(\mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt}) = (3 \text{ copies of } E^{\oplus 2}),$$

and $\mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(\mathbb{CP}^2)}$ arises from $\mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})}$ by requiring functions agree along the three edges $E \cong \mathrm{Ell}_{\mathsf{A}_i}(\mathrm{pt})$ and the point $\mathrm{pt} = \mathrm{Ell}_{\{1\}}(\mathrm{pt})$. In other words, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(\mathbb{CP}^2)} \longrightarrow \mathcal{O}_{E^2}^{\oplus 3} \longrightarrow \mathcal{O}_E^{\oplus 3} \longrightarrow \mathcal{O}_0 \to 0 \tag{34}$$



where we have also drawn the scheme corresponding to each structure sheaf. Note that the ordering of the orbits in (34) is in relation to the size of their stabilizer, not the size of the orbit, e.g. the smallest term $\mathcal{O}_0$ corresponds to the free orbit.

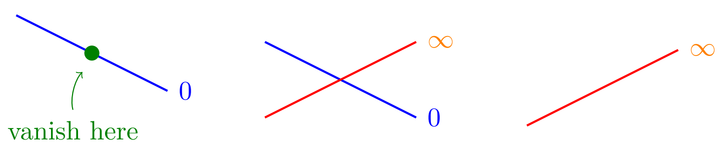
8.10

Example. A different point of view on the same computation for $\mathbb{CP}^1$ is to take a disk and glue its boundary $U(1)$ to a point. View the disk as the disk bundle $B(a)$ for the normal bundle $N_{0/\mathbb{CP}^1}$, i.e. the defining representation which we denote a. Then $\mathbb{CP}^1$ is the homotopy pushout

$$\begin{array}{ccc} U(1) & \longrightarrow & B(a) \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathbb{CP}^1 \end{array}$$

This results in the sequence

$$0 \longrightarrow \mathrm{Thom}(a) \longrightarrow \mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(\mathbb{CP}^1)} \longrightarrow \mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(\infty)} \longrightarrow 0$$



where again we have also drawn the scheme corresponding to each sheaf. Note that this is essentially the same computation, but we have made the roles of 0 and ∞ slightly asymmetric.

This asymmetric perspective is useful for $\mathbb{CP}^n$, which can be constructed in the same way by attaching $\mathbb{CP}^{n-1}$ to the normal bundle (of zero) with weights a_i/a_0. This gives

$$0 \to \mathrm{Thom}\left(\sum_i a_i/a_0\right) \to \mathcal{O}_{\mathrm{Ell}(\mathbb{CP}^n)} \to \mathcal{O}_{\mathrm{Ell}(\mathbb{CP}^{n-1})} \to 0.$$

8.11

The moral of the story is that the decomposition of a space into equivariant cells can be done in many different ways, to give different sequences defining its cohomology.

Lecture 9. Constructing stable envelopes I: the inductive strategy

Attracting manifolds, strategy for inductive construction of elliptic stable envelopes, stable envelopes as an interpolation problem, interpolation and its relations to cohomology vanishing, cohomology vanishing for line bundles on abelian varieties, Picard and Neron–Severi groups of an abelian variety.

9.1

Let X be an algebraic variety with an action of a torus A. Our perspective was that X is some moduli of vacua and A is a group of global symmetries. Associated to this is a "phase diagram", in $\mathsf{Lie}(\mathsf{A})$, recording the fixed locus in X of elements in A. Today our goal is to construct an interface between different phases, e.g. X and $X^a = X^{\mathsf{A}}$ for generic a. This will evidently depend on a choice of chamber (containing a) in the phase diagram; equivalently, this is a choice of attracting/repelling direction for the action of A.

9.2

Assume X is smooth and symplectic, and the action of A preserves the symplectic form ω. Then the desired interface is a Lagrangian in $X \times X^{\mathsf{A}}$. It is a fact that $X^{\mathsf{A}} = \bigsqcup F_i$ is also smooth, but may consist of many connected components F_i. Near such a component F_i, the desired Lagrangian consists of all attracting directions

$$\mathrm{Attr} \coloneqq \{(x, f) \mid \lim_{a \to 0} a \cdot x = f\} \tag{35}$$

in contrast to the (equal number of) repelling directions. The limit $a \to 0$ in (35) is where the choice of chamber comes in. Note that Attr is a nice Lagrangian but is not closed, since the attracting trajectory of a point in one fixed component F_1 may originate from a different fixed component F_2. The simplest example to have in mind is $X = T^* \operatorname{Gr}(n)$ with the action of a maximal torus $\mathsf{A} \subset \mathrm{GL}(n)$, where attracting manifolds are conormals to Schubert cells and are certainly not closed.

9.3

Over $\mathbb{C}$, every attracting trajectory is a $\mathbb{CP}^1$. In physics we think of it as an infinite cylinder, corresponding to the space C from §1.8, with the action of $\mathbb{C}^\times$ rotating it. The $\mathbb{C}^\times$-equivariant profile of fields on C will then look like Figure 17, going from one vacuum to another.

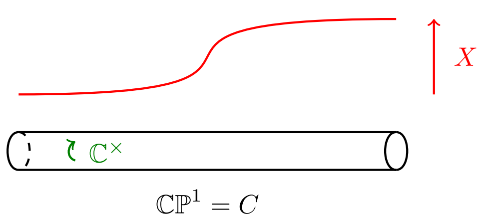


Figure 17: Attracting trajectories as physical fields

Keeping these $\mathbb{CP}^1$ trajectories in mind, the appropriate closure of Attr is obtained in an absolutely standard way: if such a trajectory flows to a fixed point, we allow it to *continue* flowing to another fixed point (cf. breaking of Morse trajectories, geodesics, etc.). In local coordinates, there is a family of attracting trajectories as in Figure 18 and some of them hit the fixed component F_i and break into two pieces. Instead of just taking the red piece alone, we must include the other blue piece as well.

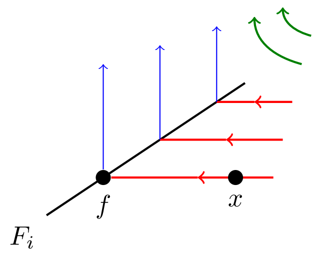


Figure 18: Attracting, repelling, and nearby trajectories for a fixed component

Definition. The *full attracting set* is

$$\mathrm{Attr}^f := \left\{ (x,f) \;\middle|\; \begin{array}{c} x \text{ connected to } f \\ \text{by a chain of attracting trajectories} \end{array} \right\} \subset X \times X^{\mathsf{A}}.$$

It is closed and proper over X, but may be singular. There is an ordering $\preceq$ on fixed components by which one is attracted to which, i.e. $F_j \preceq F_i$ if $F_j \subset \overline{\mathrm{Attr}}(F_i)$, and

$$\mathrm{Attr}^f(F_i) = \bigcup_{F_j \preceq F_i} \overline{\mathrm{Attr}}(F_j),$$

e.g. think about $X = T^* \mathrm{Gr}(n)$ and (conormals to) Schubert cells.

A good exercise is to check that for the action of a torus of any reductive group on a subvariety of projective space, $\preceq$ is actually a partial ordering.

9.4

By itself, Attr^f does not define a cobordism class. For example, again for Schubert cells, it is a basic fact that their pre-images in a resolution depend on the choice of resolution. Instead, we want a canonically-defined elliptic cohomology class supported on Attr^f. For us, this means a section of some line bundle on $\mathrm{Ell}_{\mathrm{Aut}}(X)$ where Aut is a group of automorphisms commuting with A, and to make matters simple, we choose a maximal torus $\mathsf{T} \subset \mathrm{Aut}$ containing A and consider $\mathrm{Ell}_{\mathsf{T}}(X)$. Recall that $\mathrm{Ell}_{\mathsf{T}}(X)$ is a scheme over $\mathrm{Ell}_{\mathsf{T}}(\mathrm{pt}) = E \otimes \mathrm{cochar}(\mathsf{T})$, and that sections of line bundles over it are the elliptic analogue of functions on $\mathrm{Spec}\, K_{\mathsf{T}}(X)$ or $\mathrm{Spec}\, H_{\mathsf{T}}(X)$, i.e. elements of $K_{\mathsf{T}}(X)$ or $H_{\mathsf{T}}(X)$.

9.5

The strategy is to do the construction inductively, which makes sense since Attr^f itself is naturally defined inductively. Pick some linear ordering on the fixed components F_i such that $F_i \preceq F_j$ for $i \leq j$, and set

$$X_n := X \setminus \bigsqcup_{i \geq n} \mathrm{Attr}(F_i).$$

Then by construction $\mathrm{Attr}(F_{n-1}) \subset X_n$ is closed and proper over X_n. The problem of extending a class on X_n to X_{n+1} involves studying the equivariant inclusion $X_n \hookrightarrow X_{n+1}$, which induces a map

$$\mathrm{Ell}_{\mathsf{T}}(X_n) \to \mathrm{Ell}_{\mathsf{T}}(X_{n+1})$$

which turns out to be inclusion of a closed set.

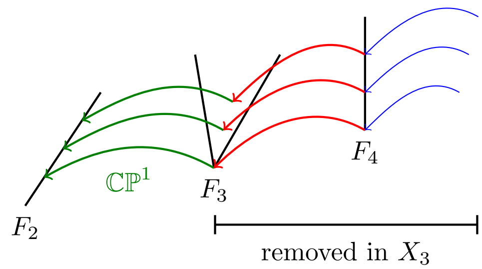


Figure 19: Stratification of X by attracting sets

Example. Recall that for $X = T^*\mathbb{P}^2$ with the action of $\mathsf{A} \ni \mathrm{diag}(1, a_2, a_3)$, the elliptic cohomology $\mathrm{Ell}_{\mathsf{A}}(X)$ consists of three copies of $E^2 \cong \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})$ corresponding to three fixed points $p_1, p_2, p_3 \in X$, and

$$X_3 = T^*\mathbb{P}^2 \setminus \mathrm{Attr}(p_3)$$
$$X_4 = T^*\mathbb{P}^2.$$

Then $\mathrm{Ell}_{\mathsf{A}}(X_3) \to \mathrm{Ell}_{\mathsf{A}}(X_4)$ is just inclusion of the p_1 and p_2 pieces.

9.6

Given a scheme Z and a line bundle $\mathcal{L}$ on it, a very general problem in algebraic geometry is *interpolation* of sections of $\mathcal{L}$, meaning to extend sections of $\mathcal{L}$ from their restrictions to a subscheme $Z' \subset Z$. The simplest case is where Z is a curve and Z' is some collection of points, and then the interpolation problem becomes whether a polynomial is uniquely determined by its values at those points. Slightly more involved, if Z is a surface and $Z' \subset Z$ is a divisor, sections of line bundles correspond to other curves on Z and then the problem becomes whether a curve is determined by its intersection with another fixed curve.

The general procedure to study this kind of interpolation problem is to take the short exact sequence $0 \to I_{Z'} \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$ and tensor with $\mathcal{L}$ to give

$$0 \to \mathcal{L} \otimes I_{Z'} \to \mathcal{L} \to \mathcal{L}\big|_{Z'} \to 0,$$

and interpolation is about studying the kernel and cokernel

$$\ker \subset H^0(\mathcal{L}) \to H^0(\mathcal{L}\big|_{Z'}) \to \mathrm{coker}\,.$$

The cokernel measures the existence of a solution to interpolation and the kernel measures uniqueness of the solution. By long exact sequence, vanishing of $H^0(\mathcal{L} \otimes I_{Z'})$ implies uniqueness and vanishing of $H^1(\mathcal{L} \otimes I_{Z'})$ implies existence.

Example. For $\mathrm{Ell}_{\mathsf{A}}(T^*\mathbb{P}^2)$, the restriction is $\mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(X_4)} \to \mathcal{O}_{\mathrm{Ell}_{\mathsf{A}}(X_3)}$ and the ideal sheaf consists of functions on $\mathrm{Ell}_{\mathsf{A}}(p_3)$ that vanish on the intersection. This is exactly the Thom sheaf of the normal bundle to $\mathrm{Attr}(F_3)$. Hence we want vanishing of $H^i(\mathrm{Thom} \otimes \mathcal{L}) = 0$ for $i = 0, 1$.

9.7

Abstractly, these are cohomology groups of a line bundle on an abelian variety, e.g. $\mathrm{Ell}_{\mathsf{A}}(p_3) = E^{\mathrm{rank}\,\mathsf{A}}$ in the above example, and so we now briefly review line bundles and their cohomology on abelian varieties.

Let $\mathcal{A}$ be an abelian variety. Then the Picard group of line bundles on $\mathcal{A}$ fits into a short exact sequence

$$0 \to \mathrm{Pic}_0(\mathcal{A}) \to \mathrm{Pic}(\mathcal{A}) \to \mathrm{NS}(\mathcal{A}) \to 0$$

where $\mathrm{Pic}_0(\mathcal{A})$ is the component consisting of line bundles algebraically equivalent to zero, and the quotient is the discrete Neron–Severi group $\mathrm{NS}(\mathcal{A})$. Both

$$\mathcal{A}^\vee \coloneqq \mathrm{Pic}_0(\mathcal{A})$$

and $\mathrm{NS}(\mathcal{A})$ are themselves abelian varieties. The main result about cohomology of line bundles on $\mathcal{A}$ is that if $\mathcal{L} \in \mathrm{Pic}_0(\mathcal{A})$ is non-trivial, then

$$H^i(\mathcal{L}) = 0 \text{ for all } i.$$

Line bundles of degree zero therefore behave like characters for $\mathcal{A}$, in the sense that $H^i(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}) = 0$ for line bundles $\mathcal{L}_1, \mathcal{L}_2$ unless $\mathcal{L}_1 \cong \mathcal{L}_2$. This is what makes Fourier–Mukai transforms work.

9.8

The degree map for line bundles takes values in $\mathrm{NS}(\mathcal{A})$, and so in our case we are interested in $\mathrm{NS}(E^r)$. When $r = 1$, every line bundle on an elliptic curve (in fact, any curve) is isomorphic to $\mathcal{O}(\sum m_i p_i)$ modulo linear equivalence, and the degree is $\sum m_i \in \mathbb{Z} \cong \mathrm{NS}(E)$. What's remarkable for abelian varieties is that the Picard functor is representable by a fine moduli space, so not only is there a space parameterizing line bundles, there is a *universal* line bundle called the Poincaré bundle. For elliptic curves, $E^\vee \cong E$ and the Poincaré bundle is

$$\mathcal{O}(\mathrm{diag} - E \times \{0\} - \{0\} \times E)$$

on $E \times E$. Its fiber over $a \in E$ is the bundle $\mathcal{O}(a - [0])$, and clearly all degree-0 line bundles on E have this form.

9.9

As a next step, take $r = 2$. Let $p_1, p_2 \colon E \times E \to E$ be projections onto each factor, and similarly let $\iota_1, \iota_2 \colon E \to E \times E$ be the inclusions. Consequently there are maps

$$\mathrm{Pic}(E)^2 \xrightarrow{p_1^* \otimes p_2^*} \mathrm{Pic}(E \times E) \xrightarrow{(\iota_1^*, \iota_2^*)} \mathrm{Pic}(E)^2$$

and the composition is the identity. Hence it suffices to compute $\ker \iota^*$ or $\operatorname{coker} p^*$. View

$$\operatorname{Pic}(E \times E)/\operatorname{Pic}(E)^2 = \left\{ \begin{matrix} \text{line bundles } \mathcal{L} \to E \times E \\ \text{trivial on } E \times \{0\} \text{ and } \{0\} \times E \end{matrix} \right\}.$$

Such a line bundle $\mathcal{L}$ is equivalently a family of line bundles on E parameterized by E. Since degree is constant in families and $\mathcal{L}|_{\{0\}\times E}$ is trivial, $\mathcal{L}$ gives a map $E \to \operatorname{Pic}_0(E) \cong E$. In addition, this map sends 0 to 0 since $\mathcal{L}|_{E\times\{0\}}$ is also trivial. Fact: a map between abelian varieties that takes 0 to 0 is a group homomorphism. Hence

$$\operatorname{Pic}(E \times E)/\operatorname{Pic}(E)^2 \cong \operatorname{Hom}(E, E)$$

which is some discrete group. (It contains at least $\mathbb{Z}$, but may be bigger if E has complex multiplication.) It follows that

$$\begin{aligned} \operatorname{Pic}_0(E^r) &= \operatorname{Pic}_0(E)^r \\ \operatorname{NS}(E^2) &= \operatorname{NS}(E)^2 \oplus \operatorname{Hom}(E, E). \end{aligned}$$

The way to phrase it invariantly is that $\operatorname{NS}(E^2) = \operatorname{Hom}_{\text{symmetric}}(E^2, E^2)$, meaning elements of the form

$$\begin{pmatrix} * & \alpha \\ \alpha^t & * \end{pmatrix} \in \operatorname{Hom}(E^2, E^2)$$

where $*$ denotes any element of $\operatorname{NS}(E) = \mathbb{Z}$ and α^t is the dual map between dual varieties. More generally,

$$\operatorname{NS}(\mathcal{A}) = \operatorname{Hom}_{\text{symmetric}}(\mathcal{A}, \mathcal{A}^\vee)$$

and symmetric here means that the dual map in $\operatorname{Hom}(\mathcal{A}^\vee, \mathcal{A}) \cong \operatorname{Hom}(\mathcal{A}, \mathcal{A}^\vee)$ is the same as the original. In particular,

$$\operatorname{NS}(E^r) \supset \{\text{symmetric } r \times r \text{ matrices with values in } \mathbb{Z}\},$$

which correspond to integral quadratic forms, and equality holds if E is not CM (which we will never need).

Remark. If $\mathcal{L}$ is a line bundle on $\mathcal{A}$ and $T_a \colon \mathcal{A} \to \mathcal{A}$ is the translation map $b \mapsto b + a$, then

$$T_a^* \mathcal{L} \otimes \mathcal{L}^{-1} \colon \mathcal{A} \to \mathcal{A}^\vee$$

is actually the degree of $\mathcal{L}$ as an element in $\operatorname{NS}(\mathcal{A})$. Exercise: check that this agrees with the usual notion of degree for $\mathcal{A} = E$.

9.10

Let A act trivially on a space Y and consider an A-equivariant vector bundle V with Chern roots $v_1, \ldots, v_r$. Recall that there is a Chern class map

$$\operatorname{Ell}_{\mathsf{A}}(Y) \xrightarrow{c} \operatorname{Ell}_{\operatorname{GL}(r)}(\text{pt}) = S^r E \ni (v_1, \ldots, v_r),$$

and there is a bundle $\Theta(V) \coloneqq c^*\mathcal{O}(D_\Theta)$ where D_Θ is the theta divisor defined by $v_i = 0$ for some i. So $\Theta(V)$ contains $\vartheta(v_1)\vartheta(v_2)\cdots\vartheta(v_r)$ as a section. One can ask for the degree of $\Theta(V)$ pulled back under the inclusion

$$\iota\colon \operatorname{Ell}_{\mathsf{A}}(\mathrm{pt}) \to \operatorname{Ell}_{\mathsf{A}}(Y)$$

for any point pt. Let $a_1, \ldots, a_n$ be coordinates on A, so they act on the fiber $V\big|_{\mathrm{pt}}$ by some monomials a^{v_i}. Then

$$\deg \prod \vartheta(a^{v_i}) = \sum \deg \vartheta(a^{v_i}) = \sum v_i^2$$

since degree is some integral quadratic form. We have proved the following.

Proposition. *In $S^2 \operatorname{char}(\mathsf{A}) \subset \operatorname{NS}(E \otimes \operatorname{cochar}(\mathsf{A}))$,*

$$\deg \Theta(V) = \sum v_i^2 = c_1(V)^2 - 2c_2(V).$$

Lecture 10. Constructing stable envelopes II: the details

Attractive line bundles, definition of elliptic stable envelopes, existence and uniqueness of elliptic stable envelopes.

10.1

The goal now is to actually construct elliptic stable envelopes. We will do the construction in a simple but representative case.

Let X be an algebraic symplectic variety with an action of a torus A preserving the symplectic form ω_X, though in general one can consider a non-abelian group. Let $\mathsf{T} \supset \mathsf{A}$ be a bigger torus, containing a $\mathbb{C}^\times$ which scales ω_X with weight $\hbar$. In other words there is an exact sequence

$$1 \to \mathsf{A} \to \mathsf{T} \xrightarrow{\hbar} \mathbb{C}^\times \to 1.$$

This $\hbar$ will be the deformation parameter of the quantum group.

For simplicity, assume the fixed locus $X^{\mathsf{A}} = \{f_i\}$ is just a finite collection of points. Then instead of a Lagrangian in $X \times X^{\mathsf{A}}$, we just need a bunch of Lagrangian subvarieties in X. The elliptic stable envelope $\operatorname{Stab}(f_i)$ for the fixed point f_i should be an elliptic cohomology class, namely a section of some line bundle $\mathcal{S}$. To specify $\mathcal{S}$, we fix the degree of its restriction to fixed points in variables $a \in \mathsf{A}$.

10.2

At a fixed point f, an attracting direction of weight w_i pairs with a repelling direction of weight $1/\hbar w_i$ because the variety is symplectic. So

$$\operatorname{Attr}(f)\big|_f = \pm \prod \vartheta(\hbar w_i), \tag{36}$$

recalling that $\vartheta(1/x) = -\vartheta(x)$. These w_i are weights of A, so $w_i = a^{\mu_i}$ for some μ_i. We computed last time that (36) therefore has degree

$$\sum \mu_i^2 \in S^2 \operatorname{char}(\mathsf{A})$$

in the variables a. In particular this is independent of which directions are attracting/repelling, because the change $w_i \mapsto 1/w_i$ corresponds to $\mu_i \mapsto -\mu_i$.

Definition. The line bundle $\mathcal{S}$ is *attractive* if

$$\deg_{\mathsf{A}} \mathcal{S}\big|_{\text{fixed locus}} = \deg_{\mathsf{A}} \operatorname{Attr}\big|_{\text{fixed locus}}.$$

Let $N_{\text{fix},<0}$ denote the repelling part of the normal bundle to the fixed locus, so that Attr is the zero section of $N_{\text{fix},<0}$. Then equivalently this degree is $\deg_{\mathsf{A}} \Theta(N_{\text{fix},<0})$, which, from last time, is $c_1^2 - 2c_2$.

If $\mathcal{S}$ is attractive, then $\mathcal{S} \otimes \mathcal{L}$ is also attractive for any line bundle $\mathcal{L}$ of degree 0 or algebraically equivalent to 0. This is a very, very, very important degree of freedom, corresponding to the necessity of dynamical variables in elliptic quantum groups. In fact we will not construct stable envelopes for one choice of attracting chamber, but for all choices; the R-matrix is obtained from changing attracting/repelling directions in Stab. Specifically, making the complete opposite choice of attracting/repelling direction changes (36) as

$$\prod \vartheta(\hbar w_i) \rightsquigarrow \prod \vartheta(w_i).$$

These have the same degree but are different bundles. Their ratio, with an aesthetic correction by an $\hbar$ factor, is

$$\frac{\vartheta(\hbar w)}{\vartheta(w)\vartheta(\hbar)}. \tag{37}$$

This is a section of the Poincaré bundle on $E \times E$, the universal degree-0 line bundle. Here we are thinking of the elliptic curve as $E = \operatorname{Ell}_{\mathbb{C}^\times}(\text{pt})$, and an element $a \in \mathbb{C}^\times$ is like a coordinate on E (modulo $q^{\mathbb{Z}}$). In (37), the coordinates on $E \times E$ are w and $\hbar$.

Equivalently, (37) is a section of

$$\Theta(\hbar w - w - \hbar) = \Theta((w-1)(\hbar - 1)),$$

where we are free to tensor by an extra factor of the trivial bundle $\Theta(1)$. From this expression it is clear the line bundle is degree-0 in either factor: setting the other variable to 1 gives the trivial bundle.

10.3

Here is a very general way to produce line bundles of degree 0. For $V \in K_T(X)$, take $z \in \mathbb{C}^\times$ for some $\mathbb{C}^\times$ that does not act on X at all, and define

$$\mathcal{U}(V, z) \coloneqq \Theta\left((V - \mathbb{C}^{\operatorname{rank} V})(z-1)\right) \in \operatorname{Pic}(\operatorname{Ell}_{\mathsf{T}}(X) \times E)$$

where E has coordinate z. This has degree 0 in either factor. We are free, and in fact forced (by the earlier discussion), to twist by such line bundles in elliptic stable envelopes.

Proposition. • $\mathcal{U}(V,z) = \mathcal{U}(\det V, z)$, *so there is no need to consider general vector bundles* V.

• $\mathcal{U}(\mathcal{L},z_1)\otimes\mathcal{U}(\mathcal{L},z_2) = \mathcal{U}(\mathcal{L},z_1 z_2)$.

Proof. An exercise in using the theorem of the cube. □

10.4

We still need at least one attractive bundle. This will be given by a choice of polarization.

Definition. A *polarization* of X is a class $T^{1/2} \in K_{\mathsf{A}}(X)$ such that

$$T^{1/2} + (T^{1/2})^\vee = TX, \tag{38}$$

i.e. it picks half of the tangent vectors.

Then $\mathcal{S} = \Theta(T^{1/2})$ has the correct degree. Note that (38) implies

$$\Theta(T^{1/2})^{\otimes 2} = \Theta(TX),$$

which is like picking a square root of the canonical bundle.

If $X = T^*M$, let $\pi\colon T^*M \to M$ be the projection. Then π^*TM is a polarization, and $\ker d\pi$ is another polarization. More generally, if $X \subset T^*(\text{stack})$ is an open subset, then the same applies. A typical such situation, which covers the case of Nakajima quiver varieties, is when X is the GIT-stable locus in a stack of the form $T^*[W/G]$, and then one can take

$$T^{1/2} = TW - \mathrm{Ad}(G)$$

where $\mathrm{Ad}(G)$ is the adjoint representation of G. Both TW and $\mathrm{Ad}(G)$ are G-equivariant and therefore descend to the quotient $T^*[W/G]$.

10.5

To summarize, we have two elliptic classes Attr and $\Theta(T^{1/2})$, which near fixed points are

$$\begin{aligned} \mathrm{Attr}\big|_f &= \pm \prod_{w_i>0} \theta(\hbar w_i) \\ \Theta(T^{1/2})\big|_f &= \prod_{w_i \in T^{1/2}_f} \theta(w_i). \end{aligned}$$

The discrepancy is therefore

$$\left.\frac{\Theta(T^{1/2})}{\Theta(N_{<0})}\right|_f = \mathcal{U}(\det T^{1/2}_{>0}, \hbar)$$

where we used Proposition 10.3 to replace $T^{1/2}_{>0}$ by its determinant. In classical terms, the attracting part $T^{1/2}_{>0}$ of the polarization is also known as the *index* of a fixed point. Hence, near the fixed locus F, the elliptic stable envelope is a section

$$\mathrm{Stab}\colon \Theta(T_F^{1/2}) \otimes \mathcal{U}(\det T^{1/2}_{>0}, \hbar)^{-1} \to \Theta(T_X^{1/2}),$$

and the twist by $\mathcal{U}(\det, \hbar)$ is forced upon us. One can additionally twist both the source and target by line bundles $\mathcal{U}(\mathcal{L}_i, z_i)$, with the effect that $\hbar$ from $\mathcal{U}(\det, \hbar)$ shifts the variables z_i.

The point is that we must be careful to pick the correct line bundle whose section will give the elliptic stable envelope. While the degree of the line bundle is fixed once and for all, the degree-0 part is free to move around; this is exactly the dynamical freedom in elliptic quantum groups.

10.6

Now that we have defined Stab near the fixed locus, it remains to extend it inductively. Fix some ordering on fixed components $\{F_i\}$ and let

$$X_n \coloneqq \bigcup_{i \ge n} \mathrm{Attr}(F_i).$$

Then there is a long exact sequence

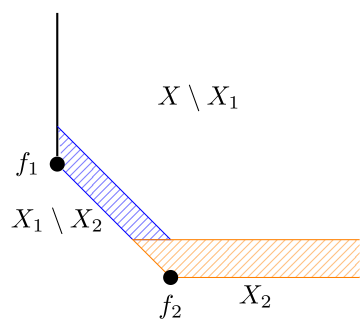


Figure 20: The inductive strata in constructing stable envelopes

$$\cdots \to H^i(X \setminus X_{n+1}, X \setminus X_n) \to H^i(X \setminus X_{n+1}) \to H^i(X \setminus X_n) \to \cdots$$

where the relative term is exactly $\mathrm{Thom}(N_{X/\mathrm{Attr}(F_n)})$. Note that $\mathrm{Attr}(F_n)$ is homotopic to F_n itself. In elliptic cohomology the sequence becomes

$$\cdots \to \Theta^i(-N_{X/F_n,<0}) \to \mathcal{O}^i_{\mathrm{Ell}_\mathsf{T}(X \setminus X_{n+1})} \to \mathcal{O}^i_{\mathrm{Ell}_T(X \setminus X_n)} \to \cdots .$$

In fact the arrow $\Theta^i \to \mathcal{O}^i$ is an embedding, because in the commutative triangle

$$\begin{array}{ccc} \Theta^i(-N_{X/F_n,<0}) & \longrightarrow & \mathcal{O}^i_{\mathrm{Ell}_\mathsf{T}(X \setminus X_{n+1})} \\ & \searrow^{\cdot \prod_{w_i<0} \vartheta(w_i)} \quad \swarrow^{\text{restriction}} & \\ & \mathcal{O}^i_{\mathrm{Ell}_\mathsf{T}(F_n)} & \end{array}$$

the multiplication map is by a non-zerodivisor. Hence the whole long exact sequence just splits into short exact sequences. In particular the zeroth such sequence is

$$0 \to \Theta(-N_{X/F_n,<0}) \to \mathcal{O}_{\mathrm{Ell}_{\mathsf{T}}(X\setminus X_{n+1})} \to \mathcal{O}_{\mathrm{Ell}_{\mathsf{T}}(X\setminus X_n)} \to 0.$$

Tensoring this with $\mathcal{S}$ gives

$$0 \to \mathcal{S}\big|_{F_n} \otimes \Theta(-N_{<0}) \to \mathcal{S}\big|_{X\setminus X_{n+1}} \to \mathcal{S}\big|_{X\setminus X_n} \to 0. \tag{39}$$

Which sections in $H^0(\mathcal{S}|_{X\setminus X_n})$ lift to $H^0(\mathcal{S}|_{X\setminus X_{n+1}})$ is therefore controlled by H^0 and H^1 of the first term in (39), which is a line bundle on $\mathrm{Ell}_{\mathsf{T}}(\mathrm{pt})$. Viewing $\mathrm{Ell}_{\mathsf{T}}(\mathrm{pt})$ as a fibration

$$\begin{array}{ccc} \mathrm{Ell}_{\mathsf{T}}(\mathrm{pt}) & \longleftarrow & \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt}) \\ \big\downarrow & & \\ \mathrm{Ell}_{\mathsf{T}/\mathsf{A}}(\mathrm{pt}), & & \end{array}$$

this line bundle is set up so that it has degree 0 on fibers. Since non-trivial degree-0 line bundles on abelian varieties have no cohomology, we expect our bundle to have no cohomology either.

10.7

To start the inductive construction, note that the stable envelope is zero on $X\setminus X_1$. To extend to $X \setminus X_2$ and get something interesting, we have carefully set it up so that the first term $\mathcal{S}|_{F_1} \otimes \cdots$ in (39) is exactly the trivial line bundle, and therefore we get something non-zero (and unique) on $X \setminus X_2$. To extend to $X \setminus X_3$, it suffices to show that $\mathcal{S}|_{F_2} \otimes \cdots$ is not trivial generically.

Lemma. *If $\mathcal{L}$ is ample, then*

$$\mathrm{wt}_a\, \mathcal{L}\big|_{F_1} \neq \mathrm{wt}_a\, \mathcal{L}\big|_{F_2}.$$

Consequently, twisting by $\mathcal{U}(\mathcal{L}, z)$ produces a result which depends non-trivially on z and is not trivial generically.

Proof. Let C be a curve connecting F_1 and F_2. Then $\deg \mathcal{L}|_C > 0$ since $\mathcal{L}$ is ample. On the other hand, by localization

$$\deg \mathcal{L}\big|_C = \frac{\mathrm{wt}\,\mathcal{L}\big|_{F_1} - \mathrm{wt}\,\mathcal{L}\big|_{F_2}}{\text{tangent weight at } F_1}.$$

So not only are the weights not equal, there is an ordering on them. □

Hence if we allow dynamical variables, then the family $\mathcal{S}|_{F_2} \otimes \cdots$ is non-trivial. Treating it as a map to $\mathrm{Pic}_0(\mathrm{Ell}_{\mathsf{A}}(\mathrm{pt}))$, the only cohomology that can appear is in the dimension of the map. Namely $H^0 = 0$ and H^1 may appear in codimension 1. That $H^0 = 0$ means elliptic stable envelopes are unique; that H^1 may appear in codimension 1 means elliptic stable envelopes may have poles in z_i and $\hbar$ along divisors where $\mathcal{S}|_{F_2} \otimes \cdots$ becomes the trivial bundle. Note that stable envelopes depend on $\hbar$ through the index term $\det T^{1/2}_{>0}$.

10.8

In the enumerative applications we have in mind, where one studies maps $f\colon C \to X$ of degree d and puts them into a generating function

$$\sum_d z^d(\text{count of maps of degree } d),$$

the variables z are exactly the dynamical variables living in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times/q^{\mathbb{Z}}$. So while dynamical variables may seem like an annoyance from the perspective of elliptic stable envelopes, from the original enumerative perspective they are very important! Note that the enumerative z is a "quantum" object while the elliptic z is a classical object. Also, the enumerative series has singularities in z related to the poles of elliptic stable envelopes, and so it is a good sign that we have discovered there are poles in z.

Lecture 11. $T^*\mathbb{P}^{n-1}$ and the hypertoric case

Elliptic stable envelopes for projective spaces, and for hypertoric varieties, Felder's elliptic R-matrix.

11.1

The plan for today is to go over some examples and properties of stable envelopes. The first example is always $X = T^*\mathbb{P}^{n-1}$. Recall that

$$\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus 0)/\mathbb{C}^\times.$$

Let $x_1, \dots, x_n$ denote the variables on $\mathbb{C}^n$, and s be the variable on $\mathbb{C}^\times$. Similarly,

$$T^*\mathbb{P}^{n-1} = \{(x,y) \in V \times V^* \mid \underbrace{x \cdot y = 0}_{\text{moment map}}, \quad \underbrace{x \neq 0}_{\text{stability condition}} \quad \}/\mathbb{C}^\times. \tag{40}$$

Here $V \cong \mathbb{C}^n$. In particular $T^*\mathbb{P}^{n-1}$ is an open set in the corresponding quotient stack

$$\{(x,y) \in V \times V^* \mid x \cdot y = 0\} = T^*[V/\mathbb{C}^\times].$$

11.2

At the level of elliptic cohomology,

$$\mathrm{Ell}_{\mathrm{eq}}(X) \subset \mathrm{Ell}_{\mathrm{eq}}(T^*[V/\mathbb{C}^\times]) = \mathrm{Ell}_{\mathrm{eq}\times\mathbb{C}^\times}(\mathrm{pt}).$$

The equivariance is chosen to consist of $\mathrm{GL}(V)$, acting on both V and $V^\times$, and also a $\mathbb{C}^\times$ scaling the cotangent vector y. Let $a_1, \dots, a_n$ be coordinates on the maximal torus $\mathsf{A} \subset \mathrm{GL}(V)$, and $\hbar$ be the coordinate on the $\mathbb{C}^\times$ (which therefore scales y by $\hbar^{-1}$). Since $\mathrm{Ell}_{\mathrm{eq}}(X) \cong \mathrm{Ell}_{\mathrm{eq}}(\mathbb{P}^{n-1})$ by the discussion in either §6.7 or §8.9,

$$\mathrm{Ell}_{\mathrm{eq}}(X) = \bigcup_i \{sa_i = 1\} \subset \mathrm{Ell}_{\mathrm{eq}\times\mathbb{C}^\times}(\mathrm{pt}) = E^n \times E \times E.$$

Note that in $\mathrm{Ell}_{\mathrm{eq}}(T^*[V/\mathbb{C}^\times])$ also sits $\mathrm{Ell}_{\mathrm{eq}}(X_{\mathrm{flop}})$, where X_{flop} is defined as in (40) but with the different stability condition $y \neq 0$. Then

$$\mathrm{Ell}_{\mathrm{eq}}(X_{\mathrm{flop}}) = \bigcup_i \{sa_i\hbar = 1\} \subset E^n \times E \times E$$

since the difference between x and y is that weights are opposite and there is a twist by $\hbar$.

11.3

The A-fixed points in X are of the form

$$f_k \coloneqq \begin{cases} y = 0 \\ x = (0, \ldots, 0, 1, 0, \ldots, 0) \end{cases}$$

where the 1 is in the k-th position. Pick the attracting order $\cdots \to f_3 \to f_2 \to f_1$, meaning that e.g. f_3 is attracted to f_2, which is attracted toward f_1.

11.4

The stable envelope $\mathrm{Stab}(f_k)$ is a section of some line bundle $\mathcal{S}$ on $\mathrm{Ell}_{\mathrm{eq}}(X)$, but it is better to write it as a restriction of a line bundle on $\mathrm{Ell}_{\mathrm{eq}}(T^*[V/\mathbb{C}^\times])$. The main ingredient in $\mathcal{S}$ is the theta bundle $\Theta(T^{1/2}X)$ so we must pick a polarization $T^{1/2}X$. From (40),

$$TX = T^*V - \mathbb{C} - \mathbb{C}$$

where the first $\mathbb{C} = \mathfrak{g}^*$ is for the moment map, and the second $\mathbb{C} = \mathfrak{g}$ is for the quotient. Here $\mathfrak{g}$ is the Lie algebra of the $\mathbb{C}^\times$ with coordinate s. So for the polarization we can take

$$T^{1/2}X = V - \mathbb{C}.$$

The trivial factor $\mathbb{C}$ does not affect $\Theta(-)$. For the remaining ingredient $\mathcal{U}(\mathcal{O}_X(1), z)$ of $\mathcal{S}$, note that $\mathcal{O}_X(1)$ comes from the defining representation of $\mathbb{C}^\times$ and is therefore just s on the stack, so its sections are like

$$\frac{\vartheta(zs)}{\vartheta(z)\vartheta(s)}.$$

Note that we are writing the group operation on abelian varieties *multiplicatively*, instead of additively.

11.5

The stable envelope therefore looks roughly like a product

$$\mathrm{Stab}(f_k) \sim \underbrace{\prod \vartheta(a_i s)}_{\Theta(V)} \cdot \frac{\vartheta(zs)}{\vartheta(z)\vartheta(s)}.$$

This has degree n in s and degree 1 in each a_i. However, it must be supported on the full attracting set of f_k. Also, the denominator $\vartheta(s)$ is problematic because it involves s, but we want a *holomorphic* section of the line bundle. However, as discussed last time, denominators in z and $\hbar$ are perfectly acceptable and form the resonant loci.

11.6

The correct answer, which one can check has the desired degrees in all variables, is

$$\mathrm{Stab}(f_k) = \frac{\vartheta(a_k s z \hbar^{n-k})}{\vartheta(z\hbar^{n-k})} \prod_{i<k} \vartheta(a_i s) \prod_{i>k} \vartheta(a_i \hbar s). \tag{41}$$

Note that $a_i s$ is the weight of the coordinate x_i, and $a_i \hbar s$ is the weight of y_i^{-1}. To check its support, it suffices to compute its restriction to f_k by substituting $s = a_k^{-1}$:

$$\mathrm{Stab}(f_k)\big|_{f_k} = 1 \cdot \prod_{i<k} \vartheta(a_i/a_k) \prod_{i>k} \vartheta(a_i/a_k \cdot \hbar).$$

We recognize these weights a_i/a_k and $a_k/\hbar a_i$ as the repelling weights at f_k, by our choice of attracting direction:

$$f_k = \begin{bmatrix} y \\ x \end{bmatrix} = \left[\begin{array}{ccc} \overbrace{0,\dots,0,}^{\text{attracting}} & 0, & \overbrace{0,\dots,0}^{\text{repelling}} \\ \underbrace{0,\dots,0,}_{\text{repelling}} & 1, & \underbrace{0,\dots,0}_{\text{attracting}} \end{array} \right]. \tag{42}$$

Indeed, the term $\prod_{i<k} \vartheta(a_i s) \prod_{i>k} \vartheta(a_i \hbar s)$ in (41) sets

$$x_1 = \cdots = x_{k-1} = 0$$
$$y_{k+1} = y_{k+2} = \cdots = y_n = 0$$

and what remains is the attracting set. Hence the formula (41) has the correct support. Finally, the remaining factor $\vartheta(a_k s z \hbar^{n-k})/\vartheta(z\hbar^{n-k})$ is uniquely specified from being a section of the required line bundle, i.e. (in older language) from having the correct *factor of automorphy*.

Remark. An important observation is that in the reasoning leading up to (41), there is no dependence on the choice of stability condition. So the same formula works for X_{flop}, and there is a certain flop invariance for stable envelopes.

11.7

Example. For $X = T^*\mathbb{P}^1$, the toric picture and full attracting sets are drawn in Figure 21. Note that

$$\mathrm{Attr}^f(f_1) = \{y_2 = 0\}$$
$$\mathrm{Attr}^f(f_2) = \{x_1 = 0\},$$

as expected. For example, in the notation of (42),

$$\mathrm{Attr}^f(f_1) = \left\{ \begin{bmatrix} y_1 & 0 \\ x_1 & x_2 \end{bmatrix} \mid x_1 y_1 = 0 \right\} = \left\{ \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right\}.$$

The first piece is the $\mathbb{P}^1$ and the second piece is the fiber over f_2.

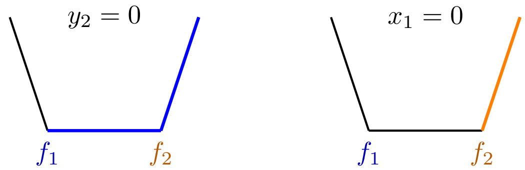


Figure 21: Toric polytope of $T^*\mathbb{P}^1$ and full attracting sets of fixed points

11.8

From $T^*\mathbb{P}^n$ one can proceed more generally to hypertoric varieties

$$X = T^*V \,/\!\!/\!\!/\!\!/\, \mathsf{S},$$

where S is a torus in $\mathrm{GL}(V)$ and $/\!\!/\!\!/\!\!/$ denotes holomorphic symplectic reduction. Namely if μ_{S} is the moment map for S, then $X = \mu_{\mathsf{S}}^{-1}(0) /\!\!/ \mathsf{S}$ where $/\!\!/$ involves some stability condition. We assume that the quotient X is smooth.

The torus A which acts on X fits into a short exact sequence

$$1 \to \mathsf{S} \to \begin{pmatrix} \text{maximal torus} \\ \text{of } \mathrm{GL}(V) \end{pmatrix} \to \mathsf{A} \to 1.$$

Let (x, y) be coordinates on T^*V. Note that we are free to change what we label as living in V vs V^*, and so without loss of generality an A-fixed point in X, viewed in T^*V like in (42), has the form

$$\begin{bmatrix} y \\ x \end{bmatrix} = \left[\begin{array}{cc} 0,0,0,0, & 0,0,0,0,0 \\ *,*,*,*, & \underbrace{0,0,0,0,0}_{\text{repelling for } \mathsf{A}} \end{array} \right] \tag{43}$$

where the unspecified coordinates $*$ correspond to a free S-orbit in T^*V. Weights of the S-action on these coordinates are the coordinates $(s_1, \ldots, s_r)$ on S. Furthermore, we take the polarization $T^{1/2}X$ to consist of the x coordinates. Let V_1 denote the free S-orbit and V_2 denote the repelling directions in $T^{1/2}X$.

11.9

With this setup, we can now repeat the same process as before to compute the elliptic stable envelope:

$$\mathrm{Stab}(\text{fixed point}) = \frac{\Theta(V_1)\big|_{s=sz}}{\prod \vartheta(z_i)} \Theta(V_2) \tag{44}$$

The theta bundle $\Theta(V_2)$ is setting $x_{r+1} = \cdots = 0$. Together with $\Theta(V_1)$ they form the theta bundle of the polarization $T^{1/2}X$, but a substitution $s \mapsto sz$ is required to produce correct factors of automorphy. Here sz denotes $(s_1 z_1, \ldots, s_r z_r)$; line bundles on X correspond to cocharacters of S, so

$$\mathrm{Pic}(X) = \mathrm{cochar}(\mathsf{S})$$
$$\mathrm{Pic}(X) \otimes \mathbb{C}^\times \ni (z_1, \ldots, z_r).$$

Note that (44) does not involve $\hbar$. This is due to our setup for the fixed point (43); for example, the stable envelope for the last fixed point of $T^*\mathbb{P}^n$ in our attracting order also has no $\hbar$.

11.10

There is a nice way to draw hypertoric varieties. As preliminary examples, consider the toric varieties $T^*\mathbb{P}^1$ and an A_n surface. Their toric polytopes in coordinates $(a, \hbar)$ are drawn in Figure 22, and the projections of these polytopes onto $\mathfrak{a}^*$ are described by the singular points of the moment map.

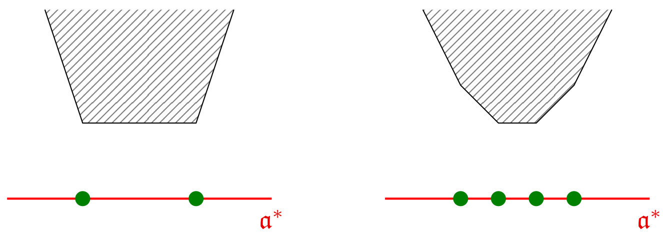


Figure 22: Projection of toric polytopes of $T^*\mathbb{P}^1$ and A_n surface onto $\mathfrak{a}^*$

For hypertoric varieties, at each fixed point there are weights w_i (of $a, \hbar$) and also weights $1/w_i\hbar$ of dual directions. The moment map takes these to a cone

$$\mathrm{Cone}\left(w_i, \frac{1}{w_i\hbar}\right) \subset \mathfrak{a}^* \oplus (\mathrm{Lie}\,\mathbb{C}^*_\hbar)^\vee.$$

Figure 23 is a 2-dimensional example. One can consider the cone as the polytope of the "toric half", in green, along with Lagrangian conormals to the toric strata, namely the 1-dimensional orbits and fixed point.

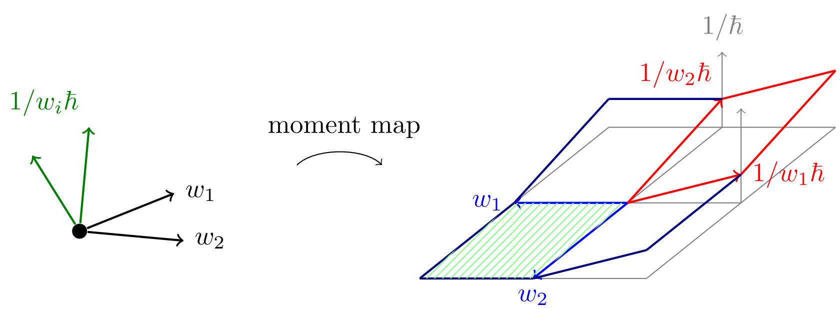
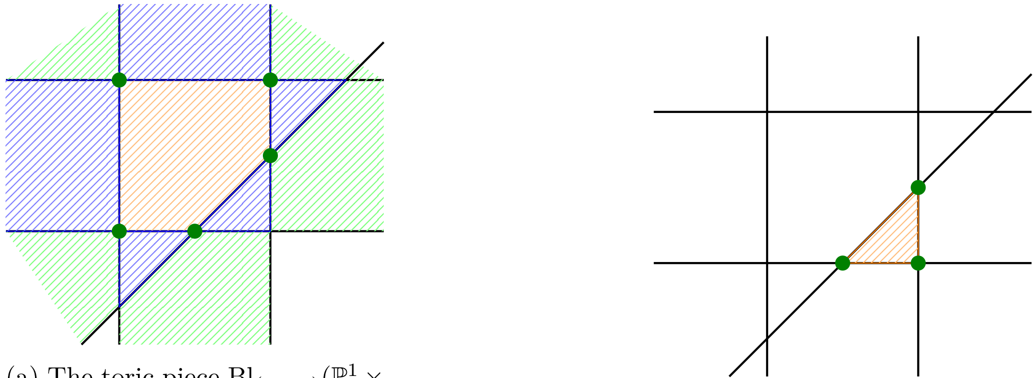


Figure 23: Image of moment map around a fixed point

The picture for a general hypertoric variety is a hyperplane arrangement in $\mathfrak{a}^*$, where hyperplanes are singular values of the moment map, with the understanding that the actual moment map is produced by "folding" along the hyperplanes like origami. For example,



(a) The toric piece $\mathrm{Bl}_{(\infty,\infty)}(\mathbb{P}^1 \times \mathbb{P}^1)$ and its conormals

(b) The toric piece $\mathbb{P}^2$

Figure 24: A hyperplane arrangement defining a hypertoric variety

Figure 24a is such a hyperplane arrangement. In orange is a toric variety which is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at (∞, ∞), and surrounding it are all its conormals. Note that this description changes depending on the point of view: for the same picture, we could center it around the toric polytope of $\mathbb{P}^2$ instead, as in Figure 24b. Regardless, we see that hypertoric varieties are therefore glued from cotangent bundles of various toric varieties.

11.11

A fixed point f is a zero-dimensional stratum of the hyperplane arrangement. The hyperplanes that meet the point f are the attracting/repelling weights at f, and half will be attracting and the other half repelling. In the stable envelope of f, the repelling coordinates are set to zero; this is the $\Theta(V_2)$ part of the formula. The remaining hyperplanes that do not meet f each produce a divisor in X that, together, span $\mathrm{Pic}(X)$. As we vary the variables z in $\Theta(V_1)\big|_{s=sz}$, we get all of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} E$. It is possible that some of these hyperplanes do not intersect the support of the stable envelope. Indeed, flops of X are obtained by rearrangements of hyperplanes, but the formula for the stable envelope remains the same.

11.12

In a different direction, $T^*\mathbb{P}^n$ corresponds to elliptic $\mathfrak{sl}_2$, and one can proceed to cotangent bundles of flag varieties and elliptic $\mathfrak{sl}_n$. Recall the space

$$TG(n) \coloneqq \bigsqcup_k T^* \operatorname{Gr}(k, n),$$

which, in the sense of geometric representation theory, is acted on by some version of $\widehat{\mathfrak{gl}}_2$. In some sense we can treat

$$TG(n) = \text{“}TG(n_1) \otimes TG(n_2)\text{”}, \quad n_1 + n_2 = n,$$

and we have

$$\begin{aligned} TG(1) &= \mathrm{pt} \sqcup \mathrm{pt} \\ TG(2) &= \mathrm{pt} \sqcup T^*\mathbb{P}^1 \sqcup \mathrm{pt}. \end{aligned} \tag{45}$$

The case $TG(1)$ corresponds to the fundamental representation $\mathbb{C}^2$ for $\widehat{\mathfrak{gl}}_2$, and $TG(2)$ is the tensor square. Observe that we now know the stable envelope of $T^*\mathbb{P}^1$, and therefore know the R-matrix (for the fundamental representation).

Remark. One can compute this R-matrix and see that it is gauge equivalent to Felder's R-matrix [Fel95]. Hence the geometric elliptic $\widehat{\mathfrak{gl}}_2$ is the same as Felder's (modulo checking Yang–Baxter, and things like $R_{V_1,V_2\otimes V_3} = R_{V_1,V_2}R_{V_1,V_3}$, but these are easy consequences of general properties of stable envelopes and we will discuss them next time).

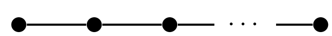
More generally, elliptic $\widehat{\mathfrak{gl}}_n$ is geometrically constructed using Nakajima varieties for the A_{n-1} quiver. The generalization of (45) is that the fundamental representation still corresponds to a collection of points, while its tensor square is a collection of points and $T^*\mathbb{P}^1$'s. It remains true that the geometric elliptic R-matrix matches Felder's for $\widehat{\mathfrak{gl}}_n$.

Lecture 12. $T^*\,\mathrm{Gr}(k,n)$ and abelianization

Cohomology of the Grassmannian, Schubert classes and interpolation Schur functions, elliptic stable envelopes for Grassmannians, abelianization of stable envelopes.

12.1

For the various quantum algebras associated to $\widehat{\mathfrak{gl}}_n$, tensor products of fundamental representations $\wedge^{k_i}\mathbb{C}^n(a_i)$ are realized geometrically as the cohomology of Nakajima quiver varieties with the quiver



called the A_{n-1} quiver. Stable envelopes in all these cases can be written down reasonably explicitly. It is good to have these explicit formulas, because we will learn eventually that, among other applications, from stable envelopes one may obtain off-shell Bethe eigenfunctions. Some integrable systems have both geometric and representation-theoretic meaning, and our particular setup corresponds to previously-studied systems whose eigenvalues/eigenfunctions are obtained from the *Bethe ansatz*. The ansatz is a formula which truly becomes an eigenfunction when its parameters satisfy the Bethe equations; the formula itself is called the off-shell Bethe eigenfunction.

12.2

We focus on the $\widehat{\mathfrak{gl}}_2$ case, so that the quiver has one node and no arrows. There is only one evaluation representation, namely $\mathbb{C}^2(a_i)$, corresponding to the cohomology of the Grassmannian

$$TG(n) := \bigsqcup_{k=0}^{n} T^* \operatorname{Gr}(k,n).$$

It carries the natural action by $\mathrm{GL}(n)$, whose maximal torus we denote A with coordinates $(a_1, \ldots, a_n)$. The evaluation parameters a_i are exactly these coordinates. Fixed points in $TG(n)^{\mathsf{A}}$ are coordinate subspaces, of which there are 2^n, and in particular

$$|T^* \operatorname{Gr}(k,n)^{\mathsf{A}}| = \binom{n}{k}.$$

It is a classical problem what the attracting subspaces look like. One way to parameterize $\operatorname{Gr}(k,n) = \{V \subset \mathbb{C}^n\}$ is to view V as the image of a map $\mathbb{C}^k \to \mathbb{C}^n$, so that

$$\operatorname{Gr}(k,n) = \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^n)_{\text{rank } k} / \operatorname{GL}(k)$$

is the open set of those maps with rank k. In other words V is some $n \times k$ matrix of full rank k, modulo column operations, and fixed points look like e.g.

$$p = \begin{pmatrix} 1 & & \\ & & \\ & & \\ & 1 & \\ & & 1 \\ & & \\ & & \end{pmatrix} \in T^* \operatorname{Gr}(k,n)^{\mathsf{A}}$$

in the coordinate basis. Its attracting manifold is the orbit of unipotent matrices applied to p via row operations:

$$\operatorname{Attr}(p) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ \ast & & & & 1 \end{pmatrix} \cdot p = \begin{pmatrix} 1 & & \\ * & & \\ * & & \\ & 1 & \\ & & 1 \\ * & * & * \\ * & * & * \end{pmatrix} \tag{46}$$

where $*$ denotes entries which may be arbitrary. For our p this attracting manifold is therefore isomorphic to $\mathbb{C}^8$.

12.3

In general, the attracting manifold is a Schubert cell. Let $\mathfrak{S}_p$ be the closure of the Schubert cell of p; these form a basis in any kind of cohomology theory. There is a very nice parameterization of this Schubert basis by partitions, obtained after compressing the matrix in (46) by removing all rows with 1's. The resulting $*$ entries form a partition fitting in a $k \times (n-k)$ box, which in the case of (46) is $(4,2,2)$. Actually it is better to think about the complementary partition inside this box, which in this case is $\lambda = (2,2)$. Denote the original partition λ^c. Then

$$|\lambda^c| = \dim \mathfrak{S}_p$$
$$|\lambda| = \operatorname{codim}_{\mathrm{Gr}(k,n)} \mathfrak{S}_p.$$

So $|\lambda|$ is the degree of the cycle in ordinary cohomology.

12.4

By the resolution of the diagonal or some other argument, the ordinary cohomology $H^*(\mathrm{Gr})$ is generated by Chern classes of the rank-k tautological bundle V. If $x_1, \ldots, x_k$ denote the Chern roots of V, then equivalently

$$H^*(\mathrm{Gr}) = \left[\text{symmetric functions in } \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} \right] /\text{ideal}.$$

For now it is not so important what the ideal is. The classical question is: what is the class of $\mathfrak{S}_\lambda$? It turns out $\mathfrak{S}_\lambda$ is the Schur function

$$s_\lambda(x_1, \ldots, x_k) = \operatorname{Symm} \frac{\prod x_i^{\lambda_i + \rho_i}}{\prod_{i<j}(x_i - x_j)} \tag{47}$$

where $\rho = (k-1, k-2, \ldots, 0)$ is the Harish-Chandra shift. The numbers $\lambda_i + \rho_i$ are the positions of 1's in (46). It is important to note that in the symmetrized expression the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$ is canceled, because a symmetric function cannot have a pole of order 1 along its diagonals. So the Schur function is a polynomial in x.

Of course there are a great many other symmetric functions, obtained by replacing the numerator in (47) by any generalization of a monomial, e.g. $\prod_i m_{\lambda_i + \rho_i}(x)$ for an arbitrary collection of functions $m_\ell(x)$ of one variable. Schur functions have $m_\ell(x) = x^\ell$, but there are many other interesting functions such as the Newton interpolation

$$m_\ell(x) = (x - a_1)(x - a_2) \cdots (x - a_\ell) \tag{48}$$

The corresponding Schur function turns out to be the *equivariant* cohomology class

$$s_\lambda^{\text{interp}} = [\mathfrak{S}_\lambda] \in H^*_{\mathsf{A}}(\mathrm{Gr}).$$

This was first observed by Lascoux–Schützenberger in [LS82], and the proof is very short: verify it has degree $|\lambda|$, and then verify it vanishes at all fixed points not in $\mathfrak{S}_\lambda$. These two properties uniquely characterize the symmetric function. Note that if a_μ denotes $(a_{\mu_1+\rho_1}, a_{\mu_2+\rho_2}, \ldots)$, then

$$s_\lambda^{\text{interp}}(a_\mu) = 0 \text{ unless } \lambda \subset \mu.$$

This comes from $m_\ell(a_m) = 0$ unless $m > \ell$.

12.5

More generally, (48) is the unique polynomial (up to scalar multiple) of degree ℓ which vanishes at $a_1, \ldots, a_\ell$. Having degree ℓ means it has a pole of order ℓ at $x = \infty$. Thinking of polynomials as sections of line bundles on $\mathbb{P}^1$, this means our section has simple zeros at $a_1, \ldots, a_\ell$ followed by a zero of order $d - \ell$ at infinity. In principle one can also replace the latter zero by simple zeros at points $b_1, b_2, \ldots, b_d$, i.e.

$$m_\ell(x) = (x - a_1)(x - a_2) \cdots (x - b_{d-1})(x - b_d).$$

We are interested in the specialization $b_k = a_k + \hbar$, treating $\hbar$ as very large. Then the associated Schur function is

$$\operatorname{Symm} \prod_{i<j} \frac{x_i - x_j + \hbar}{x_i - x_j} \prod_{i=1}^{n} m_{\ell_i}(x_i). \tag{49}$$

The first factor, in the world of symmetric functions, is a Hall–Littlewood-type function. Dividing by an overall Vandermonde factor of $\prod_{i \neq j}(x_i - x_j + \hbar)$, the cohomological stable envelope of $T^* \operatorname{Gr}(k, n)$ is

$$\operatorname{Symm} \frac{\prod_{i=1}^{n} m_{\ell_i}(x_i)}{\prod_{i<j}(x_i - x_j)(x_j - x_i + \hbar)}. \tag{50}$$

It has degree $k(n-k) = (1/2) \dim T^* \operatorname{Gr}(k, n)$, and becomes the Schubert class $\mathfrak{S}_\lambda$ when $\hbar \to \infty$. As a general principle, sending an equivariant variable to ∞ yields its fixed locus.

12.6

The distinction between the two formulas (49) and (50) comes from viewing $T^* \operatorname{Gr}$ as

$$T^* \operatorname{Gr} \underset{\text{open}}{\subset} \mu^{-1}(0) \underset{\text{closed}}{\subset} (\text{quotient stack}),$$

just like we did for $\mathbb{P}^n$. Explicitly,

$$T^* \operatorname{Gr} = \left\{ \begin{array}{l} I \in \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^n) \\ J \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^k) \end{array} \;\middle|\; JI = 0 \right\} /\!\!/ \operatorname{GL}(k)$$

and $JI = 0$ is the moment map μ. In cohomology, recall the embedding

$$\operatorname{Spec} H^*(T^* \operatorname{Gr}) \underset{\text{closed}}{\subset} \underbrace{\operatorname{Lie} \mathsf{A}}_{a_1, \ldots, a_n} \times \underbrace{\operatorname{Lie} H}_{x_1, \ldots, x_k} / W$$

given by Chern roots of the tautological bundle. Here $H \subset \operatorname{GL}(k)$ is the maximal torus. Although the formula (50) for the stable envelope is *not* a polynomial, it is nonetheless a regular function on the $T^* \operatorname{Gr}$ subset. On the other hand, (49) is a polynomial expression on the stack. The difference of the Vandermonde factor can be interpreted as the equivariant weight of the moment map $JI = 0$, noting that J is scaled by $\hbar^{-1}$ because it corresponds to the cotangent vector. Then the passage from (49) to (50) is just pushforward from $\mu^{-1}(0)$ to the whole stack.

12.7

The elliptic analogue is obtained as follows. Note that if $k = 1$ then $T^* \operatorname{Gr}(k, n)$ reduces to $T^*\mathbb{P}^1$, so our functions m_ℓ (along with the Vandermonde denominator) are just stable envelopes for $T^*\mathbb{P}^1$. Our previously-computed elliptic stable envelopes for $T^*\mathbb{P}^1$ therefore give the elliptic

$$m_\ell(s) = \vartheta(a_1 s) \cdots \frac{\vartheta(a_\ell s z \hbar^{n-\ell})}{\vartheta(z\hbar^{n-\ell})} \cdots \vartheta(a_n \hbar s),$$

where s corresponds to x^{-1} and terms such as $a_1 s$ are the multiplicative version of the terms $a_1 - x$ previously. The new piece $\vartheta(a_\ell s z \hbar^{n-\ell})/\vartheta(z\hbar^{n-\ell})$ is where the dynamical variable comes in. To get the elliptic stable envelope for $T^* \operatorname{Gr}$ it remains to symmetrize:

$$\operatorname{Symm} \frac{\prod m_\ell(x_i)}{\prod_{i<j} \vartheta(x_i/x_j)\vartheta(x_j/x_i\hbar)}.$$

This is an actual elliptic cohomology class, which upon pushforward to the whole stack $[\operatorname{Hom} \oplus \operatorname{Hom} / \operatorname{GL}(k)]$ becomes

$$\operatorname{Symm} \prod_{i<j} \frac{\vartheta(x_i/x_j\hbar)}{\vartheta(x_i/x_j)} \prod m_\ell(x_i).$$

12.8

To prove that this is indeed the stable envelope, we follow the general *abelianization* idea of D. Shenfeld [She13]. Let $\mathsf{G} = \operatorname{GL}(k)$ and $\mathsf{H} \subset \mathsf{G}$ be the maximal torus. Schematically, the idea is to examine the diagram

$$\begin{array}{ccc} \mu_{\mathsf{G}}^{-1}(0)/\mathsf{H} & \longrightarrow & \mu_{\mathsf{H}}^{-1}(0)/\mathsf{H} \\ \big\downarrow & & \\ \mu_{\mathsf{G}}^{-1}(0)/\mathsf{G} & & \end{array} . \tag{51}$$

The beauty of the moment map is that it lives in the *dual* of the Lie algebra, so the moment map directions are exactly dual to the group directions. So (51), for all intents and purposes, is a ("smooth") Lagrangian correspondence. The object $\mu_{\mathsf{H}}^{-1}(0)/\mathsf{H}$ is a hypertoric variety, assuming G acts on a vector space like $\operatorname{Hom} \oplus \operatorname{Hom}$, and hence we know its elliptic envelope explicitly. The characterization of elliptic envelopes as sections of certain line bundles is preserved under pullback and pushforward, done correctly. Inverting the pushforward to $\mu_{\mathsf{H}}^{-1}(0)/\mathsf{H}$ in (51) involves terms $1/\vartheta(x_j/x_i\hbar)$, the off-diagonal weights of the moment map, and pushing forward to $\mu_{\mathsf{G}}^{-1}(0)/\mathsf{G}$ involves terms $1/\vartheta(x_i/x_j)$, the tangent weights of the G-action. The only remaining question is why the product is $\prod_{i<j}$ as opposed to $\prod_{i \neq j}$. The answer comes from the actual geometry, where generally half of the directions are non-compact and therefore not integrated over or constrained by the moment map.

Abelianization is very general. In our $T^* \operatorname{Gr}$ example, the correspondence is between $(T^*\mathbb{P}^{n-1})^k$ and $T^* \operatorname{Gr}(k, n)$. In the more complicated setting of $\operatorname{Hilb}(\mathbb{C}^2, k)$, explicit formulas can be found in [Smi20].

Lecture 13. R-matrix from stable envelopes

Triangle lemma, polarization and index of a component of the fixed locus, dynamical Yang–Baxter equation for elliptic stable envelopes.

13.1

Recall our setup for R-matrices: associated to a torus A are Lagrangian correspondences

$$X^{\mathsf{A}} \to X \leftarrow X^{\mathsf{A}}$$

which are an improved version of attracting manifolds. One can choose A such that the fixed locus X^{A} is a product $X_1 \times X_2$, in which case the correspondences acting in cohomology give maps

$$V_1(a_1) \otimes_{\text{opp}} V_2(a_2) \xrightarrow{\text{Stab}_-} V \xleftarrow{\text{Stab}_+} V_1(a_1) \otimes V_2(a_2).$$

Their composition is the R-matrix

$$R(a_1/a_2) \coloneqq \text{Stab}_-^{-1} \circ \text{Stab}_+ .$$

While neither of $\text{Stab}_\pm$ have denominators in $a \in \mathsf{A}$, the R-matrix may. It is best to put $(a_1, a_2) \in (\mathbb{C}^\times)^2$ where the diagonal $\mathbb{C}^\times$ acts trivially, i.e. $a_1 = a_2$ corresponds to X instead of X^{A}. Then the R-matrix can be drawn as in Figure 25.

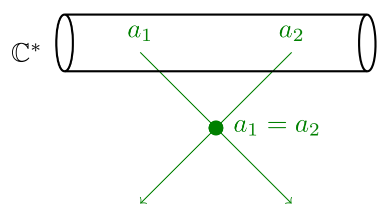
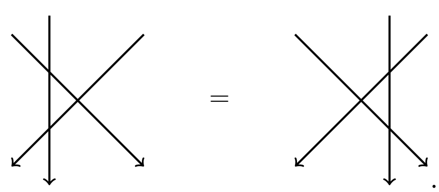


Figure 25: The R-matrix $R(a_1/a_2)$

13.2

We would like the R-matrix to satisfy the Yang–Baxter equation


(52)

This is an equation now for three parameters

$$(a_1, a_2, a_3) \in \mathsf{A} = (\mathbb{C}^\times)^3/(\text{diagonal } \mathbb{C}^\times).$$

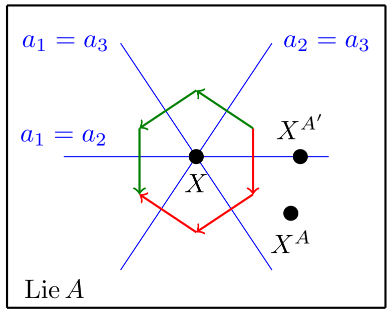


Figure 26: Fixed loci and the Yang–Baxter equation in Lie A

One can draw a "phase diagram" in $\mathrm{Lie}_{\mathbb{R}}\,\mathsf{A}$ for both this and the two-parameter case, where special hyperplanes correspond to bigger fixed loci. In the phase diagram, the Yang–Baxter equation says that the two different ways of crossing the walls in Figure 26 are equal.

Note that the picture with chambers in Lie A is more general than the Yang–Baxter picture, which is specific to root systems of type A. But in general there is some arrangement of hyperplanes in Lie A partitioning it into cones, and for every pair of adjacent cones there are maps

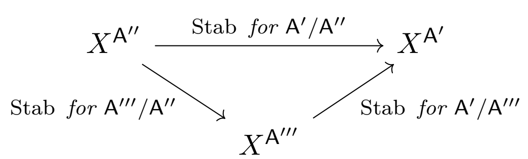
$$\mathrm{Stab}\colon X^{\mathsf{A}''} \to X^{\mathsf{A}'}$$

where A'' corresponds to the interior of the cone and $\mathsf{A}' \subset \mathsf{A}''$ corresponds to a sub-cone. Its inverse Stab^{-1} is also a stable envelope.

13.3

The key fact which makes everything work is the following.

Theorem (Triangle lemma [MO19]). *Given $\mathsf{A}' \subset \mathsf{A}''' \subset \mathsf{A}''$, the following triangle commutes:*



In other words, the maps Stab compose, i.e. form a representation of the corresponding groupoid.

Proof idea. By uniqueness of stable envelopes. □

Let C_+ and C_- be the two adjacent cones in Figure 27. The triangle lemma implies that the R-matrix

$$R_{C_+ \to C_-} \coloneqq \mathrm{Stab}_{C_-}^{-1} \circ \mathrm{Stab}_{C_+}$$

can be computed inside $X^{\mathsf{A}'}$ instead of inside all of X. Yang–Baxter in the form of Figure 26 follows immediately, because both paths are different factorizations of the same R-matrix from a cone to the opposite cone. In some sense, in the language of (52), we have defined an object corresponding to the setting where all three strands intersect at a single point.

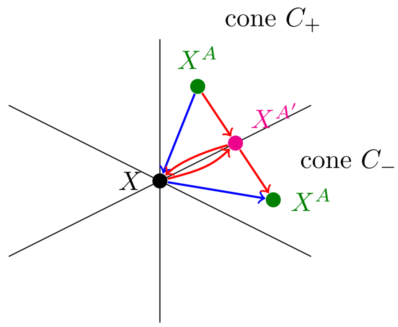


Figure 27: R-matrix computed in $X^{\mathsf{A}'} \subset X$ using triangle lemma

13.4

In fact we should be proving not the usual Yang–Baxter equation, but rather the *dynamical* Yang–Baxter equation. Remember we want a "Lagrangian" elliptic correspondence inside $X \times X^{\mathsf{A}}$. This is a section of some line bundle on $\mathrm{Ell}_{\mathrm{eq}}(X \times X^{\mathsf{A}})$. For example:

- the identity class $1 = [X]$ is a section of $\mathcal{O}_{\mathrm{Ell}_{\mathrm{eq}}(X)} = \Theta(0)$;
- the point class [pt] is a section of $\Theta(TX)$ since it is a pushforward of 1 under $\mathrm{pt} \to X$. (If X were compact then $\Theta(TX)$ is like the canonical bundle of $\mathrm{Ell}_{\mathrm{eq}}(X)$.)

Our Lagrangians are somewhere in between these two examples. They are sections of the bundle $\mathcal{S}$, which is approximately $\Theta(TX)^{1/2}$ tensor a non-trivial degree-zero bundle. Recall that this degree-zero shift is unavoidable, and stems from the attracting and repelling classes

$$[\mathrm{Attr}] = \prod \vartheta(w_i \hbar), \quad [\mathrm{Repell}] = \prod \vartheta(w_i)$$

being sections of different bundles. The compensating factor $\prod \vartheta(w_i\hbar)/\vartheta(w_i)$ has degree zero in $a \in \ker \hbar$.

13.5

We will now set up conventions to be close to existing literature, at the risk of mildly burdensome notation. Assume X has a polarization $T^{1/2}X$, i.e. a solution to

$$[T^{1/2}X] + [(T^{1/2}X)^\vee] \equiv [TX]$$

in A-equivariant K-theory. Let

$$\mathcal{S} \coloneqq \Theta(T^{1/2}X) \otimes \mathcal{U}$$

where as discussed previously $\mathcal{U} = \bigotimes \mathcal{U}(\mathcal{L}_i, z_i)$ for a basis $\{\mathcal{L}_i\}$ in $\mathrm{Pic}(X)$ and the variables z_i are dual coordinates. This $\mathcal{U}$ has a section

$$\frac{\vartheta(s_i z_i)}{\vartheta(s_i)\vartheta(z_i)}$$

where $s_i = c_1(\mathcal{L}_i)$ is the elliptic first Chern class. The fixed locus X^{A} has its own polarization

$$T^{1/2}_{X^{\mathsf{A}}} = (T^{1/2}X|_{X^{\mathsf{A}}})^{\mathsf{A}}$$

but in addition also has an *index* given by the attracting part

$$\operatorname{ind} := T^{1/2}_{X^{\mathsf{A}},>0}$$

of its polarization. Writing elliptic envelopes as maps $\mathcal{S}_{\mathsf{A}} \to \mathcal{S}$, the universal bundle in $\mathcal{S}_{\mathsf{A}} = \Theta(T^{1/2}_{X^{\mathsf{A}}}) \otimes \mathcal{U}|_{X^{\mathsf{A}}}$ must be shifted by

$$z \mapsto z - \hbar \det \operatorname{ind}.$$

Note that $\det \operatorname{ind}$ is a line bundle on X^{A}, and therefore a cocharacter of $\operatorname{Pic}(X^{\mathsf{A}}) \otimes \mathbb{C}^\times$, which is where the Kähler variables z live for X^{A}.

13.6

The triangle lemma therefore involves not just X^{A}, X, and an intermediate $X^{\mathsf{A}'}$, but also the bundles $\mathcal{S}_{\mathsf{A}}(-\hbar \det \operatorname{ind}_{X/X^{\mathsf{A}}})$, $\mathcal{S}$, and an intermediate $\mathcal{S}_{\mathsf{A}'}(-\hbar \det \operatorname{ind}_{X/X^{\mathsf{A}'}})$. The only additional thing to note is that the intermediate arrow from X^{A} to $X^{\mathsf{A}'}$ is the stable envelope for X^{A} inside $X^{\mathsf{A}'}$ but *shifted* by $-\hbar \det \operatorname{ind}_{X/X^{\mathsf{A}'}}$. In particular the dynamical Yang–Baxter equation will have these explicit shifts in Kähler (i.e. dynamical) variables.

13.7

There is a dual picture to the phase diagrams where we consider the dual polytope, namely where maximal-dimensional cones correspond to vertices and so on, see Figure 28. Since the phase diagram is in $\operatorname{Lie} \mathsf{A}$, the dual polytope lives in $(\operatorname{Lie} \mathsf{A})^*$ and has to do with weights of A. Hyperplanes $\{w_i(a) = 1\}$ in $\operatorname{Lie} \mathsf{A}$ are given by weights $w_i \in N_{X/X^{\mathsf{A}}}$, so the dual polytope is the Newton polytope of $\wedge^\bullet N_{X/X^{\mathsf{A}}}$. Such a Newton polytope is the projection of a cube of dimension $\operatorname{rank} N_{X/X^{\mathsf{A}}}$, also called a *zonotope*, since in forming the weights of the exterior algebra there is a freedom to choose whether to include or exclude each weight of $N_{X/X^{\mathsf{A}}}$.

More accurately, the w_i are Chern roots, so the zonotope lives not just in $(\operatorname{Lie} \mathsf{A})^*$ but in fact in the equivariant Picard group $\operatorname{Pic}_{\mathsf{A}}(X^{\mathsf{A}})$. For example, at a fixed point, extremal vertices correspond to $\wedge^{\mathrm{top}} T^{1/2}_{>0}$ and $\wedge^{\mathrm{top}} T^{1/2}_{<0}$; in the projection to A' one takes attracting/repelling weights with respect to A'. These are all line bundles and not just weights. Hence, when crossing a hyperplane, i.e. moving along an edge of the zonotope, the transformation is not just the R-matrix of X^{A} inside $X^{\mathsf{A}'}$, but rather it remembers the ambient geometry through precisely the shift of $\wedge^{\mathrm{top}} T^{1/2}_{>0,\mathsf{A}'}$. This is the dynamical shift we saw previously.

The moral of the story is that although zonotopes are centrally symmetric, going around the edges of our zonotope is not a centrally-symmetric operation. In analogy with 2d mirror symmetry, different edges correspond to different flops of a mirror variety, and so for example opposite edges will be almost the same transformation but not quite.

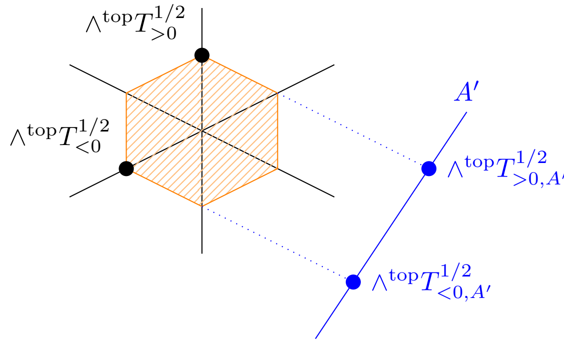


Figure 28: Zonotope for $\mathsf{A} \curvearrowright X$, with projection to $\operatorname{Lie} \mathsf{A}$

Remark. The dynamical shift may vary from situation to situation, but the total distance between the extremal vertices $\wedge^{\mathrm{top}} T^{1/2}_{>0,\mathsf{A}'}$ and $\wedge^{\mathrm{top}} T^{1/2}_{<0,\mathsf{A}'}$ is fixed by topology. Choosing a different polarization may cause the polytope to move, but its size remains fixed. Hence the polytope is fixed by topology up to translations.

Lecture 14. Properties of the geometric R-matrix

Duality for stable envelopes, rigidity in elliptic cohomology, coproduct in quantum groups in terms of R-matrices, various factorizations of R-matrices, operators of cup product as vacuum-vacuum matrix elements of geometric R-matrices.

14.1

The fixed locus X^{A} has many components in general, and its cohomology is graded by the attracting ordering on the components. The composition

$$X^{\mathsf{A}} \xrightarrow{\mathrm{Stab}} X \xrightarrow{\text{restriction}} X^{\mathsf{A}}$$

is therefore a block-triangular matrix. On the diagonal, for a component $F_i \subset X^{\mathsf{A}}$, the stable envelope near F_i is just the attracting manifold, which is cut out by repelling directions. Hence diagonal blocks are the Euler class of the repelling bundle $N_{X/X^{\mathsf{A}},<0}$. In terms of formulas, if w_k are the Chern roots of attracting directions, then these Euler classes are of the form

$$\prod \vartheta(w_k \hbar).$$

Note that these Chern roots include the data of the action of A, and are therefore never trivial. So the Euler class is a non-zerodivisor on $\mathrm{Ell}_{\mathrm{eq}}(F_i) \to \mathrm{Ell}_{\mathrm{eq}}(\mathrm{pt})$. The conclusion is that Stab is invertible after localization, and the R-matrix

$$R \coloneqq \mathrm{Stab}_{-}^{-1} \circ \mathrm{Stab}_{+}$$

has poles precisely where $\prod \vartheta(w_i) = 0$, coming from Stab_{-}^{-1}.

14.2

Lemma. *The inverse* Stab_-^{-1} *is equal to* $\mathrm{Stab}_+^{transpose}$ *with the substitutions*

$$T^{1/2} \mapsto T^{1/2}_{opp} \coloneqq TX - T^{1/2}$$
$$z \mapsto z^{-1}.$$

In other words, to get the inverse, one takes all parameters and replace them by their opposites, and then take a transpose. The transpose is with respect to some pairing like $(\alpha, \beta) = \int_X \alpha \cup \beta$, but in elliptic cohomology the map $X \to \mathrm{pt}$ has normal bundle $-TX$ so that the pushforward is $\Theta(TX) \to \mathcal{O}_{\mathrm{Ell(pt)}}$. Hence the transpose of an elliptic class $\alpha \in \mathcal{S}$ lives in some line bundle $\mathcal{S}^\nabla$ with pairing

$$\mathcal{S} \otimes \mathcal{S}^\nabla \to \Theta(TX).$$

This line bundle is therefore $\mathcal{S}^\nabla = \Theta(TX) \otimes \mathcal{S}^{-1}$. Recall that $\mathcal{S} = \Theta(T^{1/2}X) \otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i)$, so

$$\mathcal{S}^\nabla = \Theta(T^{1/2}_{\mathrm{opp}}X) \otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i^{-1}).$$

This is why we must substitute the opposite polarization and Kähler variables.

Note that the hyperplane arrangements in $\mathrm{Lie}\,\mathsf{A}$ are only centrally symmetric, and despite our schematic pictures they do not necessarily arise from reflection groups. The central symmetry relates to full flops of the variety, namely passing from one cone to its opposite cone corresponds to taking a full flop.

14.3

Proof of Lemma 14.2. We compute $\mathrm{Stab}_-^t \circ \mathrm{Stab}_+$. Each map is a correspondence, so this composition is a pushforward

$$\overbrace{X^{\mathsf{A}} \times \underbrace{X}^{\mathrm{Stab}_-} \times X^{\mathsf{A}}}_{\mathrm{Stab}_+} \xrightarrow{\text{pushforward}} X^{\mathsf{A}} \times X^{\mathsf{A}}$$

where the substitution of opposite polarization and variables means that on the X factor we get a section of $\Theta(TX)$.

The pushforward is proper even though in general X is not compact. This is because Stab_+ is supported on the full attracting set while Stab_- is supported on the full repelling set, so for points $f \in F$ and $f' \in F'$ in two fixed components, the support in X is the locus of points x such that there are (chains of) attracting trajectories $f' \to x \to f$. This locus is compact as follows. Here we must assume that X has a projective map $X \to X_0$ to an affine X_0; this is true for all the usual varieties in geometric representation theory, and is in particular true for Nakajima quiver varieties. More generally, any GIT quotient X admits a projective map to its affine quotient X_0. Or, put differently, if X is Proj of a graded algebra, then it maps to Spec of its zeroth graded component. Then any chain of attracting trajectories is collapsed under $X \to X_0$, and therefore forms a complete curve. The conclusion is that

$$\mathrm{supp}(\mathrm{Stab}_-^t)_{12} \circ (\mathrm{Stab}_+)_{23} \subset X^{\mathsf{A}} \times_{X_0} X^{\mathsf{A}}$$

is proper. Hence $\mathrm{Stab}_-^t \circ \mathrm{Stab}_+$ is regular, i.e. has no poles.

In elliptic cohomology, it is very powerful to argue by *rigidity*. The basic rigidity statement is that if $\mathcal{L}$ is a degree-0 line bundle on an abelian variety, then

$$\mathcal{L} \neq \mathcal{O} \iff H^*(\mathcal{L}) = 0.$$

In particular if a non-trivial line bundle has a section, then the section is zero. In our case, $\mathrm{Stab}_-^t \circ \mathrm{Stab}_+$ is a correspondence $F_i \to F_j$, so in elliptic cohomology it is a section

$$\Theta(T_{F_i}^{1/2}) \otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i)_{\mathrm{shifted}} \mapsto \text{(similar)}.$$

These bundles live on $\mathrm{Ell}_{\mathrm{eq}}(F_i)$, but since F_i is fixed by A, this is an $\mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})$-fibration

$$p\colon \mathrm{Ell}_{\mathrm{eq}}(F_i) \to \mathrm{Ell}_{\mathrm{eq}/\mathsf{A}}(F_i).$$

Only the shifted piece $\mathcal{U}(\mathcal{L}_i, z_i)$ is non-trivial on the $\mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})$ fibers. Hence

$$(\mathrm{Stab}_-^t \circ \mathrm{Stab}_+)\big|_p \in \text{(trivial)} \otimes \bigotimes \mathcal{U}(\mathcal{L}_k\big|_{F_i}, z_k)^{-1} \otimes \mathcal{U}(\mathcal{L}_k\big|_{F_j}, z_k).$$

There are additional shifts in z_k by factors of $\hbar$ which are unimportant. The restrictions $\mathcal{L}_k\big|_{F_i}$ mean to take weights of the A-action.

- When $F_i = F_j$, the shifts are arranged so that the resulting line bundle is trivial. Namely, the restriction is the diagonal in $F_i \times F_i$.
- When $F_i \neq F_j$, the line bundle is not trivial, because if $\mathcal{L}$ is an ample line bundle, then $\deg \mathcal{L}\big|_C > 0$ implies via localization that $\mathrm{wt}\, \mathcal{L}\big|_{F_i} > \mathrm{wt}\, \mathcal{L}\big|_{F_j}$ when F_j is attracted to F_i via a chain of attracting flows. In particular, there exists $\mathcal{L}$ such that these two weights are not equal.

The composition $\mathrm{Stab}_-^t \circ \mathrm{Stab}_+$ therefore is the identity along the diagonal $F_i \times F_i$, and zero elsewhere. □

14.4

Another cornerstone of quantum groups is that braiding a representation V_1 with a tensor product $V_2 \otimes V_3$ is the same as the composition of the two simpler braidings

$$R_{V_1, V_2 \otimes V_3} = R_{V_1, V_2} R_{V_1, V_3}.$$

Equivalently, the way coproduct Δ is defined in quantum groups is via

$$(1 \otimes \Delta) R = R_{12} R_{13}.$$

This is an example of the following general phenomenon. Let $\mathsf{A}' \subset \mathsf{A}$ be a sub-torus, corresponding to some subspace of $\mathrm{Lie}\, \mathsf{A}$. For example, recall that in the Yang–Baxter setup, the fixed loci of a hyperplane $\mathsf{A}_{12} = \{a_1 = a_2\} \subset \mathsf{A}$ is

$$X^{\mathsf{A}_{12}} = X_{12} \times X_3 \supset X_1 \times X_2 \times X_3 = X^{\mathsf{A}}$$

for some bigger space X_{12} whose cohomology is the tensor product of that of X_1 and X_2. In our more general setting, the R-matrix $R_{X^{\mathsf{A}'}}$ (which is defined in terms of A' without reference to A) can be computed in two different ways: directly passing through the origin to the opposite cone in the hyperplane containing $\operatorname{Lie}\mathsf{A}'$, or, using the data of A, a longer route in $\operatorname{Lie}\mathsf{A}$ passing through other hyperplanes in Figure 29.

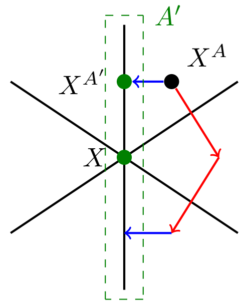


Figure 29: Computation of $R_{X^{\mathsf{A}'}}$ in the larger torus A

More precisely, in the notation of Figure 29, the following square commutes:

$$\begin{array}{ccc} X^{\mathsf{A}'} & \xleftarrow[\text{Stab}]{} & X^{\mathsf{A}} \\ {\scriptstyle R_{C'_+\to C'_-}}\downarrow & & \downarrow \prod R_{\text{walls in }\mathsf{A}} \\ X^{\mathsf{A}'} & \xleftarrow[\text{Stab}]{} & X^{\mathsf{A}} \end{array}.$$

14.5

Remark. Another interesting application (to be discussed in more detail later) of this general principle is a factorization of R-matrices associated to quivers with loops. For the quiver on one vertex with g loops, the fixed locus looks like the quiver variety with quiver $\mathbb{Z}^g$, and the original R-matrix factorizes as a product of R-matrices for this $\mathbb{Z}^g$ quiver. For example, we already mentioned that the finite A_r quiver gives a geometric construction of $\widehat{\mathfrak{gl}}(r+1)$, so when $g=1$ the resulting R-matrix factors as a product of R-matrices for $\widehat{\mathfrak{gl}}(\infty)$. Abstractly, if the torus A' of interest embeds into a bigger torus A, then the R-matrix using A' factors in terms of R-matrices using A. In schematic form,

$$R_{\text{quiver}/\Gamma} = \prod_{\Gamma}^{\rightarrow} R_{\text{quiver}}.$$

Note that such a factorization is distinct from many other factorizations, e.g. in K-theory

$$R_{U_q(\mathfrak{g})} = \prod_{\substack{\text{root sub-}\\ \text{algebras } \mathfrak{g}_\alpha}}^{\rightarrow} R_{U_q(\mathfrak{g}_\alpha)},$$

or the factorizations

$$R_{\text{elliptic}} = \prod^{\rightarrow} R_{U_q}, \quad R_{U_q} = \prod^{\rightarrow} R_{\text{Yangian}}$$

induced by writing $E = \mathbb{C}^\times/q^{\mathbb{Z}}$ and $\mathbb{C}^\times = \mathbb{C}/2\pi i\mathbb{Z}$ respectively. These latter factorizations are also very interesting because they imply (generic) flatness in terms of the elliptic curve (over $\mathbb{C}$).

14.6

In the product $R = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$, the triangularity of Stab means one entry in R can be computed without much thought. Namely, the lowest entry in the attracting order is given by the product

$$\frac{\mathrm{Euler}(N_{<0})}{\mathrm{Euler}(N_{>0})} = \prod \frac{\vartheta(\hbar w_i)}{\vartheta(w_i)}$$

of the two corresponding diagonal entries in Stab_-^{-1} and Stab_+. As a concrete example, take

$$\mathsf{A} = \mathrm{diag}(a, \underbrace{1, 1, \ldots, 1}_{n \text{ times}}) \tag{53}$$

acting on $TG(n+1) = \bigsqcup_k T^* \mathrm{Gr}(k, n+1)$, so that

$$TG(n+1)^{\mathsf{A}} = TG(1) \times TG(n).$$

Since $TG(1) = \mathrm{pt} \sqcup \mathrm{pt}$, corresponding to either $[0 \subset \mathbb{C}^1]$ or $[\mathbb{C}^1 \subset \mathbb{C}^1]$, for any given k the R-matrix has blocks for $T^* \mathrm{Gr}(k-1, n)$ and $T^* \mathrm{Gr}(k, n)$ and we know the diagonal $T^* \mathrm{Gr}(k, n)$ block:

$$\begin{pmatrix} * & * \\ * & \mathrm{Euler}(N_{T^* \mathrm{Gr}(k,n)/T^* \mathrm{Gr}(k,n+1)}) \end{pmatrix}. \tag{54}$$

Here we are assuming $[0 \subset \mathbb{C}^1]$ is lowest in the attracting order. If we call its cohomology class the *vacuum* element $\emptyset$, the entry written in (54) is the vacuum matrix element of the R-matrix.

14.7

Explicitly, at a point $[V \subset \mathbb{C}^n] \in \mathrm{Gr}(k, n)$, the tangent space is

$$\begin{aligned} T_{[V]} \mathrm{Gr}(k, n) &= \mathrm{Hom}(V, \mathbb{C}^n/V) \\ &= \mathrm{Hom}(V, \mathbb{C}^n) - \mathrm{Hom}(V, V) \end{aligned}$$

where $\mathrm{Hom}(V, \mathbb{C}^n)$ is like taking the graph of $V \hookrightarrow \mathbb{C}^n$ and $\mathrm{Hom}(V, V)$ is modding out by the $\mathrm{GL}(V)$ action. Doubling this,

$$T(T^* \mathrm{Gr}(k, n)) = \mathrm{Hom}(V, \mathbb{C}^n) + \hbar^{-1} \mathrm{Hom}(\mathbb{C}^n, V) - (1 + \hbar^{-1}) \mathrm{Hom}(V, V) \tag{55}$$

taking into account that cotangent fibers carry a weight $\hbar^{-1}$. To figure out the normal bundle $N_{T^* \mathrm{Gr}(k,n)/T^* \mathrm{Gr}(k,n+1)}$, it suffices to write $\mathbb{C}^{n+1} = \mathbb{C}^n + a\mathbb{C}$ in accordance with (53) and extract the terms with non-trivial a-dependence in (55):

$$N_{T^* \mathrm{Gr}(k,n)/T^* \mathrm{Gr}(k,n+1)} = \underbrace{aV^*}_{N_{>0}} + \underbrace{\frac{1}{a\hbar} V}_{N_{<0}}.$$

If v_i are the Chern roots of the universal bundle whose fibers are V, then the Euler class is

$$\frac{\vartheta(v_i/a/\hbar)}{\vartheta(v_i/a)}. \tag{56}$$

In ordinary cohomology, everything is additive and this becomes

$$\frac{v_i - a - \hbar}{v_i - a}. \tag{57}$$

Hence the vacuum matrix element is an operator of multiplication by characteristic classes of tautological bundles. In fact it contains them all as a function of the spectral variable a, in the sense of taking residues or linear functionals in a of the formulas (56) or (57).

14.8

Baxter's original construction of integrable systems stems from operators of the form

$$\mathrm{tr}_1(z \otimes 1) R_{12}(a),$$

where tr_1 denotes trace in the first factor, and z is a variable on the maximal torus such that $[z \otimes z, R_{12}] = 0$. These operators act on the second factor and commute for all a. In particular, the limit $z \to 0$ is like projection to the vacuum, and one obtains the vacuum matrix elements above. In general, the algebra formed by all such commuting operators is called the *Baxter subalgebra*, and a limit of the Baxter subalgebra is the algebra of multiplication in cohomology by tautological classes.

Lecture 15. Degenerations to K-theory and cohomology

Nakajima quiver varieties, Hilbert schemes of points and Heisenberg algebra, stable envelopes in K-theory and cohomology, unitarity of R-matrices, classical Yang–Baxter equation, the Lie algebra corresponding to a classical r-matrix, Langrangian correspondences, Steinberg correspondences.

15.1

So far we have been working with a slightly unspecified moduli of vacua X. Many such moduli arise as a critical locus $\mathrm{crit}(\phi)$ modulo the action of some compact gauge group U. Such a construction is what appears in supersymmetric gauge theories, a very rich world of examples. One begins with a complex group G whose maximal compact subgroup is U, along with a symplectic representation M of G which describes "matter", and produces the *algebraic symplectic reduction*

$$X = M /\!\!/\!\!/\!\!/ \mathsf{G} := \mu_{\mathbb{C}}^{-1}(0) /\!\!/_\theta \mathsf{G}. \tag{58}$$

Here $\mu_{\mathbb{C}}$ is the moment map for the G-action on M. Equivalently, using the *hyperkähler* moment map

$$\mu_{\mathbb{H}} := \mu_{\mathbb{C}} \oplus \mu_{\mathbb{R}} \colon M \to \mathfrak{g}^* \otimes (\mathbb{C} \oplus \mathbb{R}),$$

where $\mu_{\mathbb{R}}$ is the real moment map associated to the real symplectic form coming from the imaginary part of the hermitian form, we have

$$X = \mu_{\mathbb{H}}^{-1}(0 \oplus \theta)/\mathsf{U}$$

where θ is the same GIT stability parameter as in (58). With this formulation one can write

$$\phi := \langle \mu_{\mathbb{H}}, \xi_{\mathbb{H}} \rangle, \quad \xi_{\mathbb{H}} \in \mathfrak{g} \otimes (\mathbb{C} \oplus \mathbb{R})$$

as the critical function. In supersymmetric gauge theory, these $\xi_{\mathbb{H}}$ are the superpartners of the gauge fields.

15.2

A general such X is not smooth, and indeed the smooth ones are rare. This has to do with the fact that the GIT quotient in (58) means to take the (actual) quotient of the GIT semistable locus $\mu^{-1}(0)^{ss}$ by G, and this quotient usually has non-trivial finite stabilizers. To avoid stabilizers, M has to be *small*, meaning that weights appearing in M cannot be very large. In particular the weights have to be indivisible, e.g. a weight w^2 is stable if w is, but w^2 has a stabilizer of order 2. For the case $\mathsf{G} = \prod \mathrm{GL}(V_i)$, there is no known general theorem characterizing which M are permitted, but the following definitely work:

- the defining representations V_i and their duals V_i^*;
- the *bifundamental* representations $\mathrm{Hom}(V_i, V_j)$ and their duals.

Of course, for M to be symplectic, each of these representations must appear along with its dual. So in general we set

$$M = \left(\bigoplus V_i^{\oplus w_i} \oplus \mathrm{dual} \right) \oplus \left(\bigoplus \mathrm{Hom}(V_i, V_j)^{\oplus e_{ij}} \oplus \mathrm{dual} \right).$$

The resulting $X = X(v, w)$ is called a *Nakajima quiver variety* with dimension vectors $v = (\dim V_i)_i$ and $w = (w_i)_i$. The data of w_i and e_{ij} is encoded in a (framed) quiver, see Figure 30, with:

- a vertex associated to each V_i;
- edges $V_i \to V_j$ of multiplicity e_{ij};
- an additional vertex W_i (drawn as a square) connected to each V_i, corresponding to a vector space of dimension w_i, and we view $V_i^{\oplus w_i}$ as $\mathrm{Hom}(W_i, V_i)$.

In particular $\mathrm{GL}(V_i)$ does not act on W_i. An important observation is that G commutes with $\prod \mathrm{GL}(W_i) \times \prod \mathrm{GL}(E_{ij})$, where $\mathrm{GL}(E_{ij})$ is the analogue of $\mathrm{GL}(W_i)$ but for edges. Fixed points of $\mathrm{GL}(E_{ij})$ are quiver varieties whose quivers are *coverings* of the original quiver.

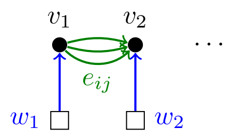


Figure 30: Framed quiver for a Nakajima quiver variety

15.3

We focus on fixed points of the $\mathrm{GL}(W_i)$, which are products of quiver varieties for the *same* quiver. For example, suppose $W = \bigoplus W_i$ is split as

$$W = W' \oplus aW''$$

by a one-parameter subgroup $\mathsf{A} \coloneqq \{(1, 1, \ldots, 1, a, a, \ldots, a)\} \subset \prod \mathrm{GL}(W_i)$. If $X(w) \coloneqq \bigsqcup_v X(v, w)$, then

$$X(w)^{\mathsf{A}} = X(w') \times X(w'').$$

The stable envelopes for this A will give the R-matrix $R(a)$ acting on the cohomology of $X(w') \times X(w'')$. Similarly, the one-parameter subgroup splitting W as

$$W = W' \oplus a_2 W'' \oplus a_3 W'''$$

gives exactly the sort of picture expressing the Yang–Baxter equation. For example, the hyperplane $a_2 = a_3$ corresponds to the enlarged fixed locus $X(w') \times X(w'' + w''')$.

15.4

From these R-matrices we get a quantum group. One goal will be to show that it is an elliptic deformation of the Hopf algebra $U(\widehat{\mathfrak{g}})$, where $\mathfrak{g}$ is a Lie algebra having something to do with the quiver. Note that $\mathfrak{g}$ is *not* the Kac–Moody algebra $\mathfrak{g}_{\mathrm{KM}}$ of the quiver. The A_{n-1} quivers already provide an example, where $\mathfrak{g}_{\mathrm{KM}} = \mathfrak{sl}_n$ but $\mathfrak{g} = \mathfrak{gl}_n$. For a more dramatic difference, take the $\widehat{A}_{n-1}$ quivers, i.e. cyclic quivers on n vertices. From $\widehat{A}_0$ we will obtain

$$\mathfrak{g} \approx \widehat{\mathfrak{gl}}_1 = \langle \alpha_n, \frac{d}{dt}, c \rangle \Big/ \begin{pmatrix} [\alpha_n, \alpha_m] = n\delta_{n+m} c \\ [\frac{d}{dt}, \alpha_n] = n\alpha_n \end{pmatrix}$$

where, viewing $\widehat{\mathfrak{gl}}_1$ as loops in $\mathfrak{gl}_1$, the element d/dt is loop rotation and c is a central extension. In contrast

$$\mathfrak{g}_{\mathrm{KM}} = \langle \alpha_{\pm 1}, \frac{d}{dt}, c \rangle / (\text{same relations})$$

is a finite-dimensional algebra. The difference becomes even more dramatic for other quivers, and in general $\mathfrak{g}$ is much bigger than $\mathfrak{g}_{\mathrm{KM}}$.

15.5

For enumerative applications, the $\widehat{A}_0$ quiver is the most important case. This is because it has an associated Nakajima quiver variety

$$X(1) = \bigsqcup_{n \geq 0} \mathrm{Hilb}(\mathbb{C}^2, n)$$

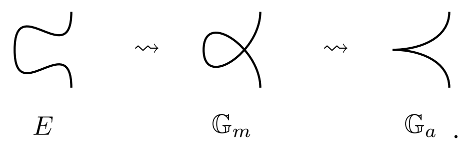
and more generally $X(r)$ is a moduli space of certain sheaves on $\mathbb{C}^2$ of rank r. The Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2, n)$ parameterizes ideals $I \subset \mathbb{C}[x_1, x_2]$ of codimension n, i.e. $\dim \mathbb{C}[x_1, x_2]/I = n$. For more general quivers, one can obtain moduli of certain sheaves on ADE surfaces.

15.6

To understand (elliptic) deformations of $U(\widehat{\mathfrak{g}})$, a good starting point is where the R-matrix is $R = 1$. In principle one could achieve this by setting $\hbar = 0$, but this is a very difficult thing to do in deformation theoretic terms. Instead, we use specializations

$$\text{(elliptic quantum groups)} \rightsquigarrow U_q(\widehat{\mathfrak{g}}) \rightsquigarrow Y(\mathfrak{g})$$

where the Yangian $Y(\mathfrak{g})$ is a graded deformation of $U(\widehat{\mathfrak{g}})$. Elliptic quantum groups are associated to elliptic curves E of $\mathrm{Ell}_{\mathsf{T}}(X)$, and the successive specializations correspond to degenerating E into a nodal and cuspidal curve respectively:



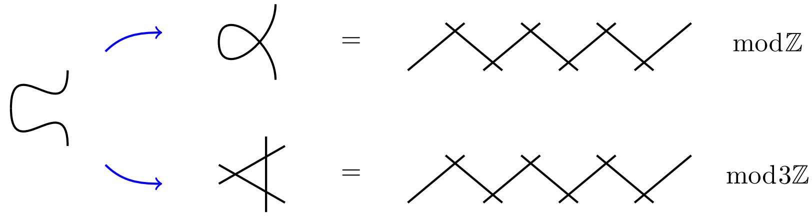
Note that while both elliptic cohomology and K-theory are $\mathbb{Z}/2$-graded, ordinary cohomology is $\mathbb{Z}$-graded.

15.7

Elliptic stable envelopes, restricted to fixed components, are sections of certain line bundles $\mathcal{S}$ on the abelian variety

$$\mathcal{E} := \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt}) \cong E^{\mathrm{rank}\,\mathsf{A}}.$$

The degeneration of the pair $(\mathcal{E}, \mathcal{S})$, as E becomes nodal or cuspidal, is done following work of [Ale02]. We first degenerate $\mathcal{E}$, in the rank 1 case. A nodal curve can be viewed as $\mathbb{P}^1$ with two points identified, but better yet it can be viewed as a chain of $\mathbb{P}^1$'s modulo the translations $\mathbb{Z}$, and a degeneration into multiple nodes can be written in the same fashion:



(59)

To illustrate the degeneration, consider the 3-torsion points

$$E[3] = \mathrm{Ell}_{\mu_3}(\mathrm{pt}) \subset \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt}) = E$$

drawn in the fundamental region of $E = \mathbb{C}^2/\Lambda$ as shown in Figure 31. Fibers at these points correspond to ordinary equivariant cohomology $H^*_{\mathbb{C}^*}(X^{\mu_3})$, except over the identity $1 \in E$ where the fiber is $H^*_{\mathbb{C}^*}(X)$. The degeneration to K-theory stretches one direction off to infinity, and each remaining clump of points should be viewed as a $\mathbb{P}^1$ in the chains in (59). Over each $\mathbb{P}^1$ lives either $K_{\mathbb{C}^*}(X)$ or $K_{\mathbb{C}^*}(X^{\mu_3})$ depending on the order of the points contained within it. Finally, degeneration to ordinary cohomology stretches both directions off to infinity, and only the fiber $H^*_{\mathbb{C}^*}(X)$ over the identity $1 \in E$ remains.

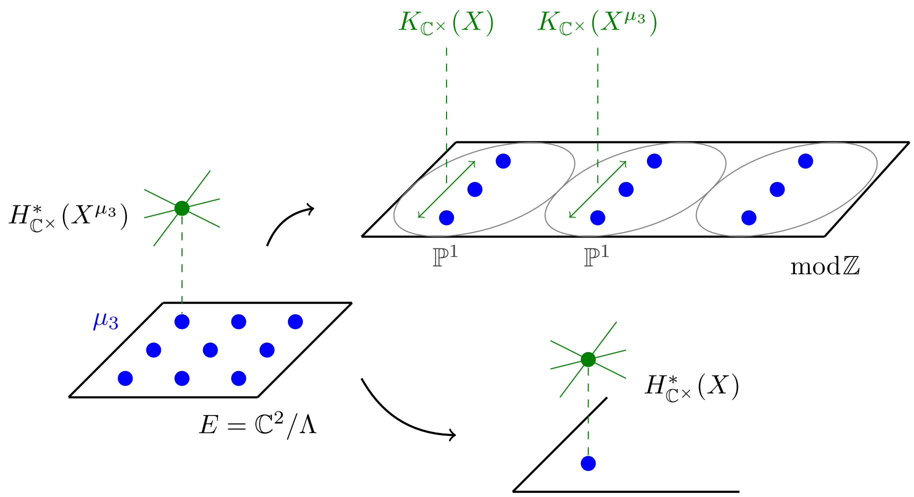


Figure 31: 3-torsion points and their fibers over $E = \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})$, degenerating to K-theory and ordinary cohomology

As for the line bundle $\mathcal{S}$, recall that a toric variety with an ample line bundle can be encoded as a polytope, e.g. $\mathbb{P}^2$ is a triangle, and integer points of the polytope are sections of the line bundle. In the case of multiple nodes, the restriction of $\mathcal{S}$ to the different $\mathbb{P}^1$'s may have polytopes of different sizes, according to $\deg \mathcal{S}$. In higher rank, for instance E^2, the data becomes a "periodic" toric variety in the sense of a tiling of the plane by polytopes, see e.g. Figure 32, which we take modulo $\mathbb{Z}^2$.

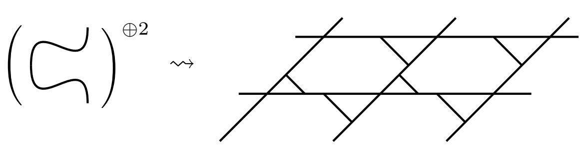


Figure 32: A degeneration of $E^{\oplus 2}$ drawn as a periodic toric variety

15.8

Summarizing, in the degeneration to K-theory, the abelian variety $\mathcal{E}$ degenerates into a union of toric varieties, each of which carry the equivariant K-theory of fixed loci of various subgroups of A. Sections of $\mathcal{S}$, restricted to fixed loci, degenerate into Laurent polynomials, whose Newton polytope must fit inside the Newton polytope of (the restriction of) $\mathcal{S}$.

15.9

In elliptic cohomology, $\mathrm{Stab}(F_i)|_{F_j}$ is a section of the bundle $\mathcal{S} = \Theta(T^{1/2}) \otimes \cdots$ with a shift by a degree-zero line bundle. Degenerating to a nodal curve, being a section of $\mathcal{S}$ becomes the constraint

$$\mathrm{Newton}(\mathrm{Stab}(F_i)\big|_{F_j}) \subset \mathrm{Newton}\left(\wedge^{\bullet} T^{1/2}\big|_{F_j}\right) + (\text{shift})$$

on Newton polytopes, where the shift comes from the degree-zero line bundle in $\mathcal{S}$. Degenerating further to the cuspidal curve, since everything escapes to infinity, all that survives is the constraint

$$\deg_a \left(\mathrm{Stab}(F_i)\big|_{F_j}\right) < \deg_a \left(\mathrm{Euler}(T^{1/2}\big|_{F_j})\right) = \frac{1}{2}\operatorname{codim} F_j.$$

The equality follows from a previous observation that $T^{1/2}|_{F_j}$ and $\mathrm{Attr}\,|_{F_j}$ have the same A-degree, and so the Euler class is a product of $\operatorname{rank} \mathrm{Attr}\,|_{F_j}$ linear terms. Stable envelopes in ordinary cohomology are extremely simple, and their existence and uniqueness is rather easy.

15.10

We discuss the setting of ordinary cohomology for today. Observe that in ordinary cohomology, stable envelopes have diagonal entries of the form $\pm \prod (w_i + \hbar)$ and off-diagonal entries $O(1/u)$ where u is the additive version of the coordinate $a \in \mathsf{A}$. It follows that the (rational) R-matrix $R_{\mathrm{rat}}(u)$ has diagonal entries

$$\prod \frac{w_i + \hbar}{w_i} + O\left(\frac{1}{u}\right)$$

and off-diagonal entries $O(1/u)$. Upon sending $u \to \infty$, we obtain

$$R_{\mathrm{rat}}(\infty) = 1.$$

This is what we originally wanted. Note that the analogous limit in K-theory produces not 1, but rather the R-matrix for $U_q(\mathfrak{g}) \subset U_q(\widehat{\mathfrak{g}})$.

15.11

We can consider the expansion of $R_{\mathrm{rat}}(u)$ in u. By other general arguments which we will not discuss, $R|_{\hbar=0} = 1$ as well, so the linear term of the expansion is of the form

$$R_{\mathrm{rat}}(u) = 1 + \frac{\hbar}{u} r + \cdots$$

for some operator r called the *classical* R-matrix. Properties of r follow immediately from properties of $R_{\text{rat}}(u)$.

- Reversing the attracting order, i.e. what is attracting/repelling, shows that

$$(12)R_{\text{rat}}(-u)^{-1}(12) = R_{\text{rat}}(u),$$

  so $[r, (12)] = 0$. Hence r is a symmetric matrix.

- The Yang–Baxter equation (with spectral parameter) for $R_{\text{rat}}(u)$ gives

$$[r_{12} + r_{13}, r_{23}] = 0 \tag{60}$$

  along with all permutations of the factors.

The relation (60) is very interesting, because we can rewrite it as $[r_{13}, r_{23}] = -[r_{12}, r_{23}]$ and take matrix elements in the first and second factors. The resulting relation implies that such matrix elements form a Lie algebra, which is precisely the $\mathfrak{g}$ that we want. Since $r \in S^2\mathfrak{g}$ is a symmetric tensor, it doesn't matter which factor we choose to take matrix elements. That (60) holds means $r \in S^2\mathfrak{g}$ is an invariant.

15.12

The operator r is an example of a *Lagrangian Steinberg* correspondence between $X(w_1) \times X(w_2)$ and $X(w_1') \times X(w_2')$, where $w_1 + w_2 = w_1' + w_2'$. Lagrangian means the correspondence is a Lagrangian subvariety, which is clear because stable envelopes are supported on the Lagrangian $\text{Attr}^f \subset X \times X^{\mathsf{A}}$.

Definition. A correspondence $Z \subset X \times Y$ is *Steinberg* if $X \times Y$ admits a proper map to an affine space which wlog we take to be a vector space V, such that in fact

$$Z \subset X \times_V Y.$$

That stable envelopes are Steinberg comes from X being a Nakajima quiver variety, which is in particular a GIT quotient $\text{Proj} \bigoplus_{n \geq 0} R_n$. So there is a projective map

$$X = \text{Proj} \bigoplus_{n \geq 0} R_n \to \text{Spec}\, R_0 =: X_0.$$

Since X_0 is affine, this map contracts all complete curves in X. Stable envelopes are only supported on such curves, and are therefore Steinberg.

15.13

In fact, Nakajima quiver varieties are examples of *symplectic resolutions*, i.e. the map $X \to X_0$ is birational onto its image. For such X, there is a general theorem that the Steinberg variety $X \times_{X_0} X$ is isotropic, so its top-dimensional components are Lagrangian. In particular

$$\dim X \times_{X_0} X \leq \dim X.$$

So in this line of work, Lagrangian correspondences come almost for free.

Note that the defining relation $(1 \otimes \Delta)R = R_{12}R_{13}$ for the coproduct Δ becomes

$$(1 \otimes \Delta)r = r_{12} + r_{13}$$

for the linear term in $1/u$. Hence $r_{12}+r_{13}$ not only commutes with r_{23}, but also with R_{23}. This is a general feature of Lagrangian Steinberg correspondences: they intertwine R-matrices.

Lecture 16. Steinberg correspondences and instanton moduli space

Equivariant symplectic resolutions, Lagrangian residues, Lagrangian Steinberg correspondences commute with R-matrices in cohomology, Nakajima–Baranovsky operators and classical r-matrix for instanton moduli spaces, general properies of the Maulik–Okounkov Lie algebras.

16.1

Let X be a Nakajima quiver variety. It is constructed as a GIT quotient $X = (\cdots) /\!\!/_\theta G$, which by definition is of the form $\operatorname{Proj} \bigoplus_{k \geq 0} R_k$. So there is always a map

$$X \to X_0 \coloneqq \operatorname{Spec} R_0$$

to the *affine* quotient, viewing R_0 as invariants of G. This map has two features: it is projective, and it is birational onto its image. The simplest example is $X = T^*\mathbb{P}^1$, where the affinization map simply blows down the $\mathbb{P}^1$ to give the cone $X_0 = \{uv = w^2\}$. A very useful axiomatization of these properties of X is as follows.

Definition. X is an **equivariant symplectic resolution** if:

1. X is algebraic symplectic, e.g. a smooth algebraic symplectic reduction;
2. $X \xrightarrow{p} X_0 \coloneqq \operatorname{Spec} H^0(\mathcal{O}_X)$ is proper and birational (for us, being proper is the most important property, but being birational also comes in handy);
3. there exists $\sigma(t) \in \operatorname{Aut}(X)$ which contracts X_0 to a point, so that X_0 looks like a cone.

In the $X = T^*\mathbb{P}^1$ example, the one-parameter subgroup σ contracts the cotangent fibers (with weight $\hbar^{-1}$).

The paper [Kal09] is recommended reading for the general theory of equivariant symplectic resolutions.

16.2

Theorem. *The Steinberg variety*

$$\operatorname{Steinberg}(X) \coloneqq X \times_{X_0} X = \{(x_1, x_2) \in X \times X : p(x_1) = p(x_2)\} \tag{61}$$

is isotropic with respect to $(\omega_X, -\omega_X)$.

Hence $\dim \mathrm{Steinberg}(X) \leq \dim X$, and the top-dimensional components of $\mathrm{Steinberg}(X)$ are Lagrangian.

This is a very useful result for us, and we will axiomatize two different aspects of it. Suppose $L \subset X \times Y$ is a correspondence, which may be singular. It can be Lagrangian, or, better, it can be *Steinberg*, meaning that

$$L \subset X \times_V Y$$

for some affine V with proper maps $X \to V$ and $Y \to V$. Of course, the Steinberg variety (61) is the prototypical example of a Steinberg correspondence.

16.3

Being a Steinberg correspondence is interesting, because Lagrangian Steinberg correspondences commute with R-matrices. We must work in ordinary cohomology for this, since L does not immediately define an operator in either K-theory or elliptic cohomology.

Definition. Let X be symplectic and A be a torus acting on X preserving the symplectic form, such that X^{A} is smooth (and symplectic). Let L be an A-invariant Lagrangian, with $L^{\mathsf{A}} = \bigsqcup L_i \subset X^{\mathsf{A}}$. Then set

$$\mathrm{Res}\, L := \sum m_i L_i$$

where the m_i a multiplicity which is zero unless $L_i \subset X^{\mathsf{A}}$ is Lagrangian. Such L_i produce Lagrangian subspaces

$$F_i \subset N_f := N_{X/X^{\mathsf{A}}}\big|_f$$

for each point $f \in L_i$, and then m_i is defined by the following.

Lemma. *In the polynomial ring $H^*_{\mathsf{A}}(N_f)$, there is some scalar m_i such that*

$$[F_i] = m_i[N_{f,<0}].$$

Proof. All coordinate Lagrangians are equal up to a sign. □

In fact it is better to take $[F_i] = m_i[T^{1/2}_{f,\neq 0}]$; it doesn't matter which Lagrangian we take, because as a corollary of the lemma, the class of *any* Lagrangian in $H^*_{\mathsf{A}}(N_f)$ is a multiple of a coordinate Lagrangian. This is absolutely false in K-theory and is one of the first things that fails there.

Note that $m_i = \pm 1$ if there is a smooth point on L_i.

16.4

Theorem. *Let L be a Lagrangian Steinberg correspondence for $X \times Y$. Then*

$$\begin{array}{ccc} X^{\mathsf{A}} & \xrightarrow{\mathrm{Stab}_{\mathfrak{C},T^{1/2}_X}} & X \\ {\scriptstyle \mathrm{Res}_{T^{1/2}_X, T^{1/2,opp}_Y} L}\downarrow & & \downarrow {\scriptstyle L} \\ Y^{\mathsf{A}} & \xrightarrow[\mathrm{Stab}_{\mathfrak{C},T^{1/2}_Y}]{} & Y \end{array}$$

commutes for all choices of attracting chamber $\mathfrak{C}$.

Proof. We show that going along $X^{\mathsf{A}} \to X \xrightarrow{L} Y \to Y^{\mathsf{A}}$ is equal to $\operatorname{Res} L$. First, use that

$$\operatorname{Stab}^{-1}_{\mathfrak{C},T_Y^{1/2}} = \left(\operatorname{Stab}_{-\mathfrak{C},T_Y^{1/2,\mathrm{opp}}}\right)^T.$$

Stable envelopes are designed so that rigidity arguments apply. Namely, the Steinberg is proper and so the resulting integral is independent of $a \in \operatorname{Lie}\mathsf{A}$. Then send $a \in \operatorname{Lie}\mathsf{A}$ to ∞ to compute the answer $\operatorname{Res} L$. $\square$

Corollary. $\operatorname{Res}(L)R_X = R_Y \operatorname{Res}(L)$ *for the R-matrices* R_X *and* R_Y *of* X *and* Y.

This is a systematic and powerful way to produce elements commuting with the R-matrix in cohomology.

16.5

Example. Consider the quiver with one vertex and one loop. Then

$$X(v_1, w_1) = \mathcal{M}(r, n)$$

are moduli spaces of torsion-free and *framed* sheaves $\mathcal{F}$ on $\mathbb{C}^2$ or $\mathbb{P}^2$ of rank r and $c_2(\mathcal{F}) = n$. Framed, here, means that at the line $\ell_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$, there is an isomorphism $\mathcal{F}|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$. So, for example,

$$\mathcal{M}(1, n) = \{\text{ideals } I \subset \mathbb{C}[x_1, x_2] \text{ of codim } n\} = \operatorname{Hilb}(\mathbb{C}^2, n).$$

Let $\mathcal{M}(r) \coloneqq \bigsqcup_n \mathcal{M}(r, n)$. The group $\mathrm{GL}(r)$ acts by changing the basis on $\mathcal{O}_{\ell_\infty}^{\oplus r}$. If $\mathsf{A} \subset \mathrm{GL}(r)$ is the maximal torus, called the *framing torus*, then

$$\mathcal{M}(r)^{\mathsf{A}} = \underbrace{\mathcal{M}(1) \times \mathcal{M}(1) \times \cdots \times \mathcal{M}(1)}_{r \text{ times}}$$

so all $\mathcal{F} \in \mathcal{M}(r)^{\mathsf{A}}$ are a sum of ideal sheaves $\mathcal{F} = \bigoplus_{i=1}^r I_i$.

Any time we work with moduli of sheaves, there are always correspondences given by short exact sequences of sheaves. Let

$$\mathcal{M}(r, n) \times \mathbb{C}^2 \xrightarrow{\alpha_{-k}} \mathcal{M}(r, n+k)$$

be the correspondence on $(\mathcal{F}_1, p, \mathcal{F}_2) \in \mathcal{M}(r, n) \times \mathbb{C}^2 \times \mathcal{M}(r, n+k)$ given by

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \begin{pmatrix} \text{torsion sheaf} \\ \text{supported at } p \end{pmatrix} \to 0.$$

In rank $r = 1$ these correspondences were first studied by Nakajima, and then in rank $r > 1$ by Baranovsky and others.

The claim is that α_{-k} is a Lagrangian Steinberg correspondence. The argument goes by embedding both sides of the correspondence into a common space $\mathcal{M}(r+1, n+k)$, by

$$\begin{array}{ccccc} \mathcal{M}(r,n) \times \mathbb{C}^2 & \longrightarrow & \mathcal{M}(r+1,n+k) & \longleftarrow & \mathcal{M}(r,n+k) \\ (\mathcal{F}_1, p) & \longmapsto & \mathcal{F}_1 \oplus \begin{pmatrix} \text{some fixed } I \\ \text{translated to } p \end{pmatrix} & & \\ & & \mathcal{F}_2 \oplus \mathcal{O} & \longleftarrow\!\shortmid & \mathcal{F}_2. \end{array}$$

The affinization map for $\mathcal{M}(r, n)$ is given by

$$\mathcal{F} \to (\mathcal{F}^{\vee\vee}, \operatorname{supp} \mathcal{F}^{\vee\vee}/\mathcal{F}),$$

mapping into the *Uhlenbeck space* $\mathcal{M}(r, n)_0$. Taking double duals removes torsion, so $\mathcal{F}^{\vee\vee}$ is a vector bundle and it is therefore clear that both $\mathcal{F}_1 \oplus I$ and $\mathcal{F}_2 \oplus \mathcal{O}$ map to the same point in $\mathcal{M}(r, n)_0$. Hence the correspondence is Steinberg, and also isotropic since it sits in a bigger space which is Steinberg and therefore isotropic. Finally, it is straightforward to compute it is middle-dimensional, to show it is therefore Lagrangian as well.

16.6

What is $\operatorname{Res} \alpha_{-k}$? Since the correspondence α_{-k} is supported on the fixed locus $\mathcal{M}(2)^{\mathsf{A}} = \mathcal{M}(1) \times \mathcal{M}(1)$, it is clear that the support of $\operatorname{Res} \alpha_{-k}$ is on the Lagrangians $\alpha_{-k} \otimes 1$ and $1 \otimes \alpha_{-k}$. In fact the multiplicities of these terms are 1, because there is a smooth point on each Lagrangian, and one can arrange the polarization such that both signs ± 1 are actually positive. Hence

$$\operatorname{Res} \alpha_{-k} = \alpha_{-k} \otimes 1 + 1 \otimes \alpha_{-k}. \tag{62}$$

Define for $k > 0$ the operators $\alpha_k \coloneqq \pm(\alpha_{-k})^T$. The sign is chosen so that the commutation relation in rank 1 is

$$[\alpha_k, \alpha_\ell] = k\delta_{k+\ell} \cdot \operatorname{diag}_{\mathbb{C}^2}.$$

The rank-r commutation relations follow directly from this and (62). It is important to keep in mind the presence of a $\mathbb{C}^2$ factor in all these correspondences; this is the $\operatorname{diag}_{\mathbb{C}^2}$ term on the rhs. One can modify the α_k to constrain the point p to lie in some equivariant cohomology classes, in which case the $\operatorname{diag}_{\mathbb{C}^2}$ will change accordingly.

16.7

The equivariant cohomology of $\mathcal{M}(1) = \operatorname{Hilb}(\mathbb{C}^2)$ is a Fock module for these operators α_k, and in particular it is irreducible. So denote it

$$\mathsf{Fock} \coloneqq H^*_{\text{eq}}(\mathcal{M}(1)).$$

The R-matrix acts on $\mathsf{Fock} \otimes \mathsf{Fock}$. Set

$$\alpha_n^{\pm} \coloneqq \alpha_n \otimes 1 \pm 1 \otimes \alpha_n,$$

so we have just shown the R-matrix commutes with all α_n^+. Decompose

$$\mathsf{Fock} \otimes \mathsf{Fock} = \mathsf{Fock}^+ \otimes \mathsf{Fock}^-$$

where $\mathsf{Fock}^{\pm}$ are the modules generated by the $\alpha_n^{\pm}$ respectively. (They commute.) Hence the R-matrix is some kind of expression in only the α_n^-.

Remark. To compare with a simpler example, take the ordinary R-matrix $R(u) = 1 - P/u$ where $P \coloneqq (12)$ swaps the two tensor factors. Then one can normalize things so that R acts trivially on the symmetric tensors and non-trivially on the anti-symmetric tensors. In our setting $\mathsf{Fock} \otimes \mathsf{Fock}$, thinking of each Fock as a boson $\varphi^{(i)}$, we just showed that our R-matrix does not act on the center of mass only acts on $\varphi^{(1)} - \varphi^{(2)}$.

Lemma. *Any expression in α_n^- is uniquely determined by its matrix elements $|\emptyset\rangle \otimes \mathsf{Fock} \to |\emptyset\rangle \otimes \mathsf{Fock}$, where the vacuum $|\emptyset\rangle$ is the generator of $H^*_{\mathrm{eq}}(\mathrm{Hilb}(\mathbb{C}^2, 0))$.*

To be clear, the R-matrix does *not* preserve the subspace $|\emptyset\rangle \otimes \mathsf{Fock}$. We are just taking matrix elements in it. In tensorial notation, these matrix elements are $R^{|\emptyset\rangle,\cdot}_{|\emptyset\rangle,\cdot}$.

16.8

We showed earlier that vacuum-vacuum matrix elements are operators of classical multiplication, i.e. cup product by characteristic classes of the tautological bundle Taut, of the form

$$\prod_{i=1}^{\mathrm{rank}} \frac{w_i + u + \hbar}{w_i + u} = 1 + \frac{\hbar}{u} \mathrm{rank} + + \frac{1}{u^2}(c_1 \cup + \mathrm{rank}) + \cdots.$$

Clearly $\mathrm{rank} = \mathrm{rank}\,\mathrm{Taut} = n$, and by an elementary observation about the Heisenberg algebra, this can be written as

$$\mathrm{rank} = \sum_{k>0} \alpha_{-k}\alpha_k$$

perhaps up to some normalization that doesn't matter at the moment. The conclusion is that

$$R(u) = 1 - \frac{\hbar}{u} \mathrm{diag}_{\mathbb{C}^2} \cdot \sum \alpha^-_{-n}\alpha^-_n + \cdots.$$

16.9

In fact, there is only one step remaining to compute the full R-matrix. Take the Yang–Baxter equation

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)\cdots$$

and consider $u_1 \to \infty$. Of course, if we literally take the limit, the resulting equation is uninteresting. Instead, consider the coefficient of $1/u_1$, which gives an equation

$$[r_{12} + r_{13}, R_{23}] = 0.$$

This we have seen already, and says that R commutes with the Nakajima–Barankovsky operators, which we also already knew. Going further, taking the coefficient of $1/u_1^2$ gives a highly non-trivial equation

$$(\cdots)\cdots R_{23} = R_{23}\cdot(\cdots)$$

involving the operator of multiplication by $c_1(\mathrm{Taut})$ on $\mathcal{M}(2)$. This operator has to do with a Virasoro algebra acting on Fock^-, of which rank $= L_0$ is the degree-zero part, so R_{23} becomes a certain Virasoro intertwiner. The Virasoro acts irreducibly on Fock^-, so this already uniquely determines the whole operator R_{23}.

16.10

As a general principle, for a Nakajima quiver variety X, one can look at the inclusions of two different fixed components

$$X(v_1, w_1) \times X(v_2, w_2) \xrightarrow{\oplus} X(v_1 + v_2, w_1 + w_2) \leftarrow X(v_1', w_1') \times X(v_2', w_2'),$$

where of course $v_1' + v_2' = v_1 + v_2$ and similarly for w. The Steinberg inside $X(v_1 + v_2, w_1 + w_2)$ gives a correspondence between the two fixed components. In particular, there is a correspondence

$$X(\alpha, w_0) \times X(v, w) \xrightarrow{\text{Act}} X(0, w_0) \times X(\alpha + v, w) = X(\alpha + v, w),$$

and this is some matrix element of the stable envelope. The upper-/lower-diagonal part of the classical part r of the R-matrix is then supported on a Lagrangian–Steinberg correspondence. This gives the action of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$ by taking appropriate matrix elements in $X(\alpha, w_0)$. To see which matrix elements to take, consider the silly correspondence

$$X(\alpha, w_0) \times X(0, w_0) \xrightarrow{P} X(0, w_0) \times X(\alpha, w_0).$$

The Yang–Baxter equation implies that $P^2 = -w_0(\alpha)P$, so up to normalization P is a projector onto the image

$$\mathfrak{g}_\alpha \left|\emptyset\right\rangle \subset H^*_{\mathrm{eq}}(X(\alpha, w_0)).$$

16.11

To make a connection with enumerative geometry, note that both P and Act are residues of the correspondence

$$\{p, q \in X \text{ both lie on a rational curve of degree } \alpha\}.$$

Moduli spaces of curves, e.g. as in Gromov–Witten theory and its relatives, are great sources of correspondences like this. It is clear that any such correspondence is Steinberg, because any complete curve is contracted by the affinization map. To define these correspondences usually requires a virtual cycle, and if one arranges the virtual dimensions correctly, they will be Lagrangian correspondences as well.

As an exercise, verify for the quiver with one vertex and one loop that $P = \alpha_{-n}\alpha_n$ in order to reproduce the formula for r.

Lecture 17. The instanton R-matrix and the Yangian $Y(\widehat{\mathfrak{gl}}_1)$

Cup product by divisor in cohomology of instanton moduli spaces, its relation to quantum Calogero–Sutherland and Benjamin–Ono integrable systems, full R-matrix for instanton moduli spaces, Yangian of $\widehat{\mathfrak{gl}}(1)$ and its relation to the Virasoro algebra and W-algebras, Kac determinant from R-matrices, the R-matrix for the Hilbert scheme of points as the reflection operator in Liouville CFT, slices and relations in quantum groups.

17.1

We continue with the example of a quiver with one vertex and one loop. Recall that, from general principles, we obtain an elliptic/trigonometric/rational R-matrix $R(u)$ acting on $\mathsf{Fock} \otimes \mathsf{Fock}$. Here the u are framing variables. Our goal is to understand the rational $R(u)$. Since many things in the theory of quantum groups are done by deformation, one can draw many conclusions about features at a general point once enough features at a special point are understood. In particular we will write $R(u)$ in terms of conformal field theory (CFT), in terms of a Virasoro algebra Vir.

Note that $R(u)$ depends on equivariant variables

$$(t_1, t_2) \in \mathsf{T}^2 \subset \mathrm{Aut}(\mathbb{C}^2). \tag{63}$$

From the perspective of the quiver, t_1 is the weight of a scaling action on the single loop, and the dual cotangent direction is $t_2 = -\hbar - t_2$, so that $\hbar := -t_1 - t_2$. Cohomology is a graded theory, so the dependence on t_1, t_2 is only up to scaling, e.g. by the degree-0 quantity

$$\kappa^2 := \frac{(t_1 + t_2)^2}{t_1 t_2}. \tag{64}$$

Another way to parameterize the scaling is by t_1/t_2.

17.2

What is the meaning of the Virasoro algebra? All the features we see in this example really find their explanation in a higher-dimensional geometry, and physically one should really be thinking about the full 11-dimensional M-theory. A particular instance where one can see higher-dimensional geometry come into play is the Alday–Gaiotto–Tachikawa (AGT) relation. Physically, for our quiver of interest, we are studying instantons on the $\mathbb{C}^2 \cong \mathbb{R}^4$ from (63). Take the 6-dimensional manifold $\mathbb{R}^4 \times C$, which could be the world-volume of an M5 brane, or, more generally, a stack of r M5 branes. These are certain extended objects in M-theory which we won't need to know much about. AGT says that to study them, we can compute either

- with instantons on $\mathbb{R}^4$, with Nekrasov partition functions (counting instantons), or
- with some CFT on C with some extended $W(\mathfrak{gl}_r) \supset \mathrm{Vir}$ symmetry,

and there is a certain correspondence between quantities derived from these two perspectives.

In this course, however, we will not discuss M5 branes at all. Instead, we focus on 3d theories, which live on world-volumes of M2 branes, and the degrees of freedom we find in things like AGT are not as intrinsic in the 3d geometries.

The quantum group we will study is the Yangian $Y(\widehat{\mathfrak{gl}}_1)$, acting on the cohomology of moduli spaces of rank-r instantons, i.e. an r-fold tensor product $\mathsf{Fock}^{\otimes r}$. In fact there is a factorization

$$\begin{array}{ccc} Y(\widehat{\mathfrak{gl}}_1) & \longrightarrow & \mathrm{End}(\mathsf{Fock}^{\otimes r}) \\ & \searrow \quad \nearrow & \\ & \overline{W(\mathfrak{gl}_r)} & \end{array}$$

through some completion of $W(\mathfrak{gl}_r)$. So all the W-algebras in the AGT relation can be understood through the image of the Yangian in its defining representation. Pictorially, generators of the Yangian are drawn in Figure 33, with the lowest row corresponding to the Heisenberg algebra $\widehat{\mathfrak{gl}}_1$ acting on Fock. The lowest r rows act irreducibly on $\mathsf{Fock}^{\otimes r}$. For example: the second-lowest row is the Virasoro Vir, which together with $\widehat{\mathfrak{gl}}_1$ act irreducibly on $\mathsf{Fock}^{\otimes 2}$; the lowest three rows form $W(\mathfrak{gl}_3)$ acting irreducibly on $\mathsf{Fock}^{\otimes 3}$; and so on.

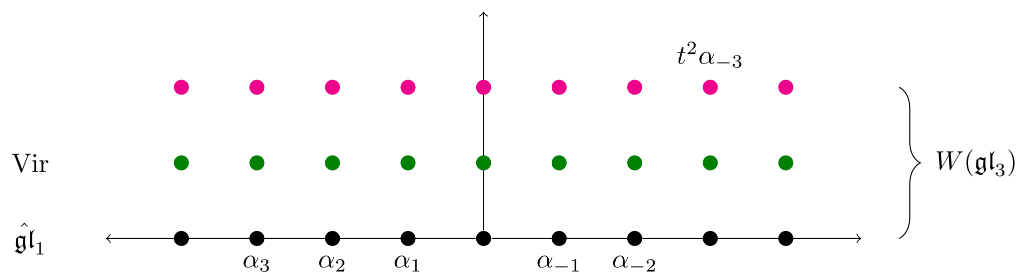


Figure 33: Generators of $Y(\widehat{\mathfrak{gl}}_1)$

17.3

For brevity, today we will work up to powers of -1, factors of 2. We also neglect cohomology labels; namely, recall that α_{-k} is a correspondence from $\mathrm{Hilb}(n) \times \mathbb{C}^2$ to $\mathrm{Hilb}(n+k)$, and one can put any cohomology class on the $\mathbb{C}^2$ factor. Though $H^*_{\mathsf{T}}(\mathbb{C}^2)$ is rather trivial, the T-equivariance does have some significance, and usually denominators of $t_1 t_2$ such as in (64) arise from integrating over this $\mathbb{C}^2$.

Recall that we already constructed $\widehat{\mathfrak{gl}}_1$ as the algebra generated by these correspondences α_n, along with a central element c and loop rotation $t\frac{d}{dt}$, with commutation

$$[\alpha_n, \alpha_m] = n\delta_{n+m} c.$$

When forming the Nakajima quiver variety Hilb, we get a tautological bundle V and a trivial framing bundle W. The element c acts by $\mathrm{rank}(W)$, which is r for rank-r instantons, and $t\frac{d}{dt}$ acts by $\mathrm{rank}(V)$, which is the instanton number. More or less, $t\frac{d}{dt}$ is the same as $L_0 \in \mathrm{Vir}$.

The element α_0 is not obtained from taking matrix elements of the coefficient $[u^{-1}]R$. In fact we will define

$$\alpha_0 \coloneqq c_1(W)$$

so that α_0 acts on $\mathsf{Fock}^{\otimes r}$ by the sum $a_1 + \cdots + a_r$ of framing variables. In particular α_0 is primitive, i.e. $\Delta\alpha_0 = \alpha_0 \otimes 1 + 1 \otimes \alpha_0$. Note that c_1 of bundles only appears starting in $[u^{-2}]R$. But we include it in $\widehat{\mathfrak{gl}}_1$ for ease in writing formulas, e.g. if we introduce

$$\alpha(t) \coloneqq \sum_n \alpha_n t^{-n},$$

we can ask about expressions like $\int :\!\alpha^2\!:$ and $\int :\!\alpha^3\!:$ and so on. Here normal-ordering $:\!(-)\!:$ means to put annihilation operators first, and integration essentially extracts the t^0 coefficient. For example

$$\frac{1}{2}\int :\!\alpha^2\!: = \frac{1}{2}\alpha_0^2 + \sum_n \alpha_{-n}\alpha_n.$$

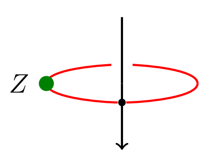
Of course, for formulas like this, the presence of α_0 makes a difference. In this language,

$$\mathfrak{r} = \frac{1}{2}\int :\!\alpha_-^2\!: .$$

17.4

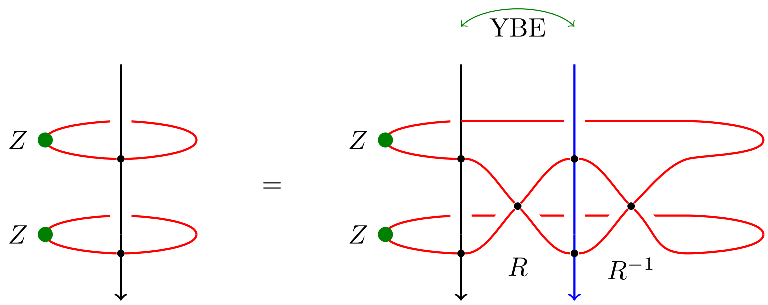
From previous discussion, it is equivalent to know any of the following: the coefficient $[u^{-2}]R(u)$, the coefficient $[u^{-2}]R(u)^{|\emptyset\rangle,-}_{|\emptyset\rangle,-}$ of the vacuum-vacuum matrix element, or the operator $c_1(V) \cup -$ where V is the tautological bundle. This is because $R(u)$ commutes with α_+, as previously discussed, and vacuum-vacuum matrix elements of $R(u)$ are operators of multiplication by characteristic classes of tautological bundles.

Remark. A quantum integrable system is some maximal collection of commuting operators in some space. The operator $c_1(V)$ is just one operator in a certain quantum integrable system consisting of operators of multiplication by *all* characteristic classes. From an R-matrix perspective, matrix elements like $[u^{-2}]R(u)^{|\emptyset\rangle,-}_{|\emptyset\rangle,-}$ form a limiting case of a *Baxter subalgebra.* This is an algebra of Baxter operators, of the form



$$= \quad \mathrm{tr}_1(Z \otimes 1)R_{12}(u) \in \mathrm{End}(\text{second factor})$$

for some Z such that $[Z \otimes Z, R] = 0$. The classical proof that such operators commute, for fixed Z and for all u, can be written as



,

by inserting R and R^{-1} and then applying the Yang–Baxter equation.

17.5

Recall that knowledge of $R(u)$ up to the u^{-2} coefficient actually recovers the whole operator $R(u)$. So now we discuss some properties of $[u^{-2}]R(u)$. All the following properties are equivalent.

17.5.1

Property 1.

$$\ln R(u) = \frac{\hbar}{2u}\int :\alpha_-^2: + \frac{\hbar}{6u^2}\int :\alpha_-^3: + O(\frac{1}{u^3}).$$

This can be seen directly from the factorization $R_{\widehat{\mathfrak{gl}}(1)} = \prod^{\rightarrow} R_{\mathfrak{gl}(\infty)}$, and one can directly compute $R_{\mathfrak{gl}(\infty)}$ and take its u^{-2} coefficient.

17.5.2

Property 2. The operator $c_1(V)$ is the (second quantized) Hamiltonian

$$H_{\mathrm{CS}} := \frac{1}{2}\in:\alpha^3: + \kappa\sum_{n>0} n\alpha_{-n}\alpha_n$$

of the quantum trigonometric Calogero–Sutherland system, with the constant κ basically the one from (64).

Remembering that $c_1(V)$ and therefore H_{CS} are vacuum-vacuum matrix elements, the term $\sum_{n>0} n\alpha_{-n}\alpha_n$ in H_{CS} comes from a vacuum-vacuum matrix element of the perfectly nice zero-mode $\int :\alpha^3:$ of a vertex operator in $R(u)$. The quantum trigonometric CS Hamiltonian is also a quantum version of the Benjamin–Ono equation, describing 1d hydrodynamics, which involves a non-local operation called the Hilbert transform. It is responsible for exactly this extra term.

Remark. In K-theory, H_{CS} becomes a Macdonald operator.

The Calogero–Sutherland system describes particles that sit on a circle, which interact with potential

$$U = c(\kappa) \sum \frac{1}{|x_i - x_j|^2}$$

for some constant $c(\kappa)$. One can look for eigenfunctions of the form

$$\prod_{i<j} (x_i - x_j)^{\text{const}} f(x_i) \tag{65}$$

where f is a symmetric polynomial. The operator H_{CS} acts on the symmetric polynomials. As the number of particles goes to infinity, the behavior of the system stabilizes and we can think about symmetric polynomials in infinitely many variables. Then there is the identification $\alpha_{-n} = \sum_{i=1}^{\infty} x_i^n$. The eigenfunctions (65) are Jack symmetric polynomials, a specific limit of Macdonald polynomials (from K-theory).

17.5.3

Property 3. (Lehn) $[c_1(V), \alpha_n] = \cdots$ is a Virasoro commutation relation.

Property 4.

$$\Delta c_1 = c_1 \otimes 1 + 1 \otimes c_1 - \hbar \sum_{n>0} n\alpha_n \otimes \alpha_{-n}.$$

Recall that quantum groups are in particular Hopf algebras. Our Hopf algebras have very few primitive elements, namely basically only the elements of the original Lie algebra, and in general the more complicated a Hopf algebra, the fewer primitive elements it has. So to understand an element such as c_1, it is productive to compute Δc_1. The commutator $[c_1, \alpha_n]$ being a Virasoro commutation relation means equivalently that $[[c_1(V), \alpha_n], \alpha_{-m}]$ is (some scalar multiple of) the primitive element c, and such a double commutator can be computed directly from a formula for Δc_1.

More generally, from property 4, the operator D of multiplication by a divisor therefore satisfies

$$\Delta D = D \otimes 1 + 1 \otimes D - \hbar \sum_{\substack{\beta>0 \\ \text{effective curves}}} (D, \beta) e_\beta \otimes e_{-\beta}$$

where $e_\beta \otimes e_{-\beta}$ is the canonical tensor in $\mathfrak{g}_\beta \otimes \mathfrak{g}_{-\beta}$.

17.5.4

Property 5. Under the identification $\alpha_{-n} \mapsto \sum x_i^n$, fixed points are Jack symmetric polynomials (determined by their triangularity and orthogonality properties).

17.6

Let's spell out in more detail how to recover the full R-matrix $R(u)$ from $[u^{-2}]R(u)$, using Yang–Baxter. The action of $c_1(V)$ on the cohomology of rank-2 instantons pulls back, via $\mathrm{Stab}_{\gtrless}$, to

$$(R_{01}R_{02})^{|\emptyset\rangle,-,-}_{|\emptyset\rangle,-,-} \quad \text{or} \quad (R_{02}R_{01})^{|\emptyset\rangle,-,-}_{|\emptyset\rangle,-,-} \tag{66}$$

depending on which chamber $\gtrless$ is taken, and the Yang–Baxter equation tells us these two operators are intertwined by the full R-matrix R_{12}. But (66) are two very explicit operators in $\mathsf{Fock} \otimes \mathsf{Fock} \cong \mathsf{Fock}^+ \otimes \mathsf{Fock}^-$, with terms

$$\cdots + \sum \alpha_n^+ \otimes L^-_{-n,\gtrless}.$$

This is not unexpected, because the Sugawara-type Virasoro for $\widehat{\mathfrak{gl}}_1$ is given by

$$\sum L_n^0 t^{-n} = :\alpha(t):^2$$

and $[u^{-2}]R(u)$ involves cubic terms in $\alpha(t)$, which in the coproduct should split into the tensor product of a degree-1 and a degree-2 term. Note, however, that our Virasoro operators coming from vacuum matrix elements has a slight correction, taking the form

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{n-k} + n\kappa\alpha_n \\ L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{k>0}\alpha_k\alpha_{-k} - \frac{\kappa^2}{2} \\ \Delta &= \frac{1}{2}\alpha_0^2 - \frac{1}{2}\kappa^2, \quad c = 1 - 12\kappa^2. \end{aligned} \tag{67}$$

The sign of κ creates a choice for

$$L_{n,\gtrless} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{n-k} \pm n\kappa\alpha_n$$

which is the dependence on the chamber for the stable envelope. The full R-matrix therefore intertwines these two, and changes $+$ to $-$. In $\mathsf{Fock}(u_1) \otimes \mathsf{Fock}(u_2) \cong \mathsf{Fock}^+ \otimes \mathsf{Fock}^-$, the module Fock^- is the Verma module for the Virasoro with lowest weight Δ and central charge c as given in (67). The eigenvalue of α_0^- in Fock^- is $u \coloneqq u_1 - u_2$. Both Δ and c are invariant under $\kappa \leftrightarrow -\kappa$, or, essentially equivalently, $u_1 \leftrightarrow u_2$. The conclusion is that the R-matrix does nothing on the Fock^+ factor, and does a *reflection* on Fock^-. This terminology comes from Liouville CFT, where the u_i are literally the momenta of two bosons, and the exchange $u_1 \leftrightarrow u_2$ is literally a reflection. The Fock^+ factor corresponds to the center of mass of the two bosons, and Fock^- is the relative position.

Remark. On one hand, we can compute $\det R(u)$ explicitly, since $R = \mathrm{Stab}_<^{-1} \circ \mathrm{Stab}_>$ and both pieces are triangular. On the other hand, $\det R(u) = 0$ whenever the two representations are not isomorphic, and unitarity implies $\det R(u)$ is therefore the isomorphism between a representation and its dual, also known as the Shapovalov form. Hence

$$\det R(u) = \det(\text{Shapovalov}) = (\text{Kac formula})$$

recovering a result of Feigin and Fuchs.

17.7

We defined quantum groups via matrix elements of R-matrices. When the R-matrix is deformed in a flat manner, e.g. from rational to trigonometric to elliptic, the generators of quantum groups obviously deform as well. It remains to ask whether relations deform. Many, many lectures ago, we discussed that relations come from operators Q that commute with R-matrices. There is a geometric source of such operators, as follows. Recall that we work with spaces $X \to X_0$ such that the action of $\mathbb{C}_\hbar^\times$ on X_0 contracts it to a single fixed point $0 \in X_0$. If we specialize other equivariant variables to depend on $\hbar$, it is possible to create other fixed points $x_0 \in X_0$, and we can look at their pre-images in X. These are called "slices" of X.

Example. Consider $X = T^*\mathbb{P}^1$ with weight u on the $\mathbb{P}^1$. Then X_0 is a cone with weights $-\hbar \pm u$, and so something special happens when $u = \pm\hbar$. In terms of representations, $H^*(T^* \operatorname{Gr}(2)) = \mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$ with $u = u_1 - u_2$, and it becomes reducible only at $u = \pm\hbar$.

17.8

Stable envelopes preserve the map $X \to X_0$, so upon specialization we get an action of the quantum group on the slice. However, we can also treat x_0 as the neighborhood of $0 \in X_0'$ for some other quiver variety $X' \to X_0'$. The inclusion of the slice into X therefore induces an intertwiner for the quantum group.

Concretely, take $\mathsf{Fock}(u_1) \otimes \mathsf{Fock}(u_2)$. It is irreducible in general, but becomes reducible if $u_2 = u_1 - nt_1 - t_2$, and there is a slice corresponding to

$$\mathsf{Fock}(u_1) \otimes \mathsf{Fock}(u_2) \to \mathsf{Fock}(u_1 - t_2) \otimes \mathsf{Fock}(u_1 - nt_1).$$

This is a screening operator for the Virasoro algebra. Commutation relations with screening operators in fact give a complete set of relations for $Y(\widehat{\mathfrak{gl}}_1)$. The book proof of this comes from looking at $\hbar = 0$, where $Y(\widehat{\mathfrak{gl}}_1)$ becomes $U(\mathfrak{gl}_\infty) \subset \operatorname{End}(\mathsf{Fock})$. The relations cutting out $U(\mathfrak{gl}_\infty)$ are Plücker relations, analogous to relations cutting out $\operatorname{End}(M) \subset \operatorname{End}(\wedge^\bullet M)$, and one can check that our commutation relations become exactly the Plücker relations at $\hbar = 0$.

17.9

Constructed geometrically in this way, these relations in quantum groups therefore persist in K-theory and elliptic cohomology. In general, it is a conjecture that this construction actually does produce a complete set of relations.

Lecture 18. Slices, intertwiners, and relations in quantum groups

Slices in quiver varieties and more general moduli problems, slices for Grassmannians and instanton moduli spaces, slices as quantum group intertwiners, screening operators and Plücker relations, quantum integrable systems from R-matrices, their relation to quantum multiplication for Nakajima varieties.

18.1

Let Q be a quiver and consider the quiver data which goes into the definition of a Nakajima quiver variety. If this data seems a bit arbitrary, equivalently one can think about representations of the *path algebra* $\mathscr{A}_Q$, whose generators are arrows of Q and relations are the moment map equations.

Remark. A nice trick to simplify the situation a little is to observe that all the framing vertices and framing maps can be combined into a single framing vertex of dimension $v_0 = 1$. Instead of having a framing vertex of dimension w_i and one arrow $w_i \to v_i$, we can instead draw w_i arrows from the new, single framing vertex.

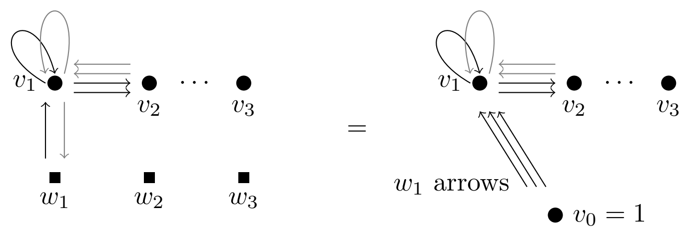


Figure 34: Replacing framing vertices in a quiver with a single additional vertex

Let R be a representation of an algebra A. Then $\mathrm{End}(R) = \mathrm{Hom}_A(R, R)$ contains automorphisms of R, first-order deformations are given by $\mathrm{Ext}^1_A(R, R)$, and obstructions to first-order deformations (i.e. whether a first-order deformation can be continued to a second-order deformation) are given by $\mathrm{Ext}^2_A(R, R)$. In principle this sequence of Exts continues, but this will not be the case for us due to the kinds of representations we consider. Let δ_i be the irreducible representation with $\mathbb{C}$ at the i-th vertex and all maps zero, so that $R = \sum v_i \delta_i$.

- $\mathrm{Hom}(R, R) = \bigoplus \mathfrak{gl}(v_i) \supset \prod \mathrm{GL}(v_i)$.
- Viewing $\mathrm{Ext}^1(A, B)$ as the space of extensions $0 \to B \to ? \to A \to 0$,

$$\dim \mathrm{Ext}^1(\delta_i, \delta_j) = (\#\text{ of arrows } i \to j).$$

  Observe that there is a duality $\mathrm{Ext}^1(\delta_j, \delta_i) = \hbar^{-1} \otimes \mathrm{Ext}^1(\delta_i, \delta_j)^\vee$, where the $\hbar^{-1}$ factor comes from the weight of the dual arrow. For our algebra, this is a special case of Serre duality.

- The obstruction map turns out to be the moment map. If we believe Serre duality,

$$\mathrm{Ext}^2(R, R) = \hbar^{-1} \otimes \mathrm{Hom}(R, R)^\vee,$$

  which is exactly where the moment map lives.

18.2

More or less, a Nakajima quiver variety X is some open subset

$$X \subset \left[\begin{array}{c} \text{quiver data +} \\ \text{moment map eq.} \end{array} \Big/ \, \mathrm{GL}(V) \right]$$

in the stack of representations of the path algebra $\mathscr{A}_Q$. Since X is a GIT quotient, it has an associated ordinary affine quotient $X_0 = \mathrm{Spec}(\text{invariants})$ and there is a blow-down map $X \to X_0$. For the action of a reductive group on an affine variety, invariant functions separate closed points. So in every fiber of $X \to X_0$ there is a unique closed $\mathrm{GL}(V)$-orbit, which is the representative of the point in X_0. In terms of representations,

$$X_0 \subset \{\text{semisimple representations}\}.$$

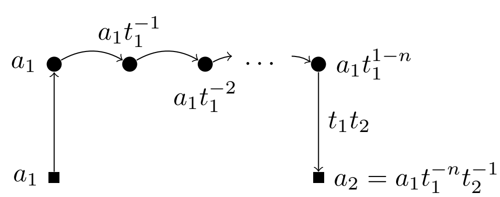
Every representation R admits a filtration $R_\bullet$ whose graded pieces R_i/R_{i-1} are semisimples. Although the filtration is not unique, the sum $\bigoplus_i R_i/R_{i-1}$ is, and the map $X \to X_0$ sends R to this sum. Another way of saying this is to map R to its K-theory class.

This puts us in a rather general framework, where if we have an algebra $\mathscr{A}$ of a certain kind, e.g. satisfying Serre duality, then its moduli space of representations will look like a quiver variety in the neighborhood of *any* point. For example this is true for moduli spaces of sheaves on K3 surfaces. So quiver varieties are not as esoteric as they may seem.

18.3

Recall that X_0 has a weight $\hbar^{-1}$ which attracts everything to the point $0 \in X_0$ where all invariants are zero. It corresponds to the representation $\bigoplus_i v_i \delta_i$. We saw last time that at certain specializations of equivariant variables, it is possible to have a fixed point R other than 0. Then in a neighborhood of R, the moduli space still looks like a quiver variety.

Example. Let Q be the quiver with one loop, of weight t_1. Take a two-dimensional framing vertex, with framing weights a_1, a_2. Then there are invariant representations of the form



as long as $a_2 = a_1 t_1^{-n} t_2^{-1}$. Note that these representations are reducible: an irreducible factor is given by the image of (the vector of weight) a_1.

18.4

Let $R = \bigoplus_i R_i^{\oplus v'_i}$ be the decomposition into irreducibles. Then, by the preceding discussion, the neighborhood of R (and its pre-image in X) will look like the quiver variety for the quiver whose vertices are the R_i, with dimensions v'_i, and arrows are $\operatorname{Ext}^1(R_i, R_j)$. This Ext we can compute in K-theory from the Euler characteristic

$$\chi(R_i, R_j) = \sum (-1)^i \operatorname{Ext}^i(R_i, R_j),$$

because we already know Ext^0 and Ext^2.

Proposition. *Suppose $R = R_0 \oplus \sum_{i \neq 0} v'_i \delta_i$ and R_0 has (equivariant) dimension $(1, \beta)$ where the first factor is the framing dimension w_0. Then the new quiver is the same, up to loops at the 0-th vertex. Let*

$$C := (1 + \hbar^{-1}) - (\textit{adjacency matrix of } Q)$$

be the equivariant Cartan matrix, which records $\chi(\delta_i, \delta_j)$. For example, for the A_2 and $\widehat{A}_0$ quivers,

$$C_{A_2} = \begin{pmatrix} 1 + \hbar^{-1} & -1 \\ -\hbar^{-1} & 1 + \hbar^{-1} \end{pmatrix}, \quad C_{\widehat{A}_0} = (1 - t_1)(1 - t_2).$$

Then the new dimension vector is v', and the new framing dimension vector is $w' = w - \hbar C \beta$.

The degree of freedom given by a loop at the 0-th vertex is a vector space, and corresponds to the freedom to move the point R in some stratum without changing what its neighborhood looks like.

Example. In the setting of Example 18.3,

$$\beta = a_1 \left(1 + t_1^{-1} + \cdots + t_1^{1-n}\right).$$

There is some cancellation and

$$w' = a_1 \left(t_2^{-1} + t_1^{-n}\right)$$

is the new framing dimension. Hence restriction to a neighborhood of a slice gives a map

$$\mathsf{Fock}(a) \otimes \mathsf{Fock}(a t_1^{-n} t_2^{-1}) \to \mathsf{Fock}(a t_2^{-1}) \otimes \mathsf{Fock}(a t_1^{-n}). \tag{68}$$

The ordering of tensor factors is important, since it determines whether we use coproduct or the opposite coproduct, and we will discuss it later.

18.5

Why do maps like (68) intertwine the quantum group actions? Recall that stable envelopes are correspondences over X_0, and a typical correspondence looks like

$$X(\beta, w_0) \times X(v, w) \xrightarrow{\text{Stab}} X(\beta + v, w_0 + w) \xrightarrow{\text{res}} X(0, w_0) \times X(\beta + v, w).$$

Suppose the component in $X(\beta, w_0)$ contracts to $0 \in X_0$, and the component in $X(v, w)$ contracts to $R \oplus 0 \in X_0$. Then on X_0 we get

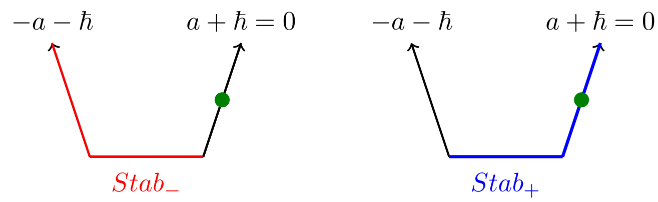
$$\{0\} \times \{R \oplus 0\} \mapsto \{R \oplus 0\},$$

and note that the dimensions of 0 add correctly. It follows that restriction to the slice at $R \oplus 0$ is a map of quantum group modules.

Example. Consider the Yangian $Y(\mathfrak{gl}_2)$ acting on $\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2)$. Geometrically,

$$\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) = H^*(T^* \operatorname{Gr}(2)^A) \xrightarrow{\text{Stab}} H^*(T^* \operatorname{Gr}(2))$$

is a map of Yangian modules. In general it is an isomorphism. Now suppose there is resonance, e.g. $a_2 - a_1 = \hbar$. Then we can take a new fixed point in $T^*\mathbb{P}^1 \subset T^* \operatorname{Gr}(2)$, and consider its interaction with some components of $\text{Stab}_\pm$:



.

- Consider the maps

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \underset{\text{Stab}_+^T}{\overset{\text{Stab}_-}{\rightleftarrows}} H^*(T^* \operatorname{Gr}(2)).$$

  As drawn, Stab_- misses the slice, so it is not surjective. On the other hand, Stab_+^T involves integration but over a component with zero weight, and therefore blows up.

- Consider the maps

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \underset{\text{Stab}_+}{\overset{\text{Stab}_-^T}{\rightleftarrows}} H^*(T^* \operatorname{Gr}(2)).$$

  Both maps are perfectly well-behaved isomorphisms.

The conclusion is that the module $H^*(T^* \operatorname{Gr}(2))$ always breaks up as

$$0 \to (3 \text{ dim}) \to H^*(T^* \operatorname{Gr}(2)) \xrightarrow{\text{slice}} (1 \text{ dim}) \to 0,$$

whereas the module $\mathbb{C}^2 \otimes \mathbb{C}^2$ has both decompositions

$$0 \to (3 \text{ dim}) \to \mathbb{C}^2 \otimes \mathbb{C}^2 \to (1 \text{ dim}) \to 0$$
$$0 \to (1 \text{ dim}) \to \mathbb{C}^2 \otimes \mathbb{C}^2 \to (3 \text{ dim}) \to 0$$

depending on the choice of sign. Put differently, these two $\mathbb{C}^2 \otimes \mathbb{C}^2$ are dual to each other; the former is the cohomology $H^*(T^* \operatorname{Gr}(2))$ while the latter is the homology $H_*(T^* \operatorname{Gr}(2))$. Generically $H^*(T^* \operatorname{Gr}(2))$ is irreducible self-dual, but at a certain point there is a difference between homology and cohomology.

In general, the ordering of tensor factors in things like (68) come from exactly this kind of situation.

18.6

Conjecture ([MO19]). *All relations in our quantum groups come from slices.*

This is true for the quiver with one loop. By deformation, it suffices to check for the Yangian: relations come from operators that commute with the R-matrix, so upon perturbing the R-matrix the number of generators can only increase while the number of relations can only decrease. In fact, for $Y(\mathfrak{gl}_1)$ it is enough to check at $\hbar = 0$, where we get all the Plücker relations. As discussed, the slices give Virasoro screening operators.

We know by different means that deformations are generically flat. There is also a result [NO] that if 0 is the only fixed representation, then $H^*(\bigsqcup_v X(v,w))$ are irreducible.

18.7

Last time we saw a quantum integrable system formed by Baxter operators $\mathrm{tr}_1(Z \otimes 1)R_{12}(a)$ for some fixed operator Z such that $[Z \otimes Z, R] = 0$. As we discussed, these operators commute for all a and fixed Z. An example of such Z is

$$Z = \prod z_i^{v_i}.$$

Since the R-matrix preserves dimensions, clearly $Z \otimes Z$ commutes with R. For $z = 0$ these are vacuum-vacuum matrix elements of R, which are operators of classical multiplication (which obviously commute). What are they for $z \neq 0$?

Conjecture (Nekrasov–Shatashvili [NS10]). *These are operators of quantum multiplication.*

Quantum multiplication is a new (assocative!) product which deformations the cup product, and encodes counts of rational curves in the variety X. In cohomology, let $(\alpha, \beta) \coloneqq \int_X \alpha \cup \beta$ be the natural pairing on $H^*(X)$. Then one defines this new (associative!) product by

$$(\alpha \star \beta, \gamma) \coloneqq \sum_C z^{[\deg C]}$$

where the sum is over rational curves $C \subset X$ with three marked points that intersect the prescribed *homology* classes $\alpha^\vee$, $\beta^\vee$, and $\gamma^\vee$. The degree of C is some element $[\deg C] \in H_2(X, \mathbb{Z})$, so this sum produces a formal series in the group algebra of the effective cone in $H_2(X, \mathbb{Z})$. The tip of the cone corresponds to $\deg C = 0$, i.e. C is just a point, and so

$$(\alpha \star \beta, \gamma) = (\alpha \cup \beta, \gamma) + O(z).$$

It is a remarkable and non-obvious fact that $\star$ is an associative product.

18.8

The Nekrasov–Shatashvili conjecture occurred before our geometric R-matrices and quantum groups, and their approach was to compute, in a completely mathematically-rigorous way, the *spectrum* of these operators of quantum multiplication in a "Bethe ansatz form". This refers to a specific way of tackling certain integrable systems, stemming from much previous work on diagonalizing Baxter's operators, in which there is a particular schematic form given for eigenvalues. Nekrasov and Shatashvili showed that the spectrum of operators of quantum multiplication matches this description, and therefore it was natural to conjecture there is an underlying quantum group and quantum integrable system.

Lecture 19. Integrable systems from enumerative geometry

Classical and quantum integrable systems in enumerative geomety, Plücker relations and Toda equations, Toda equations in the Gromov–Witten theory of $\mathbb{P}^1$, the corresponding quantum integrable system and free fermions, free fermions as the Yangian of $\widehat{\mathfrak{gl}}(1)$ at $\hbar = 0$, Donaldson–Thomas theory and its relation to GW theory, the full Yangian in the GW/DT theory of local curves in 3-folds.

19.1

Today we give a pictorial introduction to enumerative geometry and (one perspective on) its relationship with the deep and multi-dimensional subject of integrable systems. Namely, our goal in this course is to understand some aspects of the theory of membranes and M-theory, and the way that these subjects manifest themselves is via the connection between M-theory and Donaldson–Thomas (DT) theory. There is also a bridge to Gromov–Witten (GW) theory called the GW/DT correspondence [MNOP06], and in the world of GW counts there are a lot of approaches to obtaining various integrable systems.

19.2

Rather than starting from M-theory, we begin with the KdV equation. Generally, to study enumerative geometry means to study intersection theory on some moduli space, e.g. Schubert calculus on the Grassmannian. A well-studied moduli space is $\overline{\mathcal{M}}_{g,n}$, the Deligne–Mumford moduli space of (connected) stable n-pointed curves of genus g. These are curves C that may have singularities that are at worst nodal, with n distinguished points $p_1, \ldots, p_n$.

The cohomology ring $H^*(\overline{\mathcal{M}}_{g,n})$ is very interesting but very complicated. Mumford's conjecture [Mum83], later proved by Madsen and Weiss in [MW07], says that the only cohomology classes that are "stable" as $g \to \infty$, i.e. those that can be defined at "infinite" genus, are the characteristic classes of the line bundles $T^*_{p_i}C$ on $\overline{\mathcal{M}}_{g,n}$. These have fibers which are the cotangent spaces at the i-th marked point, as the curve C varies. (We prefer the cotangent line bundle over the tangent line bundle because it is more ample.) As a line bundle, its only non-trivial characteristic class is the first Chern class $c_1(T^*_{p_i}C)$, and we define

$$\langle \prod_i \tau_{k_i} \rangle \coloneqq \int \prod_i c_1(T^*_{p_i}C)^{k_i}.$$

There is no need to indicate the genus of C because it is found by a dimension constraint, namely this integral vanishes unless

$$3g - 3 + n = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = \sum k_i.$$

Though thinking about connected curves is more economical, the generating series which count disconnected curves often have better formal properties, and the two kinds of generating series are related by

$$(\text{disconnected curves}) = \exp(\text{connected curves})$$

where denominators like $n! = |S_n|$ come from a permutation symmetry of n connected components. One lesson that mathematicians have learned from theoretical physicists is that when enumerative data is packaged in an appropriate way, intricate structures appear. For example, a remarkable conjecture [Wit91] of Witten's is that

$$\left\langle \exp\left(\sum_{k=0}^{\infty} \tau_k t_k\right)\right\rangle = (\tau \text{ function of KdV}), \tag{69}$$

where the lhs is expanded as a series in the formal variables t_k. We will not spend too much time discussing this particular conjecture, now a theorem; many giants of mathematics, including Kontsevich, Mirzakhani, etc., have contributed to our understanding of it and nearby subjects.

19.3

Is there some direct geometric meaning of the KdV equation? In the absence of a direct connection between two objects which are known to be related, it is often productive to ask whether the relation is a degenerate case of something much more general, where the connection may be more direct. The perspective we will take is that the KdV equation arises as a specialization

$$(\text{2-Toda equation}) \rightsquigarrow (\text{KP equation}) \rightsquigarrow \text{KdV},$$

Many brilliant people have contributed to our understanding of the 2-Toda system, but in particular we will focus on the work of the Kyoto school (see e.g. [MJD00]), where the 2-Toda system arises as follows. Consider the group $\mathrm{GL}(\infty)$, which acts on the Fock module. There are two different flavors of Fock module, bosonic and fermionic, and the one we want sits in the semi-infinite wedge $\wedge^{\frac{\infty}{2}}\mathbb{C}^\infty$ of the defining representation of $\mathrm{GL}(\infty)$. To be more precise, this $\mathrm{GL}(\infty)$ action is by determinants, but in the infinite-dimensional limit some regularization is necessary in order to take determinants, creating an action by a central extension of $\mathrm{GL}(\infty)$ instead. After this central extension, there is an identification

$$U(\mathfrak{gl}(\infty)) \cong Y(\widehat{\mathfrak{gl}}(1))\big|_{\hbar=0}.$$

Under this identification, the Nakajima operators α_n we discussed become matrices of 1's along some sub-diagonal and zeroes everywhere else, called *infinite Töplitz matrices*. Let $\{e_n\}_{n\geq 0}$ be a basis for $\mathbb{C}^\infty$, and let

$$|n\rangle \coloneqq e_n \wedge e_{n-1} \wedge e_{n-2} \wedge \cdots \in \bigwedge^{\frac{\infty}{2}} \mathbb{C}^\infty.$$

Then the semi-infinite wedge can be presented as

$$\bigwedge^{\frac{\infty}{2}} \mathbb{C}^\infty \coloneqq \bigoplus_{n\in\mathbb{Z}} U(\mathfrak{gl}(\infty))\,|n\rangle\,.$$

In fact it suffices to act on $|n\rangle$ using only the raising operators α_{-k} for $k > 0$, i.e.

$$U(\mathfrak{gl}(\infty))\,|n\rangle \cong \mathbb{C}[\alpha_{-1}, \alpha_{-2}, \ldots]\,|n\rangle\,.$$

As discussed for the Yangian, the image of $\mathrm{GL}(\infty)$ is cut out by equations

$$[g \otimes g, (\text{screening op.})\big|_{\hbar=0}] = 0,$$

which are bilinear equations on matrix elements of g. More specifically, they are Plücker relations among the minors of g. At $\hbar = 0$, the screening operators are explicitly

$$(\text{screening op.})\big|_{\hbar=0} = \sum_i \psi_i \otimes \psi_i^*$$

where $\psi_i \coloneqq e_i \wedge -$ and its adjoint is $\psi_i^* \coloneqq \partial/\partial e_i$. In this setup, the 2-Toda equation is an equation on a sequence of functions

$$\tau_n(x, y) \coloneqq \left\langle n \,\middle|\, \exp\left(\sum_{k>0} \frac{\alpha_k}{k} x_k\right) g \exp\left(\sum_{k<0} \frac{\alpha_k}{k} y_k\right) \middle|\, n \right\rangle \tag{70}$$

for some arbitrary operator g. We will discuss why this is called a τ function later. Taking partials with respect to x and y, we get *all* possible matrix elements of the operator in (70), not just the $\langle n| \cdot |n\rangle$ matrix element. Plücker relations between all these matrix elements form the 2-Toda PDE. The operator g is like an initial condition.

Remark. A valid question to ask if why the 2-Toda PDE is interesting if there is already an explicit formula (70) for its solutions; when we talk about a *solvable* integrable system, there is a suggestion that it may not actually be solvable. But in reality the underlying theory often just directly yields formulas for the solution, and we should actually talk about a solved and integrated system.

19.4

This formulation of the 2-Toda system can be seen directly within the Gromov–Witten theory of $\mathbb{P}^1$. This is the study of the moduli space

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1) = \{f : C \to \mathbb{P}^1\}$$

for some *stable maps* f. Roughly one can imagine that C is smooth, of genus g with n marked points, but f is a rational map with zeros and poles at certain points on C. Note that it can very well be the case that f contracts entire components of C. On $\mathbb{P}^1$ there is the standard $\mathbb{C}^\times$ action of scaling by q, but we will work in cohomology with the variable $\epsilon \coloneqq \log q \in \mathsf{Lie}(\mathbb{C}^\times)$. In addition to the old discrete invariants g and n, there is a new invariant $\deg f$. The genus g is, again, constrained by a virtual dimension

$$\operatorname{vir}\dim \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1) = 2g + 2\deg - 2 + n.$$

Note that $2g + 2\deg - 2$ is the number of branch points of a generic f.

Let u be a formal variable keeping track of the genus g, and similarly let Q keep track of $\deg f$. Work in [OP06] shows that

$$\left\langle \prod \tau_{k_i}(0) \prod \tau_{\ell_i}(\infty) Q^{\deg f} u^{2g-2} \right\rangle_{\mathbb{P}^1} = \left\langle 0 \,\middle|\, \prod \tau_{k_i}(0) e^{\alpha_1} \left(\frac{Q}{u^2}\right)^{L_0} e^{\alpha_{-1}} \prod \tau_{\ell_i}(0)^* \,\middle|\, 0 \right\rangle \tag{71}$$

where now these $\tau_{k_i}(0) \in \mathfrak{gl}(\infty)$ are explicit, commuting operators. They depend on u, and are of the form

$$\tau_k(0) = W\left(\frac{u^k}{(k+1)!}\alpha_{k+1} + (\text{explicit linear function})(\alpha_k, \ldots, \alpha_1)\right) W^{-1}$$

where the operator W is also explicit, and in fact triangular. The explicit linear function here is responsible for a linear shift of the variables x_k, y_k in the solution (70) to the 2-Toda equation.

19.5

Take the point of view of the target space $\mathbb{P}^1$, which consists of two "halves" around 0 and ∞ respectively glued together along a circle S^1. The two halves of the formula (71) come from maps to each of these halves, particularly if one computes via localization. The Fock space is the space of states over this S^1.

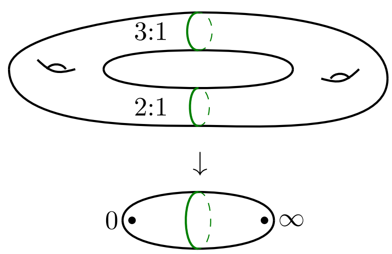


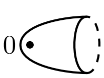
Figure 35: A map $f\colon C \to \mathbb{P}^1$ corresponding to a partition $(3, 2)$

Given a map $f\colon C \to \mathbb{P}^1$, the pre-image $f^{-1}(S^1)$ may consist of many circles, and the topological data for the k-th S^1 upstairs is the (winding) number n_k of times it covers the downstairs S^1. These numbers $\{n_k\}$ then form a partition of size $\deg f$, since $\deg f$ is exactly the total winding number.

Example. The map f which is d copies of the identity map $\mathbb{P}^1 \to \mathbb{P}^1$ is recorded by a term $Q^d u^{-2d}$, which is exactly the $(Q/u^2)^{L_0}$ operator in (71).

So we can really interpret

$$\langle 0| \prod \tau_{k_i}(0) e^{\alpha_1} = \; 0 \tag{72}$$



as vectors. The rhs consists of some Hodge integrals on the GW side. More generally, one can put different insertions of $\tau_k(0)$, and view this "cap" as an operator

$$(\text{polynomials in } \tau_k(0)) \to \mathsf{Fock}.$$

19.6

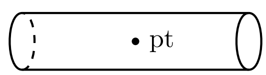
The non-equivariant GW theory of $\mathbb{P}^1$ is a certain limit of the equivariant theory, and is described by:

- the usual Toda system for $\tau_k(\mathrm{pt})$ since $0, \infty \in \mathbb{P}^1$ are the same point in non-equivariant cohomology;
- the Dubrovin–Zhang system for $\tau_k(1)$, where 1 is the non-equivariant limit of $([0] - [\infty])/\epsilon$.

The former is much more basic:

$$\left\langle \prod \tau_{k_i}(\mathrm{pt}) Q^{\deg} u^{\chi} \right\rangle = \left\langle 0 \middle| e^{\alpha_1} (\text{diagonal}) \left(\frac{Q}{u^2}\right)^{L_0} e^{\alpha_{-1}} \middle| 0 \right\rangle. \tag{73}$$

Here (diagonal) means some diagonal matrices in $\mathfrak{gl}(\infty)$. They come from the "tube" operator



$\tau_k(\mathrm{pt})$ (74)

where the point is an insertion $\tau_k(\mathrm{pt})$. These operators commute with each other and form a quantum integrable system, but not a very complicated one because they are all diagonal matrices. It is a very degenerate case of the commutative algebra in the Yangian $Y(\widehat{\mathfrak{gl}}_1)$ of Baxter type.

We can expand (73) more explicitly. The creation operator e^{α_1} creates all possible partitions λ, each with weight $\dim \lambda / |\lambda|$, and the annihilation operator $e^{-\alpha_1}$ annihilates all of them with the same weight. The diagonal matrices are diagonal precisely in the basis of monomials, i.e. wedges of the e_i, or, in the language of the α's, the basis of Schur functions. Hence (73) becomes

$$\sum_\lambda \left(\frac{\dim \lambda}{|\lambda|}\right)^2 (\text{sym. poly.})(\lambda_i - i).$$

The weight $(\dim \lambda/|\lambda|)^2$ is some multinomial coefficient, so this is a finite analogue of integrals like

$$\int_{\mathbb{R}^n} \prod (x_i - x_j)^2 P(x) e^{-\sum x_i^2}.$$

19.7

At this point, it is natural to ask:

- is there a deeper reason we see partitions here?
- what about $\hbar \neq 0$, i.e. $Y(\widehat{\mathfrak{gl}}_1)$ in its full glory?

The answers are found in the GW/DT correspondence for local curves in 3-folds. Note that the case of a general curve reduces to the case of $\mathbb{P}^1$ by degeneration. A *local curve* in a 3-fold means to take a (target) curve B and promote it to a 3-fold, by taking a formal neighborhood of the curve B in an ambient 3-fold. Specifically, take two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over B; the resulting 3-fold is the total space

$$X \coloneqq \mathrm{tot}(\mathcal{L}_1 \oplus \mathcal{L}_2 \to B).$$

In principle we can take any rank-2 vector bundle over B, but GW theory and any enumerative theory is deformation-invariant, and by deformation we can break any rank-2 vector bundle into the sum of two line bundles. A nice feature of having two line bundles is we gain two more automorphisms, scaling fibers of $\mathcal{L}_1$ and $\mathcal{L}_2$ by t_1 and t_2 respectively.

In general, any kind of theory for a vector bundle recovers the theory for the original base in some kind of limit, but here the passage back is very easy: if $\mathcal{L}_2 = \mathcal{L}_1^\vee$, so in particular $t_2 = -t_1$, then there is a relation due to Mumford that says the GW theory of X reduces to the GW theory of B. The parameter $\hbar$ in the Yangian is the combination $\hbar = -t_1 - t_2$, as one may expect, and so we expect the full $Y(\widehat{\mathfrak{gl}}_1)$ for X, while for B we get the $\hbar = 0$ specialization. Operators like the "tube" earlier will form the Baxter sub-algebra. The parameter Z in the Baxter operator turns out to be e^{iu}. The "cap" (72) can also be understood in the same way.

19.8

Now we address why partitions show up, via the GW/DT equivalence. DT theory is an enumerative theory of sheaves on 3-folds. For smooth 3-folds, the category $\mathsf{Coh}(X)$ of coherent sheaves behaves nicely, but DT theory also encompasses the study of objects in related categories that behave very similarly. Note that if X is a fibration

$$\begin{array}{ccc} S & \longrightarrow & X \\ & & \downarrow \\ & & B \end{array}$$

by surfaces S over a base curve B, e.g. like for local curves, then a sheaf on X is like a map

$$B \to (\text{moduli of sheaves on } S). \tag{75}$$

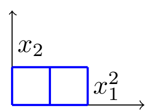
This is actually true if the sheaf on X in question is actually flat over B, but in general for DT theory we should consider the non-flat sheaves too. These will induce singularities in the map (75). Such singularities are analogous to ones appearing in GW theory when the rational map $f\colon C \to X$ must be resolved by degenerating the source curve C. In contrast, for DT theory, the curve B is part of the target space and does not degenerate, and instead we get singularities. This turns out to work better in many aspects than the compactification, via degeneration, specified by GW theory.

19.9

Example. Figure 36a shows an example of a sheaf on a specific local curve. As drawn, it has degree 2 over the base $\mathbb{P}^1$ and therefore corresponds to a (rational) map

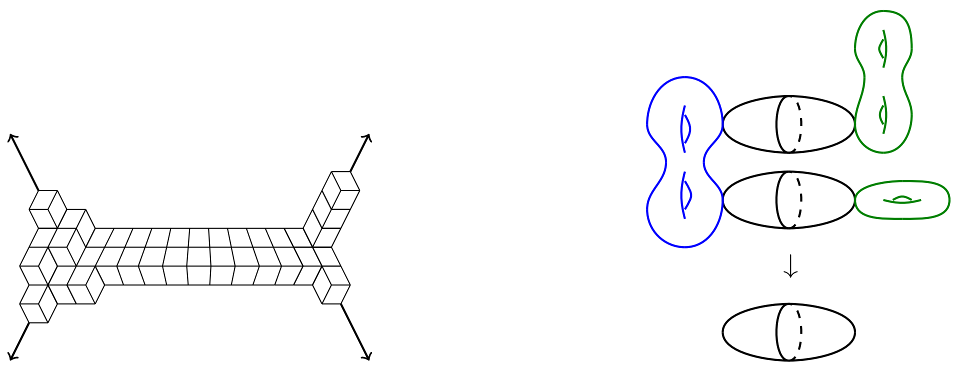
$$\mathbb{P}^1 \dashrightarrow \mathrm{Hilb}(\mathbb{C}^2, 2).$$

Away from $0, \infty \in \mathbb{P}^1$, this map is the constant map to the point



$\in \mathrm{Hilb}(\mathbb{C}^2, 2),$

i.e. the ideal generated by x_1^2 and x_2. The singularities are at $0, \infty \in \mathbb{P}^1$. In comparison, in GW theory, such singularities involve components of the source curve contracted by f, as in Figure 36b. Contributions of the contracted components are Hodge integrals, which become equivalent to "boxcounting" in DT theory (which is especially simple in PT theory).



(a) A monomial ideal sheaf on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ of degree 2 over $\mathbb{P}^1$

(b) A degree-2 map $C \to \mathbb{P}^1$ with contracted components over $0, \infty \in \mathbb{P}^1$

Figure 36: Elements in GW vs DT theory with singularities

Regardless of the (combinatorial) nature of singularities, both GW and DT theories can be solved explicitly in the language of quantum groups.

19.10

So far we have discussed Yangians; what about other quantum groups? For example, $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ has to do with the K-theory of DT moduli spaces, for X a local curve. While cohomology has to do with the space of states in X, K-theory has to do with the space of states in $X \times S^1$ for an extra circle which physically represents time. The base $B \times S^1$ becomes a real 3-fold, and is the world-volume of an M2 brane in M-theory. M-theory is an 11-dimensional theory with some extended degrees of freedom: M5-branes and M2-branes. We will not discuss M5-branes, but M2-branes take the form $B \times S^1$ for a Riemann surface B, and to compute in the

Hamiltonian formulation of the theory of M2-branes means to deal with the K-theory of some moduli of objects on B. Specifically, the theory with $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ symmetry should be the theory of a stack of M2-branes on $B \times S^1$ inside the 11-dimensional $Z \times S^1$, for a Calabi–Yau 5-fold X. The number of M2-branes in the stack is the analogue of $\deg f$, or the operator L_0. One of the many motivations for studying $2+1$-dimensional theories like we do in this course is to better understand M2-branes. In particular, the actual moduli space of M2-branes is still under construction [NO16].

Lecture 20. Virtual fundamental classes in enumerative geometry

Lecture by Chiu-Chu Melissa Liu.

20.1

Consider a moduli space $\mathfrak{X}$, over $\mathbb{C}$, on which we have enumerative problems of interest. In all the problems considered so far in this course, $\mathfrak{X}$ has been a Deligne–Mumford stack which is not smooth but instead *virtually* smooth. This means $\mathfrak{X}$ is equipped with a perfect obstruction theory E, which is a certain object in the derived category $D(\mathfrak{X})$ locally of the form $[E^{-1} \to E^0]$ for vector bundles E^{-1}, E^0. From the perfect obstruction theory we get a virtual tangent bundle

$$T^{\mathrm{vir}} := (E^0)^\vee - (E^1)^\vee,$$

a replacement for the tangent bundle T in the smooth case. Its virtual dimension is

$$\operatorname{vir}\dim T^{\mathrm{vir}} := \operatorname{rank} T^{\mathrm{vir}} = \operatorname{rank} E^0 - \operatorname{rank} E^{-1}.$$

20.2

Given a virtually smooth DM stack $(\mathfrak{X}, E)$, a construction [BF97] of Behrend and Fantechi yields a *virtual fundamental class*

$$[\mathfrak{X}, E]^{\mathrm{vir}} \in A_d(\mathfrak{X}; \mathbb{Q})$$

in the Chow group of $\mathfrak{X}$ (of the virtual dimension d). Note that, since $\mathfrak{X}$ is a stack, we are forced to take coefficients in $\mathbb{Q}$ as opposed to $\mathbb{Z}$. Here are some properties of $[\mathfrak{X}, E]^{\mathrm{vir}}$ which will be verified later.

- If $\mathfrak{X}$ is actually a scheme, then $[\mathfrak{X}, E]^{\mathrm{vir}} \in A_d(\mathfrak{X}; \mathbb{Z})$ is actually an integral class.
- If $\mathfrak{X}$ is smooth, then the (canonical choice of) perfect obstruction theory $E = [0 \to \Omega_{\mathfrak{X}}]$ has just one term, namely the cotangent bundle of $\mathfrak{X}$. Equivalently, $T^{\mathrm{vir}} = T_{\mathfrak{X}}$ is just the tangent bundle. The virtual dimension becomes $d = \dim \mathfrak{X}$, and $[\mathfrak{X}, E]^{\mathrm{vir}} = [\mathfrak{X}] \in A_d(\mathfrak{X})$.

In this course, we would also like to work in K-theory. Following [Lee04], $(\mathfrak{X}, E)$ also yields a *virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X},E} \in D(\mathfrak{X}).$$

As with $[\mathfrak{X}, E]^{\mathrm{vir}}$, if $\mathfrak{X}$ is smooth and $T^{\mathrm{vir}} = T_{\mathfrak{X}}$, then this is just the usual structure sheaf $\mathcal{O}_{\mathfrak{X}}$.

If $\mathfrak{X}$ is proper, enumerative invariants arise from the pushforward to a point of virtual classes like $[\mathfrak{X}, E]^{\mathrm{vir}}$ or $\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X},E}$, in cohomology or K-theory respectively. More generally we can take proper pushforwards to other targets, not just a point.

20.3

We now describe a perfect obstruction theory, and the Behrend–Fantechi construction [BF97] in more detail.

Definition. Let $D^{[-1,0]}(\mathfrak{X}) \subset D(\mathfrak{X})$ consist of only those complexes E with $h^i(E) = 0$ unless $i = 0, -1$. Then a *perfect obstruction theory* is an element $E \in D^{[-1,0]}(\mathfrak{X})$ with a morphism

$$\begin{array}{rcl} E & = & [\cdots \to E^{-1} \to E^0] \\ & & \quad\quad\downarrow \phi \\ L_{\mathfrak{X}} & = & [\cdots \to L^{-1}_{\mathfrak{X}} \to L^0_{\mathfrak{X}}] \end{array}$$

where $L_{\mathfrak{X}}$ is the cotangent complex of $\mathfrak{X}$, to be discussed later. We require:

- $h^0(\phi)$ is an isomorphism;
- $h^{-1}(\phi)$ is surjective.

20.4

Objects on stacks can be understood using an atlas of the stack. For ordinary schemes $\mathfrak{X}$, an atlas is a map $\bigsqcup_\alpha U_\alpha \to \mathfrak{X}$ where the charts $U_\alpha = \operatorname{Spec} R_\alpha$ are affine schemes, and the maps $U_\alpha \to \mathfrak{X}$ are open embeddings. For DM (resp. Artin) stacks, the maps

$$U_\alpha \to \mathfrak{X}$$

are now allowed to be étale (resp. smooth). Restriction of the object in question to each U_α chart allows us to understand it in terms of commutative algebra.

Take $\phi\colon E \to L_{\mathfrak{X}}$ and restrict it to an étale chart $U \to \mathfrak{X}$ to get

$$\phi_U \colon E_U \to L_U. \tag{76}$$

Let $i\colon U \to W$ be a closed embedding of the affine U, which may be singular, into a non-singular scheme W. Let I be the ideal sheaf of the embedding, and

$$C_{U/W} \coloneqq \operatorname{Spec}\left(\bigoplus_{n\geq 0} I^n/I^{n+1}\right)$$

$$N_{U/W} \coloneqq \operatorname{Spec}\left(\operatorname{Sym} I/I^2\right)$$

be the normal cone and normal sheaf respectively; see Fulton [Ful84]. In this situation, the (truncation of the) cotangent complex $\tau_{\geq -1} L_U$ is very explicit, and (76) becomes

$$\begin{array}{ccccc} E_U & = [E_U^{-1} & \longrightarrow & E_U^0] \\ & \downarrow & & \downarrow \cong \\ L_U & = [I/I^2 & \longrightarrow & L_{\mathfrak{X}}^0]. \end{array}$$

Taking Spec, let $(E_1)_U \coloneqq \operatorname{Spec}(\operatorname{Sym} E^{-1})$, which can be identified with the vector bundle $(E_U^{-1})^\vee$. The condition that $h^{-1}(\phi)$ is surjective means there are embeddings

$$C_{U/W} \subset N_{U/W} \subset (E_1)_U. \tag{77}$$

Note that $(E_1)_U$ is an actual vector bundle, while $N_{U/W}$ is an abelian cone and $C_{U/W}$ is just a cone. Moreover, there is a map

$$i^* T_W \cong (E_U^0)^\vee \to (E_1)_U.$$

Take the quotient of (77) by the image of this map to get

$$\mathfrak{C}_U \subset \mathfrak{N}_U \subset \mathfrak{E}_U \tag{78}$$

where $\mathfrak{C}_U \coloneqq [C_{U/W}/i^* T_W]$ and similarly for $\mathfrak{N}_U$ and $\mathfrak{E}_U$. These are now stacky vector bundles and (abelian) cones. In particular, the cone $\mathfrak{C}_U$ is actually independent of the choice of embedding $i\colon U \to W$, and has pure dimension 0 since $C_{U/W}$ has pure dimension $\dim W$.

20.5

All the objects in (78) glue appropriately across different affine charts, i.e. they descend to the stack $\mathfrak{X}$, and we denote the resulting objects by

$$\mathfrak{C}_{\mathfrak{X}} \subset \mathfrak{N}_{\mathfrak{X}} \subset \mathfrak{E}_{\mathfrak{X}}.$$

Definition. $\mathfrak{C}_{\mathfrak{X}}$ (resp. $\mathfrak{N}_{\mathfrak{X}}$) is the *intrinsic normal cone* (resp. *sheaf*).

Both $\mathfrak{C}_{\mathfrak{X}}$ and $\mathfrak{N}_{\mathfrak{X}}$ are determined by $\mathfrak{X}$ itself, but $\mathfrak{E}_{\mathfrak{X}}$ depends on the choice of perfect obstruction theory. It has rank

$$\operatorname{rank} \mathfrak{E}_{\mathfrak{X}} = \operatorname{rank} E^{-1} - \operatorname{rank} E^0 = -\operatorname{vir}\dim.$$

20.6

If E be a vector bundle of rank r over a scheme X, then in the classical setting of Fulton's intersection theory there is a Gysin map

$$0_E^!\colon A_d(E;\mathbb{Z}) \to A_{d-r}(X;\mathbb{Z}).$$

Now if we have a vector bundle stack $\mathfrak{E}$ of rank r over a DM stack $\mathfrak{X}$, then Kresch [Kre99] constructs an analogous Gysin map

$$0_{\mathfrak{E}}^!\colon A_d(\mathfrak{E};\mathbb{Q}) \to A_{d-r}(\mathfrak{X};\mathbb{Q}).$$

Definition. The *virtual fundamental class* of $(\mathfrak{X}, E)$ is

$$[\mathfrak{X}, E]^{\mathrm{vir}} \coloneqq 0^!_{\mathfrak{E}}[\mathfrak{C}_{\mathfrak{X}}] \in A_{\mathrm{vir}\dim}(\mathfrak{X}).$$

To understand this more concretely, we return to the simpler case over a scheme instead of a DM stack by restricting to affine charts in an atlas (namely, the same atlas considered earlier). Let $\mathfrak{E}_U \to U$ be the restriction of $\mathfrak{E}_{\mathfrak{X}} \to \mathfrak{X}$. While U is now a scheme, $\mathfrak{E}_U$ is still a vector bundle stack, and so to further simplify things we pass to the smooth cover

$$\pi\colon (E_1)_U \to \mathfrak{E}_U = [(E_1)_U/i^*T_W].$$

Clearly π is smooth of relative dimension $d_0 \coloneqq \dim W = \operatorname{rank} E^0$. Let $d_1 \coloneqq \operatorname{rank} E^{-1}$ and $d \coloneqq d_0 - d_1 = -\operatorname{rank} \mathfrak{E}_U$ be the virtual dimension. By construction,

$$\pi^*[\mathfrak{C}_U] = [C_{U/W}] \in A_{d_0}((E_1)_U)$$

and there is a commutative diagram

$$\begin{array}{ccc} A_{d_0}((E_1)_U) & \xrightarrow{0^!_{(E_1)_U}} & A_{d_0-d_1}(U) \\ {\scriptstyle \pi^*}\uparrow & & \| \\ A_0(\mathfrak{E}_U) & \xrightarrow{\quad 0^!_{\mathfrak{E}_U} \quad} & A_d(U). \end{array}$$

Hence

$$[U, E_U]^{\mathrm{vir}} = 0^!_{(E_1)_U}[C_{U/W}] = 0^!_{\mathfrak{E}_U}[\mathfrak{C}_U] \in A_d(U).$$

The intuition is that, inside the total space of $(E_1)_U$, we are intersecting the normal cone $C_{U/W}$ with the zero section U.

For the virtual structure sheaf, the idea is analogous:

$$\mathcal{O}^{\mathrm{vir}}_{U,E_U} \coloneqq \mathcal{O}_U \otimes^L_{\mathcal{O}_{(E_1)_U}} \mathcal{O}_{C_{U/W}} \in D(U)$$

where $\otimes^L$ is the *derived* tensor product. In terms of the stacky objects, this is equivalent to $\mathcal{O}_U \otimes^L_{\mathcal{O}_{\mathfrak{E}_U}} \mathcal{O}_{\mathfrak{C}_U}$. These descend to $\mathfrak{X}$ to give

$$\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X},E_{\mathfrak{X}}} = \mathcal{O}_{\mathfrak{X}} \otimes^L \mathcal{O}_{\mathfrak{C}_{\mathfrak{X}}} \in D(\mathfrak{X})$$

when $\mathfrak{X}$ is a virtually smooth DM stack.

20.7

Example. If U is a smooth scheme, for the closed embedding into something smooth one can take the identity map $i = \mathrm{id}\colon U \to U$. Then $C_{U/U} = U$ is the zero bundle and there is an isomorphism

$$\begin{array}{ccc} E_U & = & [0 \to \Omega^1_U] \\ \| & & \\ L_U & = & [0 \to \Omega^1_U]. \end{array}$$

The stacky cones are $\mathfrak{C}_U = \mathfrak{E}_U = [U/T_U]$, the Gysin map $0^!\colon A_d(U;\mathbb{Z}) \to A_d(U;\mathbb{Z})$ is the identity, and so

$$[U, E_U]^{\text{vir}} = 0^![C_{U/U}] = [U] \in A_d(U;\mathbb{Z}).$$

The same computation applies for a smooth DM stack $\mathfrak{X}$, but in $A_d(\mathfrak{X};\mathbb{Q})$, and similarly for virtual structure sheaves.

20.8

The next step, important for this course, is to make everything equivariant. Let $\mathfrak{X}$ be a DM stack with the action of an algebraic group G. It is good to be careful about what a group action on a stack means; we mean in the sense of [Rom05]. Then $L_{\mathfrak{X}} \in D_G(\mathfrak{X})$ is a complex of G-equivariant sheaves. The obstruction bundle

$$E \in D_G^{[-1,0]}(\mathfrak{X})$$

must also be G-equivariant. The result is equivariant classes

$$[\mathfrak{X}, E]^{\text{vir}} \in A_d^G(X), \quad \mathcal{O}_{\mathfrak{X},E}^{\text{vir}} \in D_G(\mathfrak{X}).$$

20.9

When $G = (\mathbb{C}^\times)^k$ is a torus, there is the torus localization of virtual fundamental classes [GP99, Beh02] as follows. Let $\iota\colon \mathfrak{X}^G \hookrightarrow \mathfrak{X}$ be the inclusion of the fixed locus. In K-theory, there is a splitting

$$E|_{\mathfrak{X}^G} = E^f \oplus E^m$$

into G-fixed and G-moving parts. The G-fixed part is used to define the virtual fundamental class on $\mathfrak{X}^G$. So, similarly,

$$T_{\mathfrak{X}}^{\text{vir}}|_{\mathfrak{X}^G} = T_{\mathfrak{X}^G}^{\text{vir}} \oplus N^{\text{vir}}$$

where the moving part N^{vir} is called the *virtual normal bundle*. Then, with this notation, the virtual localization formula is

$$[\mathfrak{X}, E]^{\text{vir}} = \iota_* \left(\frac{[\mathfrak{X}^G, E^f]^{\text{vir}}}{e_T(N^{\text{vir}})} \right) \in A_*^G(\mathfrak{X}) \otimes_{R_G} S_G. \tag{79}$$

Here $R_G \coloneqq A_G^*(\text{pt}) = \mathbb{Q}[t_1, \ldots, t_k]$ is the G-equivariant Chow ring of a point, generated by universal first Chern classes, and $S_G = \mathbb{Q}(t_1, \ldots, t_k)$ is its fraction field.

Remark. The original proof of the virtual localization formula (79) required a G-equivariant embedding of $\mathfrak{X}$ into something smooth, but this assumption has since been removed by [CKL17].

20.10

Finally, we need a relative version of the virtual classes. Let $\mathfrak{X}$ be a DM stack, and let $\Pi_{\mathfrak{X}/\mathfrak{M}}\colon \mathfrak{X} \to \mathfrak{M}$ be a projection to a smooth Artin stack of dimension m. Suppose $\Pi_{\mathfrak{X}/\mathfrak{M}}$ is virtually smooth of relative dimension d, in the sense that there is a relative perfect obstruction theory, i.e. an object

$$E_{\mathfrak{X}/\mathfrak{M}} \in D^{[-1,0]}(\mathfrak{X})$$

with morphism $\phi\colon E_{\mathfrak{X}/\mathfrak{M}} \to L_{\mathfrak{X}/\mathfrak{M}}$ to the relative cotangent complex such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is a surjection. Again, we consider étale charts U and closed embeddings into schemes W

$$\begin{array}{ccc} U & \xrightarrow{\text{étale}} & \mathfrak{X} \\ & \searrow \quad \swarrow & \\ & \mathfrak{M} & \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{i} & W \\ & \searrow \quad \swarrow & \\ & \mathfrak{M} & \end{array}$$

except now everything is over $\mathfrak{M}$, and $W \to \mathfrak{M}$ is smooth of relative dimension d_0. The relative intrinsic normal cone is

$$\mathfrak{C}_{U/\mathfrak{M}} \coloneqq [C_{U/W}/i^*T_{W/\mathfrak{M}}]$$

and has pure dimension $\dim \mathfrak{M}$. Similarly there is the relative intrinsic normal sheaf $\mathfrak{N}_{U/\mathfrak{M}}$ and there are embeddings like in (78). The relative virtual fundamental class, on U, is then

$$[U, E_{U/\mathfrak{M}}]^{\mathrm{vir}} \coloneqq 0^!_{\mathfrak{E}_{U/\mathfrak{M}}}[\mathfrak{C}_{U/\mathfrak{M}}] \in A_{\dim \mathfrak{M}+d}(U).$$

Note that $\mathfrak{E}_{U/\mathfrak{M}}$ is a vector bundle of rank $-d$, and $[\mathfrak{C}_{U/\mathfrak{M}}] \in A_{\dim \mathfrak{M}}(\mathfrak{E}_{U/\mathfrak{M}}; \mathbb{Q})$. Similarly, we also get the virtual structure sheaf.

20.11

Example (Gromov–Witten theory)**.** Let X be a non-singular projective variety. In physics, we consider two-dimensional sigma models to X. This means to consider maps of curves into the target X. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class, and let

$$\widetilde{\mathfrak{X}} \coloneqq \mathfrak{M}^{\mathrm{pre}}_{g,n}(X, \beta)$$

be the moduli of genus-g, n-pointed prestable maps of degree β to X. Denote points in $\widetilde{\mathfrak{X}}$ by

$$f\colon (C, x_1, \ldots, x_n) \to X.$$

Two such prestable maps are isomorphic if there is an isomorphism $\phi\colon C \to C'$ of source curves such that

$$\begin{array}{ccc} (C, x_1, \ldots, x_n) & & \\ \downarrow & \overset{f}{\searrow} & \\ & & X \\ & \underset{g}{\nearrow} & \\ (C', x_1', \ldots, x_n') & & \end{array}$$

commutes. The actual moduli space of interest

$$\mathfrak{X} = \overline{\mathfrak{M}}_{g,n}(X, \beta) \subset \widetilde{\mathfrak{X}}$$

is the moduli of *stable* maps to X. While $\widetilde{\mathfrak{X}}$ is usually a singular Artin stack, the stability condition $|\operatorname{Aut}| < \infty$ implies that $\mathfrak{X}$ is a proper DM stack. But actually $\widetilde{\mathfrak{X}}$ already has a relative perfect obstruction theory; once we describe what it is, the Behrend–Fantechi construction will produce the virtual fundamental class.

Let $\mathfrak{M} \coloneqq \mathfrak{M}_{g,n}$ be the moduli of genus-g n-pointed prestable curves. It is a smooth Artin stack of dimension $3g - 3 + n$. The forgetful map

$$\Pi_{\widetilde{\mathfrak{X}}/\mathfrak{M}} \colon \widetilde{\mathfrak{X}} \to \mathfrak{M}$$

is virtually smooth, by the following relative perfect obstruction theory. Let $\pi_{\mathfrak{M}} \colon C_{\mathfrak{M}} \to \mathfrak{M}$ be the universal curve, and form the Cartesian diagram

$$\begin{array}{ccc} C_{\widetilde{\mathfrak{X}}} & \longrightarrow & C_{\mathfrak{M}} \\ \big\downarrow{\scriptstyle \pi_{\widetilde{\mathfrak{X}}}} & & \big\downarrow{\scriptstyle \pi_{\mathfrak{M}}} \\ \widetilde{\mathfrak{X}} & \longrightarrow & \mathfrak{M} \end{array}$$

so that $C_{\widetilde{\mathfrak{X}}}$ is the universal curve over $\widetilde{\mathfrak{X}}$. One can identify

$$\widetilde{\mathfrak{X}} = \Gamma(\mathfrak{M}, C_{\mathfrak{M}} \times X / C_{\mathfrak{M}})$$

over $\mathfrak{M}$, because once a specific curve $\xi \coloneqq (C, x_1, \ldots, x_n)$ has been fixed, the remaining data is just that of the map $C \to X$, which corresponds to its graph in $C \times X$. There is a universal evaluation map

$$\operatorname{ev}_{\widetilde{\mathfrak{X}}} \colon C_{\widetilde{\mathfrak{X}}} \to \mathfrak{X}$$

which on fibers is just the map $f \colon C \to X$. With this setup, the relative perfect obstruction theory is

$$E_{\widetilde{\mathfrak{X}}/\mathfrak{M}} \coloneqq \left(R^\bullet \pi_{\widetilde{\mathfrak{X}},*} \operatorname{ev}_{\widetilde{\mathfrak{X}}}^* T_X\right)^\vee.$$

Concretely, the cohomologies of $E_{\widetilde{\mathfrak{X}}/\mathfrak{M}}$, over the point ξ, are

$$\begin{aligned} h^0(E^\vee_{\widetilde{\mathfrak{X}}/\mathfrak{M}})\big|_\xi &= H^0(C, f^*T_X) \\ h^1(E^\vee_{\widetilde{\mathfrak{X}}/\mathfrak{M}})\big|_\xi &= H^1(C, f^*T_X), \end{aligned}$$

measuring infinitesimal deformations (and obstructions thereof) of a map from a fixed domain curve C to X. Finally, $\mathfrak{M}$ is a smooth Artin stack whose tangent space is a two-term complex

$$T_{\mathfrak{M}}\big|_\xi = \left[\operatorname{Ext}^0_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C)\right]$$

where $D \coloneqq x_1 + \cdots + x_n$ is the divisor of the marked points. The first term sits in degree -1, meaning that there are no obstructions to deforming the domain C, but there exist infinitesimal automorphisms.

One can compute the relative virtual dimension

$$\operatorname{vir}\dim \widetilde{\mathfrak{X}}/\mathfrak{M} = \chi(C, f^*T_X) = \int_\beta c_1(T_X) + (\dim X)(1-g)$$

by Riemann–Roch, and add it to the dimension of $\mathfrak{M}$ to get

$$\operatorname{vir}\dim \overline{\mathfrak{M}}_{g,n}(X,\beta) = \int_\beta c_1(T_X) + (\dim X - 3)(1-g) + n.$$

Lecture 21. Self-duality and quasimaps

Exact and approximate self-duality of obstruction theories, symmetrized virtual structure sheaf, the quest for proper moduli spaces with self-dual obstruction theory, quasimaps to GIT quotients, quasimaps to projective spaces and to the Hilbert schemes of points.

21.1

Recall that our original story began with a moduli space X of vacua in a supersymmetric $(2+1)$-dimensional theory. At large scales, i.e. in the infrared, this theory behaves like a theory of modulated vacua. In other words, we consider a theory of maps $f\colon C \to X$ from a Riemann surface C. Since there is an extended supersymmetry, only supersymmetric configurations f will contribute. We can arrange it so that these are precisely the f which are holomorphic, for a fixed choice of complex structure on C and for the one already present on X (which is actually almost hyperkähler). The moduli of supersymmetric states is therefore, approximately, a moduli of such holomorphic maps f.

21.2

Recall from the very first lecture that in order to take indices in this setting, we require an *index sheaf* on this moduli of maps. It should behave like the index sheaf for the Dirac operator $\not{D}$. For example, for a Kähler manifold M, the Dirac operator is

$$\not{D} = \overline{\partial} \curvearrowright K_M^{1/2} \otimes \bigoplus_i \Omega^{0,i}(M) \tag{80}$$

where K_M is the canonical bundle and $\Omega^{0,i}(M)$ are anti-holomorphic i-forms. Note that there is a $\mathbb{Z}/2$-grading by whether i is even or odd. Analogously, let

$$T_{\text{vir}} = \underbrace{TM}_{\text{deformations}} - \underbrace{\text{Obs}}_{\text{obstructions}}$$

be the tangent-obstruction theory (in K-theory). Then, in the construction of the virtual structure sheaf, we should also twist by $K^{1/2}$ and define

$$\widehat{\mathcal{O}}_{\text{vir}} \coloneqq \mathcal{O}_{\text{vir}} \otimes K_{\text{vir}}^{1/2}$$

where

$$K_{\mathrm{vir}} \coloneqq \det(T_{\mathrm{vir}})^{-1}$$

is the *virtual canonical*. There is no reason a priori why a square root of K_{vir} exists; we will discuss this later.

21.3

Suppose the obstruction sheaf Obs is actually a vector bundle. Then

$$\mathcal{O}_{\mathrm{vir}} = \mathcal{O}_M \otimes \sum(-1)^i \wedge^i (\mathrm{Obs}^\vee) \tag{81}$$

arises from the Koszul complex of Obs. Comparing with (80), and remembering that $\Omega^{0,*}(M)$ is dual to TM, we see that we should require

$$\mathrm{Obs} \approx (\mathrm{deformation})^\vee.$$

The $\approx$ is because it is also OK if, instead,

$$\mathrm{Obs} = (\mathrm{deformation})^\vee \otimes (?) \tag{82}$$

where (?) is something we can control. In particular, if obstructions are dual to deformations in this way, then

$$K_{\mathrm{vir}} = \det(\mathrm{Obs}^\vee - \mathrm{Obs})^{-1} = \det(\mathrm{Obs})^2.$$

So indeed K_{vir} will have a square root. However this is not a proof that K_{vir} always has a square root, since usually our moduli spaces are not cut out in M by a section of a vector bundle, Obs is in general some sheaf, and (81) does not hold.

21.4

To recap, we want a moduli space of maps $f \colon C \to X$ with a perfect obstruction theory that is almost self-dual, in the sense of (82). In algebraic geometry, we would also like this moduli space to be proper, but this cannot happen when X itself is not proper. Better, recall that X has a projective map $\pi \colon X \to X_0$ to its affinization, and so the composition

$$C \xrightarrow{f} X \to X_0$$

maps C to a point (assuming C is connected). We would like the moduli to be proper *over* the analogous moduli of maps $C \to X_0$. Namely, each fiber $\pi^{-1}(p) \subset X$ is projective, and the subspace of maps $C \to \pi^{-1}(p)$ should be proper.

21.5

Neither the self-duality nor the properness property is easy to achieve! To compactify a space of maps $f \colon C \to X$, it is necessary to add maps which are singular. The simplest possible example is to take a conic in $\mathbb{P}^2$, which equivalently is a degree-2 map $\mathbb{P}^1 \to \mathbb{P}^2$, and degenerate it into a union of two lines. There is no way to parameterize the union of two lines by a map $\mathbb{P}^1 \to \mathbb{P}^2$. In the moduli space of stable maps, for Gromov–Witten theory, the solution is to allow C to grow extra components called *bubbles* and become a nodal (singular) curve. This indeed results in a proper moduli space, but the presence of bubbles spoils the self-duality of the perfect obstruction theory even for the best of targets X. We can observe this as follows. Suppose X is smooth, and consider the deformation theory of the map f. For every point in the image $f(C) \subset X$, there is a freedom to deform it along any tangent vector, so

$$\mathrm{Def} = H^0(C, f^*TX).$$

To identify the obstructions, we can work very concretely. Cover C by affine charts $C \setminus \{\mathrm{pt}\}$ and D where D is a formal disk around pt. (These charts are glued along the formal punctured disk). Maps from affine varieties can be deformed in any way, but to glue the deformations together requires them to agree on the overlap of the two charts. This is measured exactly by the Čech complex. In particular

$$\mathrm{Obs} = H^1(C, f^*TX).$$

For us, X is a symplectic variety and is therefore always self-dual. Then

$$\mathrm{Obs}^\vee = H^1(C, f^*TX)^\vee = H^0(C, (f^*TX)^\vee \otimes K_C) = H^0(C, f^*TX \otimes K_C)$$

by Serre duality. This canonical bundle K_C is the canonical bundle of the *whole* curve, including any bubbles. Since it is some divisor on C, it is something we can control as long as we can control C itself. Unfortunately, the bubbles affect K_C in predictable but a basically unmanageable way.

21.6

One approach to K-theoretic Gromov–Witten theory is to modify the index sheaf, via some insertions, to restore an approximate self-duality; see [Liu19] for a discussion of this approach. The basic challenge is, for $X = T^*G/B$, to recover Macdonald operators. In the degeneration $\hbar \to 0, \infty$ of the cotangent weight, this should recover the Toda operators for $X = G/B$ found in [GL03].

21.7

We take a different approach to compactifying the moduli of maps, using that X comes from a supersymmetric *gauge* theory, and is a Nakajima quiver variety.

Remark. Note that T^*G/B is *not* a Nakajima quiver variety for $G \neq \mathrm{GL}(n)$. This is not a new issue in integrable systems. It is known that Macdonald polynomials and other objects, have features in type A which are different from those in other types. Here we have another manifestation of this fact.

In particular, X is the Higgs branch of the supersymmetric gauge theory, whose ingredients are a gauge group G and a symplectic representation M containing matter in the theory. Then M has a moment map $\mu\colon M \to \mathfrak{g}^\vee$, and X is the stable locus inside the quotient stack

$$X \subset \mathfrak{X} \coloneqq [\mu^{-1}(0)/G].$$

Instead of considering maps $f\colon C \to X$, we first think about maps $f\colon C \to \mathfrak{X}$. Singular points will be where the image of f lands in the *unstable* locus inside $\mathfrak{X}$ instead of the stable locus X.

21.8

To understand maps to stacks, we begin with the simple case of $[\mathrm{pt}/G]$. By definition,

$$(f\colon C \to [\mathrm{pt}/G]) \equiv (\text{principal } G\text{-bundle } P \to C).$$

In more generality it would be important to discuss what topology (analytic, Zariski, étale, etc.) to take on G and C for the principal G-bundles, but in our setting we are doubly safe: they are all equivalent for $G = \mathrm{GL}(n)$ or a product of such, and also they are all equivalent for a smooth curve C.

In more generality, a map $f\colon C \to [Y/G]$ is the data of the principal G-bundle $P \to C$ along with a choice of section of the associated bundle $P \times_G Y \to C$. For instance, if $G = \mathbb{C}^\times$ acts on $Y = \mathbb{C}^n$, the space of maps is

$$\{\text{line bundle } \mathcal{L} \to C \text{ and }, s \in H^0(C, \mathcal{L}^{\oplus n})\}.$$

This is almost equivalent to a map $f\colon C \to \mathbb{P}^{n-1}$, which is given by

$$f = [f_1(x) : f_2(x) : \cdots : f_n(x)]$$

for a coordinate $x \in C$. On $\mathbb{P}^{n-1}$, these coordinates $f_i(x)$ are not functions but rather sections of $\mathcal{O}(1)$. Equivalently, setting $\mathcal{L} = f^*\mathcal{O}(1)$, these f_i are sections $f_i \in H^0(C, \mathcal{L})$. But to actually be a map to $\mathbb{P}^{n-1}$, they must have no *base points*, i.e. there cannot exist $x \in C$ such that

$$f(x) = [0 : \cdots : 0].$$

In our previous stacky language, we are viewing $\mathbb{P}^{n-1} \subset_{\text{stable}} [\mathbb{C}^n/\mathbb{C}^\times]$ and

$$\mathrm{Maps}(C \to \mathbb{P}^{n-1}) \subset \mathrm{Maps}(C \to [\mathbb{C}^n/\mathbb{C}^\times])$$

consists of those $\mathcal{L}$ and n sections *with no base points*.

21.9

The moduli $\mathrm{Maps}(C \to \mathbb{P}^{n-1})$ is not proper, because the condition that something doesn't vanish is an *open* condition. Instead, there is an intermediate space

$$\mathrm{Maps}(C \to \mathbb{P}^{n-1}) \subset \mathrm{QMaps}(C \to [\mathbb{C}^n/\mathbb{C}^\times]) \subset \mathrm{Maps}(C \to [\mathbb{C}^n/\mathbb{C}^\times])$$

of *stable quasimaps* where we allow a zero-dimensional locus of base points. Eventually we will also think about nodal C, in which case the base points should be disjoint from the nodes (and any marked points) of C as well. Actually, QMaps is a very simple space: for a fixed line bundle $\mathcal{L}$, it is just

$$\mathbb{P}(H^0(C, \mathcal{L}^{\oplus n})) = (H^0(C, \mathcal{L})^{\oplus n} \setminus \{0\})/\mathbb{C}^\times.$$

The projectivization is necessary because every line bundle $\mathcal{L}$ has $\mathrm{Aut}(\mathcal{L}) = \mathbb{C}^\times$, from an overall scaling, which also acts on sections. Allowing $\mathcal{L}$ to move, we see that QMaps is just a projective space bundle.

Definition. A **stable quasimap** in $\mathrm{QMaps}(C \to [Y/G])$ is:

- a principal G-bundle $\mathcal{P} \to C$;
- a section f of the associated Y-bundle $\mathcal{P} \times_G Y \to C$ whose base locus $f^{-1}(Y_{\text{unstable}})$ is zero-dimensional and disjoint from nodes (and marked points, if any) of C.

21.10

Example. What are stable quasimaps to $\mathrm{Hilb}(\mathbb{C}^2, n)$? Recall that $\mathrm{Hilb}(\mathbb{C}^2, n)$ is a Nakajima quiver variety consisting of representations

$$W = \mathbb{C} \underset{B}{\overset{A}{\rightleftarrows}} V \circlearrowleft x_1, x_2. \tag{83}$$

The moment map is $\mu = [x_1, x_2] + AB \in \mathfrak{gl}(V)^\vee$, and

$$\mathrm{Hilb}(\mathbb{C}^2, n) \subset_{\text{stable}} [\mu^{-1}(0)/\operatorname{GL}(V)].$$

The stability condition we impose (among two possible conditions) is that the vectors $x_1^i x_2^j A \cdot 1$ span V. This turns out to imply $B = 0$. To move to quasimaps, in general the only thing that changes is that all vector spaces in (83) are now replaced with vector bundles over the curve C:

$$\mathcal{W} = \mathcal{O}_C \underset{B}{\overset{A}{\rightleftarrows}} \mathcal{V} \circlearrowleft x_1, x_2 .$$

As with $\mathrm{Hilb}(\mathbb{C}^2)$, the bundle $\mathcal{V}$ varies as we move in the moduli space, while $\mathcal{W}$ is fixed and in general is acted upon by $\mathrm{GL}(\mathcal{W})$, or $\mathrm{GL}(W)$ if it really is a trivial bundle. The maps are now maps of bundles, e.g. $x_i \in \mathrm{Hom}_C(\mathcal{V}, \mathcal{V})$. A base point is where $\{x_1^i x_2^j A\}_{i,j}$ fail to generate the fiber of $\mathcal{V}$. Since by stability B vanishes away from the base locus, i.e. away from finitely many points, B is actually still identically zero.

21.11

The data of $\mathcal{V}$ and the two operators x_1, x_2 is equivalent to the data of a sheaf $\mathcal{F}$ on $C \times \mathbb{C}^2$. Namely, given $\mathcal{F}$, pushing forward along the projection $\pi\colon C \times \mathbb{C}^2 \to C$ gives a sheaf $\mathcal{V} = \pi_* \mathcal{F}$ on the base along with an action of the two coordinates $x_1, x_2 \in \mathbb{C}^2$. In addition, $\mathcal{V}$ is not just a sheaf on C, but rather a *bundle*, i.e. it is locally free. On curves, locally free is equivalent to torsion-free, meaning that there are no zero-dimensional subsheaves, and this is the property that lifts to $\mathcal{F}$. In other words, $\mathcal{F}$ has no zero-dimensional subsheaves; it is *pure* of dimension 1. Finally, the image of $x_1^i x_2^j A \cdot \mathcal{O}_C$ is exactly $\mathcal{O}_{C\times\mathbb{C}^2}$, and therefore the map

$$\mathcal{O}_{C\times\mathbb{C}^2} \to \mathcal{F} \tag{84}$$

has non-trivial cokernel supported at exactly the base locus in C.

Definition. Pairs like (84), on threefolds in general, where $\mathcal{F}$ is pure 1-dimensional and $\dim \operatorname{coker} = 0$, are known as *Pandharipande–Thomas (PT) pairs* [PT09].

We have just verified that

$$\operatorname{QMaps}(C \to \operatorname{Hilb}(\mathbb{C}^2, n)) = \operatorname{PT}(C \times \mathbb{C}^2).$$

So quasimaps really correspond, on the nose, to some sort of three-dimensional sheaf counting.

21.12

A variation is to take a non-trivial $\mathbb{C}^2$-bundle over C instead of the trivial bundle $C \times \mathbb{C}^2$. In quasimap language, this corresponds to a twist of the quiver data by $\operatorname{Aut}(X) \supset \operatorname{GL}(W) \times \operatorname{GL}(\text{edge}) \times \mathbb{C}^\times_\hbar$. For example, we could make

$$x_i \in \operatorname{Hom}\left(\mathcal{V}, \mathcal{V} \otimes (\text{line bundle on } C)\right).$$

Explicitly, if the bundle is $\mathcal{L}_1 \oplus \mathcal{L}_2 \to C$, then this line bundle could be $\mathcal{L}_i^\vee$.

21.13

To summarize, in Gromov–Witten theory we take the target X to be smooth but otherwise fairly arbitrary, e.g. projective-over-affine to achieve properness, and the compactification happens via bubbling of the source curve C. In quasimap theory, bubbling is replaced by the presence of base points, and, importantly, the notion of base point *does* depend on the ambient stack $\mathfrak{X}$. Also, for properness, the target X must embed as an open (stable) locus in a quotient stack $\mathfrak{X} = [V/G]$ where V is affine and G is reductive. For example, this will not be the case for (the obvious realization of) T^*G/B, since the Borel subgroup B is *not* reductive. Furthermore, to get a perfect obstruction theory for quasimaps, V itself must already have a perfect obstruction theory, which implies V must be a local complete intersection (lci). This means the number of equations cutting out V in its ambient space is equal to its codimension. One can verify that $\mu^{-1}(0)$, for Nakajima quiver varieties, is indeed lci. For $\operatorname{Hilb}(\mathbb{C}^2)$, this is precisely why it is important to have the map B, even though on the stable locus B is identically zero.

21.14

In principle, these conditions on the quasimap target X fit into a larger world of X of the form
$$X = [\text{crit}(f)/\text{reductive}],$$
where f is a function on a smooth ambient space with certain scaling properties. For Nakajima quiver varieties, we can introduce an extra variable $\xi \in \mathfrak{gl}(V)$ and define the function
$$f = \langle \xi, \mu(x) \rangle.$$
Then $\text{crit}(f) = \{\mu = 0,\ \xi \cdot x = 0\}$, and on the stable locus $\xi \cdot x = 0$ implies $\xi = 0$. This illustrates the general idea that equations like $\xi = 0$ can be traded for extra variables in a defining function like f. In different work, see e.g. [CLLL17], these extra variables are called *P-fields*.

Lecture 22. Quasimaps to $\text{Hilb}(\mathbb{C}^2)$ as PT theory

Donaldson–Thomas counts of subschemes in 3-folds, quasimaps to $\text{Hilb}(\mathbb{C}^2)$ *and Pandharipande–Thomas counts for rank 2 bundles over curves, torus fixed points in the Hilbert scheme of curves and the PT spaces, twisted quasimaps, evaluation maps, relative quasimaps, accordions, nodes and generation formula, the glue matrix.*

22.1

Recall that $\text{Hilb}(\mathbb{C}^2, n)$ parameterizes 0-dimensional subschemes of $\mathbb{C}^2$ of length n. In quiver language, there is a surjection
$$\mathbb{C}[x_1, x_2] \cdot 1 = \mathcal{O}_{\mathbb{C}^2} \twoheadrightarrow V = \mathcal{O}_{\text{subscheme}}.$$
To take a map $C \to \text{Hilb}(\mathbb{C}^2, n)$ means to have a flat family (over C) of such surjections, so it follows that
$$\begin{pmatrix} \text{nonsingular map} \\ f\colon C \to \text{Hilb}(\mathbb{C}^2, n) \end{pmatrix} \equiv \begin{pmatrix} \text{subscheme } Z \subset \mathbb{C}^2 \times C \\ \text{flat over } C \end{pmatrix}.$$
Flatness here can be thought of as the condition that the fiber over $c \in C$ is the limit of nearby fibers, i.e. the whole subscheme Z is a closure of its generic fibers. As we discussed previously, this condition is *not* a closed condition; a family of regular maps can converge to something with a singularity. For us, the easiest example to look at the family of automorphisms of $C = \mathbb{P}^1$ which scales everything toward $0 \in C$. In the limit, we get a map which is constant away from $0 \in \mathbb{P}^1$, namely it takes the value $f(\infty)$ everywhere else, and clearly this is not flat.

22.2

Last time we resolved this issue by considering quasimaps instead of maps. Quasimaps to $\text{Hilb}(\mathbb{C}^2, n)$ relax the condition that
$$\mathcal{O}_{C \times \mathbb{C}^2} \to \mathcal{O}_Z \to 0.$$

has to be surjective, so that $\mathcal{O}_Z$ becomes something which is no longer the structure sheaf of a subscheme. Let $\pi\colon C \times \mathbb{C}^2 \to C$ be the projection. Recall that the vector bundle $\mathcal{V}$ in the quiver data of a quasimap $C \to \mathrm{Hilb}(\mathbb{C}^2, n)$ can be identified as $\pi_* \mathcal{F}$ for a sheaf $\mathcal{F}$. Then there are equivalences

$$\mathcal{V} \text{ is a vector bundle} \iff \begin{pmatrix} \mathcal{F} \text{ has no} \\ \text{0-dimensional subsheaves} \end{pmatrix}$$

$$\begin{pmatrix} x_1^i x_2^j \cdot 1 \text{ generate } \mathcal{V} \\ \text{except at finitely many base points} \end{pmatrix} \iff \dim \operatorname{coker}(\mathcal{O}_{C\times\mathbb{C}^2} \xrightarrow{s} \mathcal{F}) = 0.$$

So on one hand we allow the map $\mathcal{O} \to \mathcal{F}$ to not be surjective, but on the other hand we require $\mathcal{F}$ must be *pure* of dimension 1. In an enumerative context, such pairs were first studied by Pandharipande and Thomas [PT09] and are called PT pairs.

22.3

In some sense, for $\mathrm{Hilb}(\mathbb{C}^2)$, the quasimap stability condition equals the PT stability condition. More generally, PT pairs fall into the more general framework of "Donaldson–Thomas counts", which count stable objects in categories that look like sheaves on 3-folds, meaning that there is a Serre duality

$$\mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2) \cong \mathrm{Ext}^{3-i}(\mathcal{F}_2, \mathcal{F}_1 \otimes K)^\vee \tag{85}$$

for objects in the category. In particular, the deformations $\mathrm{Ext}^1(\mathcal{G}, \mathcal{G})$ are roughly dual to the obstructions $\mathrm{Ext}^2(\mathcal{G}, \mathcal{G})$. Examples of such categories include:

- actual sheaves on 3-folds;
- representations of quivers (which form a "2d" category) in sheaves over a curve (which provides the extra dimension).

In the former example, note the presence of a potentially non-trivial K in (85); the 3-fold does not need to be Calabi–Yau. Quasimaps fall into the latter example. The earliest enumerative theory of this flavor is from [MNOP06], studying ideal sheaves of curves in 3-folds. In that setting, there is an actual surjection

$$\mathcal{O}_{\text{3-fold}} \to \mathcal{O}_{\text{curve}} \to 0,$$

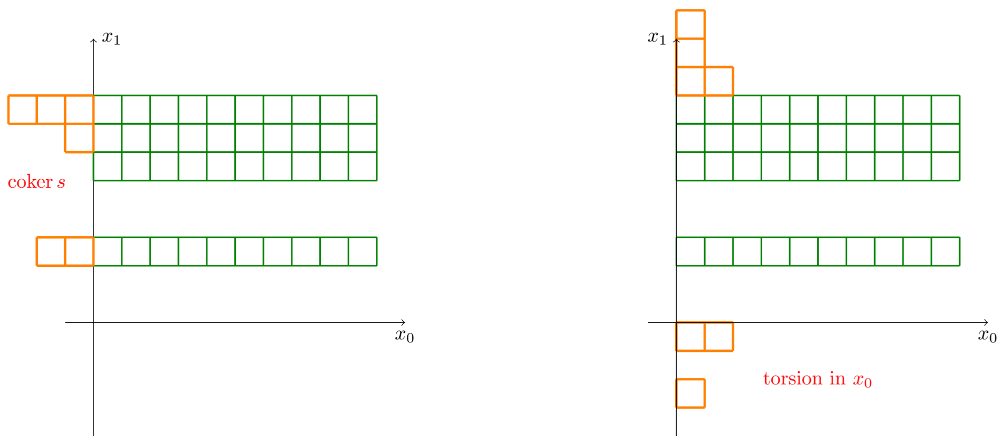
which in terms of quiver data means we give up on $\mathcal{V}$ being a vector bundle (cf. stability for PT pairs). There are many, many other possibilities for stability chambers.

22.4

A general expectation, supported by many theorems and computations, is that all stability conditions give equivalent counts in the sense that there is some dictionary which converts between the counts. This is known for local curves in cohomology, and in K-theory is an area of active research. In cohomology, DT counts are also equivalent to GW counts for local curves, and this was the original motivation of [MNOP06]. It is unknown what happens to GW/DT in K-theory, one issue being the loss of self-duality. In this course, many of our K-theoretic computations will rely crucially on self-duality properties.

22.5

Example. We can draw what PT and DT objects look like for $\mathbb{C}$ instead of $\mathbb{C}^2$. On $\mathbb{C}$ it is fairly easy to draw arbitrary subschemes, e.g. we can depict $Z = \{(x-a)^3(x-b) = 0\}$ as a single "box" at a and three boxes at b, and then $\operatorname{rank} \mathcal{O}_Z = 1 + 3 = 4$. The operators $(x-a)$ and $(x-b)$ are nilpotent on $\mathcal{O}_Z$, and our boxes are actually the Jordan blocks of these operators. Now consider a map $\mathbb{C} \to \operatorname{Hilb}(\mathbb{C}^1, 4)$ which is constant and maps to Z away from $0 \in \mathbb{P}^1$. Let x_0 be the coordinate on the source curve $\mathbb{C}$, and x_1 be the coordinate on the target $\mathbb{C}^1$.



(a) A PT-stable configuration

(b) A DT-stable configuration

Figure 37: A comparison of PT and DT stability conditions for maps $\mathbb{C}_{x_0} \to \operatorname{Hilb}(\mathbb{C}_{x_1}, 4)$ which generically land in $\{(x_1-a)^3(x_1-b) = 0\}$.

To have a vector bundle means coordinates act freely, so PT configurations are free along the x_0 direction, but the action of x_0 (starting at 1) may not generate the whole sheaf. Additional boxes may appear in negative powers of x_0, and correspond to singularities. In contrast, in DT configurations, x_1 need not act freely and boxes which are torsion in x_1 correspond to singularities. However, the boxes $x_0^j x_1^0 \cdot 1$ must generate everything under the action of x_1; this is the condition for the sheaf to be an ideal sheaf.

22.6

In 3d, there are now three coordinates x_0, x_1, x_2, and we can draw torus-fixed points for the torus $(\mathbb{C}^\times)^3$ scaling these coordinates. Such torus-fixed $\mathcal{F}$ are therefore graded by $\mathbb{Z}^3$, with $\deg x_1 = (1,0,0)$, $\deg x_2 = (0,1,0)$ and $\deg x_3 = (0,0,1)$. Consider slices in x_0 direction. For PT configurations, multiplication by x_0 is always injective, so slices "increase" in positive powers of x_0. For DT configurations, multiplication by x_0 can be torsion but must generate everything, so slices "decrease" in positive powers of x_0.

Remark. These combinatorial descriptions get a lot more interesting when $\mathbb{C}^2$ is replaced with an ADE surface, see [Liu21] for a description of Bryan–Steinberg configurations.

Instead of a trivial bundle $C \times \mathbb{C}^2$, we can also take a bundle $\mathcal{L}_1 \oplus \mathcal{L}_2 \to C$. For quasimaps, this means quiver maps like x_1 become an element of $\mathrm{Hom}(\mathcal{V}, \mathcal{V}) \otimes \mathcal{L}_1^\vee$. As a general principle, if $X \subset [(\text{prequotient})/G]$, there could be a bigger group

$$1 \to G \to \widetilde{G} \to G_{\mathrm{Aut}} \to 1$$

where G_{Aut} contains additional automorphisms of the Nakajima quiver variety, e.g. for $\mathrm{Hilb}(\mathbb{C}^2)$ we have $\mathrm{GL}(2) \subset G_{\mathrm{Aut}}$ acting on $\mathbb{C}^2$. Then, for quasimap data, instead of a principal G-bundle $\mathcal{P}$, we can consider a principal $\widetilde{G}$-bundle $\widetilde{\mathcal{P}}$ with given image in G_{Aut}-bundles. For $\mathrm{Hilb}(\mathbb{C}^2, n)$, this means we can take an arbitrary rank-2 bundle over C.

22.7

Now that we have discussed moduli spaces, it is time to set up enumerative counts of curves in X. For us, "curves" will mean quasimaps; this is our choice of compactification. The counts will be tensors in $K_{\mathrm{eq}}(X)^{\otimes \cdots}$, where a vector in $K_{\mathrm{eq}}(X)$ remembers the notion of how the curve meets a prescribed cycle at a certain point, and the $(-)^{\otimes \cdots}$ means there may be multiple such points. More generally, we should also allow for the possibility that these tensors take values in $K_{\mathrm{eq}}(\mathfrak{X})^{\otimes \cdots}$. The spaces $K_{\mathrm{eq}}(X)$ and $K_{\mathrm{eq}}(\mathfrak{X})$ are usually very big, and it is impractical to write down every entry of the tensors. Rather, to compute the counts means to identify the tensors in terms of quantum groups.

22.8

Consider a stable quasimap $f\colon (C, p_1, \ldots, p_n) \to X$ where the source curve C carries marked points $p_1, \ldots, p_n$. Then there is an *evaluation map*

$$\mathrm{ev}\colon f \mapsto (f(p_1), f(p_2), \ldots, f(p_n)) \in \mathfrak{X}^n$$

since at base points $f(p_i)$ may not land in $X \subset \mathfrak{X}$. More generally, for $\mathsf{G}_{\mathrm{Aut}}$-twisted stable quasimaps, we can also remember the data of the principal bundle:

$$(\mathcal{P}_{\mathrm{Aut}} \to C, f(p_1), \ldots, f(p_n)) \in \mathsf{Bun} \times \mathfrak{X}^n.$$

With respect to this data, the K-theoretic counts form what is called an "extended" K-theoretic cohomological field theory (cohFT). Here "extended" means that we remember $\mathcal{P}_{\mathrm{Aut}}$, and "cohFT" means the collection of K-theoretic tensors and how they behave under nodal degenerations of the curve C. In the moduli of nodal curves $(C, p_1, \ldots, p_n)$, there is a stratification by how many nodes C has, and degenerations tell us how the K-theory classes behave as we move from one stratum into deeper strata. In particular, we want some kind of gluing operator at nodes, and therefore nodes *must* be disjoint from base points. For this and other reasons, it is useful to have an evaluation map not to $\mathfrak{X}$, but to $X \subset \mathfrak{X}$. Another reason is that the K-theory of X is finite-dimensional, while the K-theory of $\mathfrak{X}$ is something nice but infinite-dimensional.

22.9

To get an evaluation map to X and not $\mathfrak{X}$, we need a moduli space that resolves the following map. Inside the moduli of all quasimaps sits the open locus

$$\mathrm{QMaps} \supset \mathrm{QMaps}_{\substack{\text{nonsing} \\ \text{at } p}} \coloneqq \{f(p) \in X\},$$

which by definition admits an evaluation map to X. Hence there is a rational map $\mathrm{QMaps} \dashrightarrow X$, and we should blow up QMaps in a specific way to get a regular proper map

$$\mathrm{QMaps}_{\substack{\text{relative} \\ \text{at } p}} \to X.$$

In more detail, imagine a one-dimensional family of quasimaps, for a family of curves C_t. A marked point p on each C_t forms a section $p(t)$ of the family. Suppose there are other sections corresponding to base points in each C_t, which generically are disjoint from $p(t)$ but at some special $t = t_{\text{bad}}$ hits $p(t)$. The standard algebro-geometric way to handle this is to blow up the intersection (perhaps with some weighted blow-up, base changes, etc.). The result is that $C_{t_{\text{bad}}}$ can grow several new components, i.e. the exceptional divisor, and these new components separate the marked point $p(t)$ from the base points, see Figure 38.

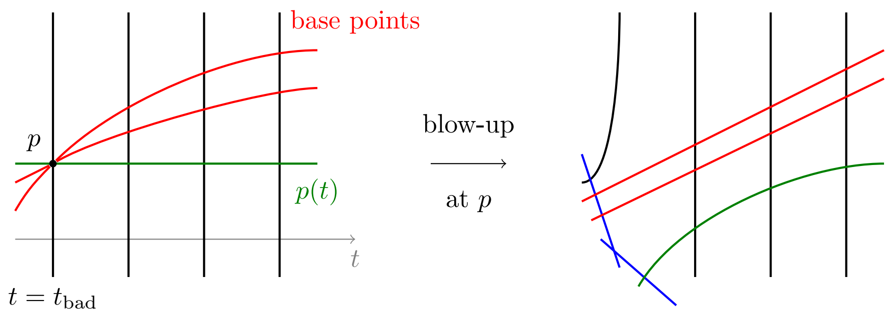


Figure 38: Bubbling for relative quasimaps, to separate the marked point $p(t)$ from base points.

This is not unlike what happens in ordinary 2d field theories, where we could have some insertion of a point in the worldsheet which we would like to keep away from other operators, or defects. As the other operators approach, our point goes off in a new bubble, thereby escaping from the operators.

22.10

Importantly, in this resolution we construct, curves C only bubble in *chains* as opposed to in *trees*. While the original component of C is fixed and parameterized, and therefore has no automorphisms, the new components form a chain of $\mathbb{P}^1$'s which themselves have automorphisms. Namely, each bubble has $\mathrm{Aut} = \mathbb{C}^\times$, and so overall there is an automorphism group

$$\mathrm{Aut} = (\mathbb{C}^\times)^{\#\text{ of bubbles}}.$$

The presence of these automorphisms means we must take the new stability condition

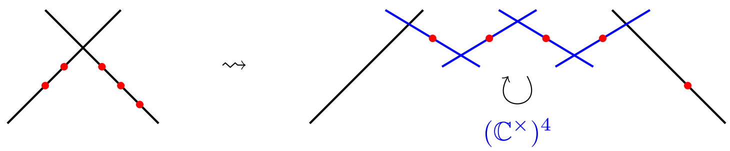
$$\{\dim(\text{base locus}) = 0 \text{ and } \dim(\text{stabilizer}) = 0\},$$

where the new condition $\dim(\text{stabilizer}) = 0$ prevents unnecessary bubbles/blow-ups.

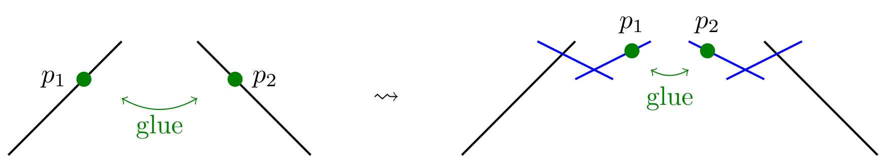
Note that the moduli space of relative quasimaps is a *DM stack*, and in particular localization on it is much more complicated. For example, fixed loci are not isolated points anymore.

22.11

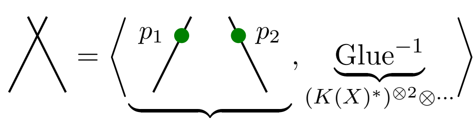
We conclude with a discussion of K-theoretic CohFT via quasimaps. How should gluing work, i.e. what are quasimaps from a nodal curve $C_1 \cup_p C_2$? The condition to impose is that nodes are disjoint from base points, where, to keep nodes away from base points, we perform the exact same procedure as for marked points. After blow-up, the node "opens" into a chain of $\mathbb{P}^1$'s, which we call an accordion:



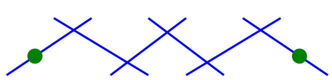
The accordion swallows all the base points, in the sense that base points now live on each of the bubbles, and again we have an overall automorphism by $(\mathbb{C}^\times)^{\#\text{ of bubbles}}$. On the other hand, we can take quasimaps on C_1 relative to a point p_1, and quasimaps on C_2 relative to a point p_2, and "just glue" them.



This means to pair with the diagonal in $X \times X$ via the evaluation map. But this will overcount, compared to the accordion, since we must make a choice of where to cut the accordion into two pieces. It turns out the correct formula is, schematically,



where

Glue :=  $= (\text{diagonal}) + (\text{corrections}) \in K(X)^{\otimes 2}$

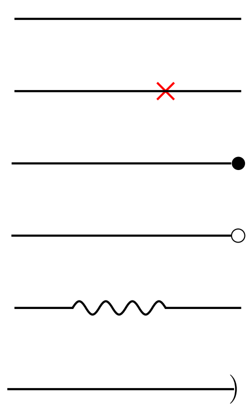
is a sum over accordions of all possible lengths.

Lecture 23. Non-singular and relative boundary conditions

Diagrammatic notation for different flavors of insertions/boundary conditions in enumerative problems, relative moduli spaces in DT theory, expanded degenerations, degeneration formulas, correspondence between different boundary condition, degeneration and algebraic cobordism, relative counts in GW theory, their correspondence with relative DT counts.

23.1

We introduce some useful shorthand for quasimaps:



= component of source curve C

= base point

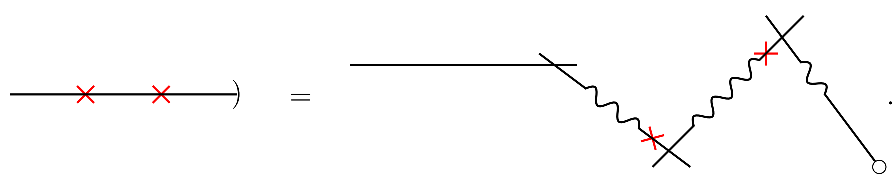
= a point, evaluates to the stack $\mathfrak{X}$

= a nonsingular point, evaluates to $X \subset \mathfrak{X}$

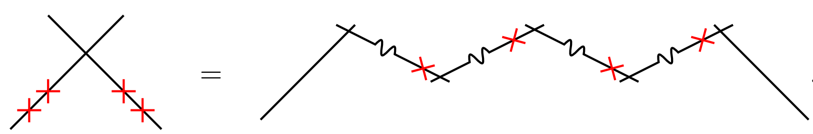
= a $\mathbb{P}^1$ component modulo $\mathbb{C}^\times$ rescaling

= a relative point, with proper evaluation to X.

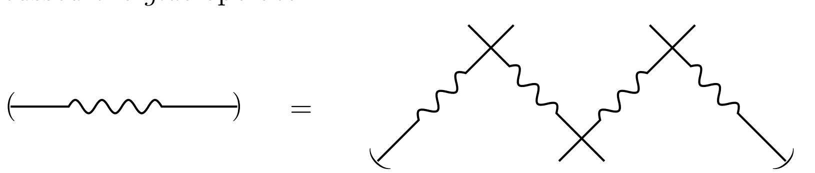
Recall that base points are outside of our control; they appear, and we try to deal with them. When they approach a point which we wish to remain non-singular, recall we blow up and bubbles isomorphic to $[\mathbb{P}^1/\mathbb{C}^\times]$ appear, and we get relative quasimaps. In our shorthand,



In general it is difficult to allow source curves with arbitrary singularities, but we discussed how to allow for *nodal* singularities:



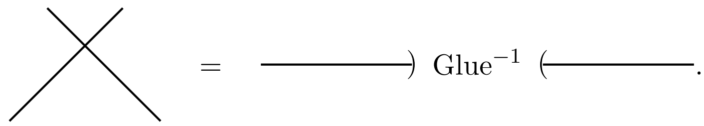
Finally, we discussed the *glue* operator:



Importantly, the bubbles form chains (called accordions) as opposed to trees. This is for control over the canonical class ω_C, which gives us the crucial property of self-duality.

23.2

Theorem.



If we think of Glue as an operator G_{ab} with two lower indices, it sort of makes sense that the actual operator which glues two vectors should be $G^{ab} = (G^{-1})_{ab}$.

Proof. See [Oko17, Section 6.5]. □

This equality is true not just for a fixed source curve, but also in moduli. Namely, the lhs defines a divisor in the moduli of (nodal) source curves, and, more precisely, it is a sheaf on this divisor. On the rhs, we have the same kinds of sheaves but on a different moduli space. The equality is of K-theory classes of these sheaves. It therefore defines a K-theoretic gluing rule analogous to the gluing rule in cohFTs. It would be very interesting to extend the very powerful Givental–Teleman classification of cohFTs to K-theory, which would allow us to sidestep many units of complexity in enumerative computations.

Remark. The presence of a non-trivial operator Glue^{-1} in K-theory was first recognized by Givental in K-theoretic Gromov–Witten theory [Giv00]. In cohomology, Glue is trivial for dimensional reasons, but such dimensional vanishing is not available in K-theory.

23.3

Recall that quasimaps to $\mathrm{Hilb}(\mathbb{C}^2)$ are the same as PT theory of a local curve in a threefold. It is good to rephrase non-singular points, relative points, etc. in the language of the threefold.

Let Y be a threefold, e.g. Y is a $\mathbb{C}^2$-fibration over a curve C. The analogue of a point on C is a divisor $D \subset Y$. We can impose enumerative conditions on D. For example, a curve in Y can be viewed in either DT or PT theory, as the support of a sheaf $\mathcal{F}$, and it could hit D in various ways. If x is the coordinate normal to $D \subset Y$, then in PT theory the analogue of the evaluation map is

$$\mathrm{ev}\colon [\mathcal{O}_Y \xrightarrow{s} \mathcal{F}] \mapsto [\mathcal{O}_D \xrightarrow{s \bmod x} \mathcal{F} \otimes \mathcal{O}_D].$$

The map $s \bmod x$ could be anything; the only thing we know is that $\mathcal{F}$ has no zero-dimensional subsheaves. We would like to impose some conditions on the original $[\mathcal{O}_Y \to \mathcal{F}]$ so that its image under the evaluation map is still a PT pair:

- $\mathcal{F} \otimes \mathcal{O}_D$ has no zero-dimensional subsheaves iff $\mathcal{F}$ has no 1-dimensional components in D;
- the image of $\mathcal{O}_D$ under $s \bmod x$ generates iff there are no base points of $\mathcal{F}$ in D, i.e. $\mathrm{coker}(s)$ must be disjoint from D.

This defines the threefold version of ———○. Similarly, the relative condition ———) means that there can be an accordion at the divisor D, where bubbles can contain base points and entire 1-dimensional components of $\mathcal{F}$, but the intersection of $\mathcal{F}$ with any copy of D must still be nice (like how quasimaps must be non-singular at nodes). Each bubble is a component isomorphic to $\mathbb{P}(N_{Y/D} \oplus \mathcal{O}_D)$, a $\mathbb{P}^1$-bundle over D. Such pictures were first studied by Jun Li in [Li01] under the name of *expanded degenerations*.

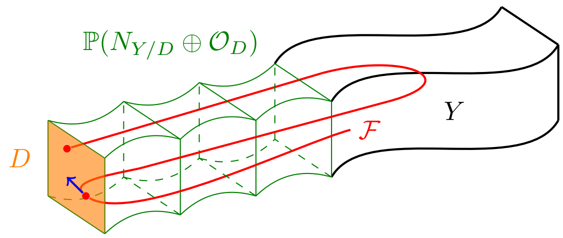


Figure 39: An expanded degeneration of (Y, D) with three accordions. The restriction of $\mathcal{F}$ to any of the four copies of D has no base points.

23.4

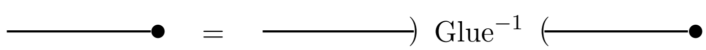
It turns out that there are some building blocks from which all other enumerative counts can be recovered. Explicitly, consider a threefold Y containing a cycle $Z \subset Y$, and impose some constraints on how our curves meet Z. It is still valid to blow up $Z \times \{t = 0\} \subset Y \times \mathbb{C}_t$, to get a new threefold (at $t = 0$) of the form

$$Y' \coloneqq Y \cup_{\mathbb{P}(N_{Y/Z})} \mathbb{P}(N_{Y/Z} \oplus \mathcal{O}_Z).$$

Here $\mathbb{P}(N_{Y/Z})$ is a divisor in Y, recording all possible ways in which the curve can approach Z. The new threefold Y' consists of the original Y along with the bubble, attached by a nodal singularity at $\mathbb{P}(N_{Y/Z})$. The original cycle Z lives on the bubble. Counting curves on Y' therefore involves gluing relative counts on Y with relative counts on the bubble $\mathbb{P}(N_{Y/Z} \oplus \mathcal{O}_Z)$. In other words, *any* count on Y can be written in terms of

- counts relative to a divisor in Y, and
- whatever counts we originally wanted, but in the model geometry $\mathbb{P}(N_{Y/Z} \oplus \mathcal{O}_Z)$.

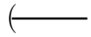
This model geometry is special in comparison to Y, e.g. because it admits a $\mathbb{C}^\times$ action by scaling either the bundle $N_{Y/Z}$ or $\mathcal{O}_Z$, and then equivariant localization applies. In our quasimap shorthand, we have shown

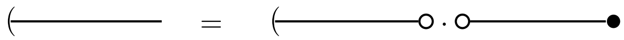


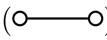
(86)

where on the lhs the line and • represents an arbitrary geometry and "boundary condition", but on the rhs the second piece is a projective bundle.

23.5

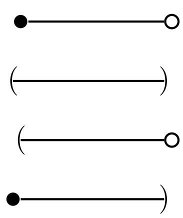
There is an interplay between non-singular and relative counts, of the following kind. Note that in (86), $\mathbb{C}^\times$ acts on the relative counts and not on the gluing operator. The $\mathbb{C}^\times$-fixed points are constant and non-singular on  except possibly at the point •. In particular, this means at $\mathbb{C}^\times$-fixed points there is a decomposition

 (87)

where $\mathbb{C}^\times$ still acts on both pieces in the rhs. Between the two non-singular points we must insert ()$^{-1}$ by the same reasoning as for Glue^{-1}, but this operator is much, much simpler.

23.6

The conclusion is that there are dictionaries between "boundary conditions" given by (86) and (87), where in (87) the boundary condition • could be anything, including a relative point. If X is a variety of interest, it is therefore important to know tensors of the form



$$\in \quad K_{\mathrm{eq}}(\mathfrak{X}) \otimes K_{\mathrm{eq}}(X) \otimes (\text{series in } z)$$
$$\overset{\text{(Thm)}}{=} \quad \mathrm{Glue}$$
$$\in \quad K_{\mathrm{eq}}(X)^{\otimes 2} \otimes (\text{series in } z)$$
$$\in \quad K_{\mathrm{eq}}(\mathfrak{X}) \otimes K_{\mathrm{eq}}(X) \otimes (\text{series in } z).$$

The second line is a non-trivial theorem. The formal variable z records the degree of the quasimap.

23.7

The first and most crucial order of business is then to describe these tensors. Our goal is to describe all of them in terms of quantum groups acting on X. The analogy to have in mind is with Chern–Simons theory. It takes place on a *real* threefold, which is easy to break into pieces. Then Chern–Simons theory reduces to computations on a few different pieces, which one then computes in terms of (ordinary) quantum groups. In our setting, we encounter not just quantum loop groups, but quantum double loop groups.

23.8

Our degeneration arguments are a special case of the following more general phenomenon. Let Y be a smooth threefold, and suppose it degenerates into $Y_1 \cup_D Y_2$ where both Y_1 and Y_2 are smooth and D is a divisor. This means that there is a flat family in which the general fiber is Y and the special fiber is $Y_1 \cup_D Y_2$. In such a degeneration, $N_{Y_1/D} \otimes N_{Y_2/D} = \mathcal{O}_D$. Jun Li's theory of expanded degenerations works equally well in this setting.

Theorem ([LP09])**.** *The relations*

$$[Y] = [Y_1] + [Y_2] - [\mathbb{P}(\mathcal{O}_D \oplus N_{X_1/D})]$$

generate algebraic cobordism.

We have already discussed the *smooth* cobordism ring; algebraic cobordism is more complicated to define, but its representatives are smooth algebraic varieties over a base. Over $\mathrm{pt} = \operatorname{Spec}\mathbb{C}$, smooth cobordism is the Lazard ring, which, when tensored with $\mathbb{Q}$, is

$$\mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \ldots].$$

In particular, in dimension 3, there are only three classes in smooth cobordism: $(\mathbb{P}^1)^3$, $\mathbb{P}^2 \times \mathbb{P}^1$, and $\mathbb{P}^3$. However, this argument does not immediately reduce all counts to toric counts, because our enumerative counts depend not just on X but also on a smooth divisor $D \subset X$, but smoothness of the divisor is not necessarily preserved under degeneration. Nonetheless, Theorem 23.8 is still very powerful.

Remark. It is useful to read Theorem 23.8 in both directions. Sometimes it is good to add extra smooth components to the variety and relate the original counts to counts on the new, bigger variety.

23.9

There is a notion of relative invariants in Gromov–Witten theory. For a divisor $D \subset X$ and a map $f\colon C \to X$, the non-singularity condition is that

$$f^*D = \sum_i \mu_i [p_i]$$

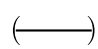
for some *fixed* μ_i, and any points $p_i \in C$. So there cannot be any 1-dimensional components of C lying in D, and C must intersect D with prescribed multiplicities. When this fails to hold in a degeneration, bubbling occurs. The evaluation map produces points with *fractional* multiplicities, since the pullback of a point $x \in X$ can be a non-trivial multiple of a point $p \in C$, or, equivalently, the image of $p \in C$ can be some non-trivial fraction of $f(p) \in X$. Hence the evaluation map really takes values in the orbifold cohomology of

$$\exp(D) \coloneqq \bigsqcup_n [D^n/S(n)]. \tag{88}$$

Conjecture. *Relative counts in GW are equivalent to relative counts in PT/DT/$\cdots$.*

This general conjecture is known for ADE surface fibrations over a curve, in cohomology. But the orbifold (88) is actually *derived equivalent* to $\bigsqcup_n \operatorname{Hilb}(D, n)$ [BKR01, Hai01], not just isomorphic in cohomology or K-theory. So it would be nice to sort out at least the case of K-theoretic GW theory.

23.10

Gromov–Witten theory of target curves with relative insertions is very nice, e.g. it has Virasoro constraints. For a more general relative divisor, this is something that is not so well-understood and it is even unclear how to create operators that form something like a Virasoro algebra. But for target curves, basic tensors like  are very close to things like double ramification cycles (DRC). This is where one studies moduli of a curve and a function with prescribed zeros and poles, which is nothing more than a map from the curve to $\mathbb{P}^1$. The DRC is then the pushforward of the virtual class for this moduli to the moduli of curves. As a cycle in the moduli of curves, it is a different object of study, but when it is paired with other cycles we get exactly counts from relative GW theory.

Lecture 24. Building blocks of quasimap counts and their q-difference equations

Basic building blocks of quasimap counts, vertex with descendants, its expression as an elliptic stable envelope, relative counts and q-difference equations, q-Gamma functions, vertex with descendants and integral solutions to quantum difference equations, integral solutions and Bethe Ansatz, residues in the integral vs. localization formulas for vertices with descendants.

24.1

We continue the discussion of basic building blocks of the enumerative theory of maps $C \to X$. Already we have reduced to the case of $C = \mathbb{P}^1$ with marked points at 0 and ∞, treated equivariantly with respect to a $\mathbb{C}^\times$ acting with a weight q. Those familiar with topological field theories may expect that more generally we need a 3-pointed sphere, but this equivariance is a substitute for the third point in a precise way. Physically, if one can understand some field theory on flat space equivariantly with respect to all of its diffeomorphisms, then actually one understands it on any manifold. This may be familiar from instanton counting, where Nekrasov partition functions really live on flat space, and from them one extrapolates to more general manifolds. This is also what we are doing here. The analogue of flat space for us is the Riemann sphere; objects live on $\mathbb{C}$ with the origin 0, equivariantly with respect to rotation by q, and with boundary conditions at infinity. In instanton counting, these boundary conditions at infinity are also present, from the Uhlenbeck compactification.

24.2

Recall that we had three kinds of boundary conditions: none, non-singular, and relative. The points 0 and ∞ can carry various combinations of these boundary conditions. For example, $\bullet\!\!-\!\!-\!\!-\!\!\circ$ corresponds in the PT picture to configurations as in Figure 40. Highlighted boxes correspond to the fiber at 0, which is a point in the ambient stack $\mathfrak{X}$ instead of X.

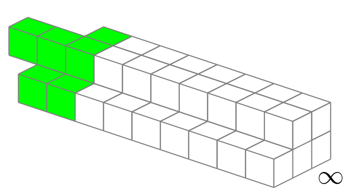


Figure 40: A PT (1-leg) vertex configuration with ev_∞ landing in $\mathrm{Hilb}(\mathbb{C}^2, 5)$.

In the lingo of Donaldson–Thomas theory, this is the 1-leg vertex with descendants (at 0).

Descendants refer to the characteristic classes of $\mathcal{F} \otimes \mathcal{O}_0$, where $\mathcal{F}$ is the universal sheaf. To get 2- and 3-leg vertices, we must replace $\mathrm{Hilb}(\mathbb{C}^2)$ with $\mathrm{Hilb}(A_n)$.

24.3

Another building block is (——○. By degeneration,

●——○ = ●——) G^{-1} (——○.

We can think of ●——) G^{-1} as the matrix which relates (——○ and ●——○. Explicitly, if $p(n) = \operatorname{rank} K(\mathrm{Hilb}(\mathbb{C}^2, n))$ is the number of partitions of size n, then (——○ is a $p(n) \times p(n)$ matrix, and ●——○ is an $\infty \times p(n)$ matrix. Here ∞ is because K-theory of the stack is infinitely generated.

Another relation comes from localization:

(——) = (——○ · ○——) .

So we have a bunch of interrelated tensors in $K(X)^{\otimes 2}$, or $K(\mathfrak{X}) \otimes K(X)$, etc.

24.4

The hardest building block is ●——○. It can be viewed as

$$\mathrm{ev}_{0,*}\left(\mathrm{ev}_\infty^*(-) \otimes \widehat{\mathcal{O}}_{\mathrm{vir}} z^{\deg}\right) : K(X) \to K(\mathfrak{X})_{\mathrm{loc}}[[z]]. \tag{89}$$

The evaluation $\mathrm{ev}_{0,*}$ is not a proper pushforward, so it requires localization with respect to $\mathbb{C}_q^\times$ acting on the source $\mathbb{P}^1$. While a general quasimap can have base points located anywhere on $\mathbb{P}^1$, fixed loci of this $\mathbb{C}_q^\times$ can only have base points at the origin $0 \in \mathbb{P}^1$. Hence fixed loci are proper. The result therefore lives in localized K-theory for this $\mathbb{C}_q^\times$. As a series in z, it is in fact convergent and meromorphic.

24.5

The claim is that this map (89), with suitable q-Gamma function factors, is the elliptic stable envelope in the context of $X \subset \mathfrak{X}$. Recall that stable envelopes arise as an extension problem from an open subset, specifically

$$\mathrm{Attr} \subset \left(\text{component of } X^A \times \left(X \setminus \begin{matrix}\text{lower attracting} \\ \text{manifolds}\end{matrix}\right)\right).$$

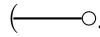
There is a stratification of X by Attr(components of X^A), and we used a long exact sequence in elliptic cohomology to inductively extend the definition of stable envelopes to all strata. In comparison, our setting now is that we have the GIT stable locus $X \subset \mathfrak{X}$, and the complement $\mathfrak{X} \setminus X$ has a stratification (studied by Bogomolov, Hesselink, Kempf, Ness, Rousseau, etc.) which reduces to the attracting stratification in the case that the group is a torus.

Our $\mathfrak{X}$ are quotients by $\mathsf{G} = \prod \mathrm{GL}(V_i)$, which may be reduced to tori by a trick to be discussed later. In general G should be reductive and connected. Let $\mathfrak{X} = [M/\mathsf{G}]$, and take elliptic cohomology for an elliptic curve $E = \mathbb{C}^\times/q^{\mathbb{Z}}$. This q is our equivariant variable q from earlier. The elliptic stable envelope we want is a map

$$\mathrm{Stab}\colon \Theta(T^{1/2}M\big|_X) \otimes \cdots \to \Theta(T^{1/2}M) \otimes \cdots .$$

Here $\cdots$ denotes some degree-0 line bundle which depends on a coordinate $z \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times$. This z is also our variable z from earlier. As we have repeatedly emphasized, Stab is an interpolation problem, and there is an map in the reverse direction which is just restriction. Since the prequotient M is a vector space, the image of Stab lives in sections of a line bundle on $\mathrm{Ell}_{\mathsf{G}\times\mathsf{G}_{\mathrm{Aut}}}(\mathrm{pt})$, which is just a product of elliptic curves modded out by a Weyl group. On the other hand, the source lives over something 0-dimensional over $\mathrm{Ell}_{\mathsf{G}_{\mathrm{Aut}}}(\mathrm{pt})$. So Stab is a finite $\times\, \infty$ matrix.

24.6

Now we discuss . This is a fundamental solution of a flat q-difference connection (for the same q as before) in both the Kähler variables $z \in \mathrm{Pic}(X) \otimes \mathbb{C}^\times$ and equivariant varaibles in $\mathrm{Aut}(X)$. The mechanism which constructs the q-difference equation itself is very simple and will be explained in due course, but the actual identification of the equation is very complicated. We can do it in terms of the quantum group $U_\hbar(\widehat{\mathfrak{g}})$.

- The q-difference equation in equivariant variables is a generalization of qKZ to a cocharacter (with some technical restrictions)

  $$\sigma\colon \mathbb{C}^\times \to \mathrm{Aut}(X), \quad a \mapsto \sigma(q)a.$$

  Namely, there is an "R-matrix" associated to σ, and it is the q-difference operator.

- The q-difference equation in Kähler variables is a generalization of the dynamical equation (e.g. for KZ). This must be the case since the overall q-difference connection is flat, and therefore the two q-difference operators must commute.

Note that q is a free parameter! This is surprising because in the usual KZ setup, how much to shift by is related to the parameter $\hbar$ of the quantum group (and the level). But q and $\hbar$ are independent in our setting.

24.7

A very classical question is whether one can solve this q-difference equation (in both sets of variables) by an integral. KZ and related equations generalize things like hypergeometric equations, which in turn generalize things like spherical harmonics, so having an integral representation of solutions generalizes a very broad range of questions about special functions in mathematical physics. In the usual hypergeometric world, there are two kinds of representations:

- an "Euler" integral $\int_0^1 x^{\cdots}(1-x)^{\cdots}(z-x)^{\cdots}\,dx$;
- a "Mellin–Barnes" integral $\int \mathrm{poly}(x)\frac{\Gamma(\cdots)}{\Gamma(\cdots)}\,dx$.

In the q-difference world, these two become the same. Namely,

$$(1-x)^m \rightsquigarrow (1-x)(1-qx)\cdots(1-q^{m-1}x) = \frac{(x)_\infty}{(q^m x)_\infty}.$$

It is better to use the second formula since m is not necessarily an integer. Here

$$(x)_\infty \coloneqq \frac{1}{\Gamma_q(x)} \coloneqq \prod_{i=0}^{\infty}(1-q^i x), \quad |q|<1.$$

Note that $\Gamma_q(x)$ is therefore the character of the polynomial ring in infinitely many generators, of weights x, qx, q^2x, and so on. This is exactly

$$\mathcal{O}_{\mathrm{Maps}(\mathbb{C}\to\mathbb{C})}$$

where q acts on the source and x^{-1} acts on the target. Explicitly, if $t\in\mathbb{C}$ is the coordinate on the source, then the action is

$$t^k \mapsto x^{-1}(q^{-1}t)^k = q^{-k}x^{-1}.$$

Then recall that weights of functions are dual to weights of the vectors themselves. In this way, q-Gamma functions are ubiquitous in the study of moduli spaces of maps.

24.8

The usual form of integral representations is as follows. For $\alpha,\beta\in K_{\mathrm{eq}}(X)$,

$$\left\langle \alpha, \begin{matrix}\text{fundamental}\\ \text{solution}\end{matrix} \cdot\beta \right\rangle = \int_{\mathrm{cycle}(\beta)} f_\alpha(x,\ldots)\mathrm{weight}(x,\ldots)\,dx.$$

Here x is a dummy integration variable, and $\ldots$ denote the actual variables of interest in the q-difference equation. Which coordinate α we want is specified by a function f_α in the integrand, and which solution β we want is specified by a choice of contour cycle(β). But in the q-difference situation (and for a torus) we can do better, by fixing the cycle and writing

$$\left\langle \alpha, \begin{matrix}\text{fundamental}\\ \text{solution}\end{matrix} \cdot\beta \right\rangle = \int_{|x_i|=1} f_\alpha(x,\ldots)g_\beta(x,\ldots)\mathrm{weight}(x,\ldots)\prod\frac{dx_i}{2\pi i x_i}. \tag{90}$$

Here g_β is some *elliptic* function. The idea is that functions constant with respect to q-shifts are elliptic functions, and inserting them into the integrand is like moving the contour. In our situation,

- the x_i are coordinates of the maximal torus in G,

- f_α is the K-theoretic stable envelope of the class α,
- g_β is the elliptic stable envelope of the class β, and
- the overall integral expression is the inner product on $K(\mathfrak{X})$ given by

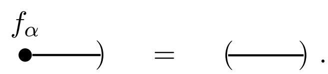
$$(\mathcal{F}_1, \mathcal{F}_2) \mapsto \chi(M, \mathcal{F}_1 \otimes \mathcal{F}_2)^{\mathsf{G}}$$

  on the prequotient M. The weight function includes the Weyl integration formula, for taking G-invariants, and some q-Gamma factors coming from $\mathrm{Maps}(\mathbb{C} \to M)$, as before.

K-theoretic and elliptic elements are combined as follows. Recall that elements of equivariant elliptic cohomology, associated to $E = \mathbb{C}^\times/q^{\mathbb{Z}}$, pull back to equivariant K-theory, associated to the pre-quotient $\mathbb{C}^\times$. This is "reduction mod q", and sections of line bundles are re-interpreted as solutions of some abelian q-difference equation.

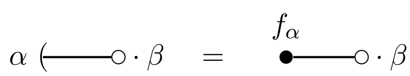
24.9

More explicitly, and returning to enumerative problems, suppose we can find f_α such that



(91)

Then the matrix elements of the our fundamental solution become exactly the vertex with descendent f_α:



$$= \chi\left(\mathrm{QMaps}, \mathrm{ev}_0^*(f_\alpha)\, \mathrm{ev}_\infty^*(\beta) \widehat{\mathcal{O}}_{\mathrm{vir}} z^{\deg}\right)$$
$$= \chi\left(\mathfrak{X}, f_\alpha \otimes (\text{elliptic Stab})(\beta) \otimes \mathrm{ev}_{0,*}(\text{maps } \mathbb{D} \to X)\right)$$

where the last equality follows from push-pull with respect to ev_0. The elliptic stable envelope arises by our discussion of (89). The term $\mathrm{ev}_{0,*}(\text{maps } \mathbb{D} \to X)$, involving maps from the formal disk $\mathbb{D}$, is the origin of Γ_q factors in the formula (90). Finally, we will see that the solution to the problem of finding f_α satisfying (91) is solved by stable envelopes in K-theory.

24.10

Now suppose an expression of the form in (90) solves an equation

$$\Psi(qz) = M(z)\Psi(z).$$

Recall that q is a free parameter. What will happen if $q \to 1$? Let ψ_i be an eigenvector of $M(z)$ with eigenvalue $\lambda_i(z)$. Then in the limit,

$$\Psi(z) \sim e^{\frac{1}{\ln q}\int \lambda_i} \psi_i(z).$$

Hence the limit provides both eigenvalues and eigenvectors of the operator $M(z)\big|_{q=1}$.

The exponential behavior of $\lim_{q\to 1}\Psi(z)$ comes from the weight term in the integral representation, while the eigenvector comes from $f_\alpha(x,\ldots)$. The equation $\nabla_x \cdots = 0$ for the critical point is the Bethe equation, and the vector $f_\alpha(x,\ldots)$ is called the *off-shell Bethe eigenvector*. Substituting in solutions x gives the eigenvector. Many people study these objects and it is useful to have a geometric realization of the off-shell Bethe eigenvector in terms of K-theoretic stable envelopes.

24.11

One can compute the integral representation (90) by residues, especially in specific examples. The singularities are in the weight function.

Example. For $\mathrm{Hilb}(\mathbb{C}^2, n)$, there are poles at

$$x_i = q^{d_i} t_1^{\cdots} t_2^{\cdots}$$

where d_i is the number of boxes stacked in the negative direction at x_i, and t_1, t_2 are equivariant weights of $\mathbb{C}^2$.

More generally, the weight function looks like

$$\frac{\cdots}{\varphi(1/x_i)\varphi(x_i/x_j)\varphi(qx_i/\hbar x_j)},$$

so in addition to the poles $x_i = q^{d_i}$ coming from the first two factors, there are poles coming from $\varphi(qx_i/\hbar x_j)$ that we don't want. The former poles are exactly Chern roots of $\mathcal{F} \otimes \mathcal{O}_0$. By an elliptic transformation,

$$g_\beta(q^{d_i}x) = z^{\sum d_i} g_\beta(x) = z^{\deg} g_\beta(x).$$

The poles we don't want are killed by g_β. This is because g_β is supported on the open subset $\mu^{-1}(0) \subset \mathfrak{X}$, meaning that $g_\beta\big|_{\mathfrak{X}\setminus\mu^{-1}(0)} = 0$; recall that

$$\mathrm{Ell}(\mathfrak{X} \setminus \mu^{-1}(0)) \hookrightarrow \mathrm{Ell}(\mathfrak{X})$$

is closed. The embedding generalize the "wheel conditions" for functions in $\mathrm{Ell}(\mathfrak{X})$. These are specific vanishing conditions, i.e. zeros of g_β, at exactly the poles we want to cancel away.

Lecture 25. The q-shift operator, via explicit localization

Difference equations in equivariant variables, twisted quasimaps, equivariant localization, formula for the K-theory class of the virtual tangent bundle, edge and vertex contributions in localization formulas, pure edge and q-Gamma functions, q-analog of the Iritani class, the degree of a twisted map, its relation to Kähler line bundles in elliptic cohomology.

25.1

Both ●——○ and (——○ require localization to define. On the other hand, objects like (——) and ●——) are still not exactly proper integrals, but it turns out they are rational functions in all variables. Most importantly, they are rational functions of the Kähler variable $z \in \operatorname{Pic}(X) \otimes \mathbb{C}^\times$ recording the degree of the quasimap. But both of these types of objects are born as series of the form

$$\sum_d z^d(\text{rational function})$$

because we sum over maps of all possible degrees, and then for each degree we take some Euler characteristic of some sheaf on a finite-dimensional variety, and in equivariant variables these will all be rational functions.

25.2

Recall that, up to normalization, ●——○ is basically the elliptic stable envelope. Today we will discuss (——○, which is the fundamental solution to a certain q-difference equation in all variables.

Remark. Proofs of all these statements require different geometric inputs at multiple different steps, and is a sign of a real synergy between geometry and representation theory. They do not appear to stem from a single geometric fact from which everything else follows by formal manipulations in quantum groups. For example, we will use localization and degeneration arguments, which are fairly general and “soft” arguments (modulo the technicalities of constructing virtual classes, etc.), but also we will use rigidity, which really requires a careful examination of the moduli space for the enumerative problem at hand.

25.3

Where do the difference equations come from? We begin with q-difference equations in equivariant variables, which were computed first in the current logic of the subject. Let $X = \text{prequotient} /\!\!/ \mathsf{G}$ be a Nakajima quiver variety, e.g. $\operatorname{Hilb}(\mathbb{C}^2, n)$. Let $\widetilde{\mathsf{G}}$ be a symmetry group acting on the prequotient, in which G is a normal subgroup, and equivariant variables live in the quotient G_{Aut} in

$$1 \to \mathsf{G} \triangleleft \widetilde{\mathsf{G}} \to \mathsf{G}_{\text{Aut}} \to 1.$$

Most of the time $\widetilde{\mathsf{G}}$ really is split, since we work with reductive groups. But it is good to remember what is a sub-object and what is a quotient.

In addition to counting quasimaps to X, we can also count *twisted* quasimaps $C \to X$, consisting of the data of:

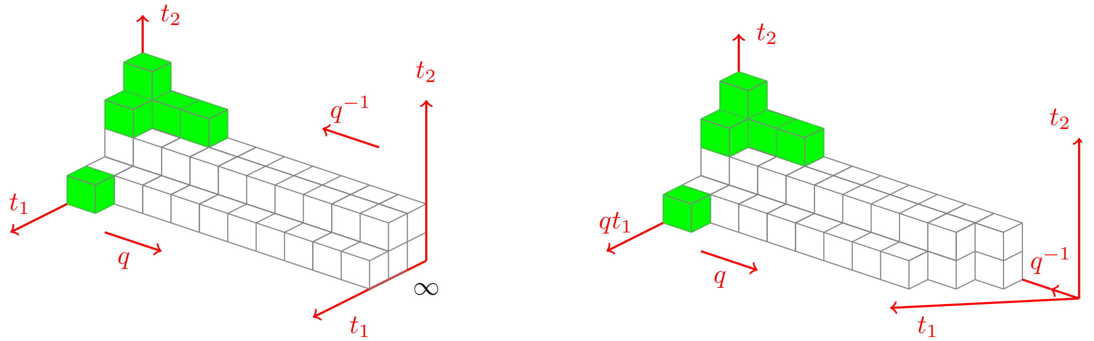
- a principal $\widetilde{\mathsf{G}}$-bundle $\widetilde{\mathcal{P}}$ (whose G_{Aut} structure is fixed);
- a section $C \to \widetilde{\mathcal{P}} \times_{\widetilde{\mathsf{G}}} (\text{prequotient})$.

It is stable if it evaluates to a stable point away from a finite set of points in C.

Example. Ordinary quasimaps $C \to \mathrm{Hilb}(\mathbb{C}^2, n)$ form the PT moduli space of $\mathbb{C}^2 \times C$. There is an action of $\mathrm{GL}(2)$ on $\mathbb{C}^2$ that can be used to twist this trivial product. Then

$$\Big(\text{twisted quasimaps } C \to \mathrm{Hilb}(\mathbb{C}^2, n)\Big) = \mathrm{PT}(Y \to C)$$

where Y is some twisted $\mathbb{C}^2$-bundle over C, e.g. $\mathcal{O}(1) \oplus \mathcal{O} \to \mathbb{P}^1$ as in Figure 41b (but drawn for DT, not PT configurations).



(a) For $\mathcal{O} \oplus \mathcal{O} \to \mathbb{P}^1$ (b) For $\mathcal{O}(1) \oplus \mathcal{O} \to \mathbb{P}^1$

Figure 41: An untwisted vs. twisted DT (1-leg) configuration.

In general, for non-trivially twisted quasimaps, we pick the linearization so that the twist is equivariantly trivial at infinity, and so there is a non-trivial q-shift in equivariant variables at 0. The shift happens by $\sigma(q)$ where $\sigma \colon \mathbb{C}^\times \to \widetilde{\mathsf{G}}$ is a cocharacter, and we use σ as a clutching function for a non-trivial X-bundle over $C = \mathbb{P}^1$.

25.4

We can now define the twisted versions $\bullet\!\overset{\sigma}{\text{——}}\!\circ$ and $(\!\overset{\sigma}{\text{——}}\!\circ$, and ask how they compare with the original untwisted counts. The first observation is that the combinatorics remain the same, and in fact there is an obvious bijection on the actual moduli spaces. However, the localization weights of fixed points are different, but in a very controllable way.

25.5

We briefly discuss what localization looks like, in particular the weights. For concreteness we focus on DT/PT theory, but in general objects can be in any category of "sheaves on 3-folds" or representations of quivers valued in vector bundles on a curve C. Quasimaps in general are of the latter kind. Let $\mathcal{E}$ be such an object. Then in general

$$T^{\mathrm{vir}}_{\mathcal{E}}(\text{moduli}) = -(\text{automorphisms}) + (\text{deformations}) - (\text{obstructions}) + \cdots .$$

We would like there to be no automorphisms for stable $\mathcal{E}$, and by some sort of Serre duality we would like all the higher obstructions $\cdots$ to also vanish. For example, for an object in a category of sheaves, this would concretely become

$$-\operatorname{Hom}(\mathcal{E},\mathcal{E}) + \operatorname{Ext}^1(\mathcal{E},\mathcal{E}) - \operatorname{Ext}^2(\mathcal{E},\mathcal{E}) + \cdots = -\chi(\mathcal{E},\mathcal{E}).$$

This is a very accessible object in K-theory. Namely, the K-theory class $[\chi(\mathcal{E},\mathcal{E})]$ only depends on the K-theory class $[\mathcal{E}]$ and not $\mathcal{E}$ itself.

For DT/PT, objects are $\mathcal{E} = \{\mathcal{O}_Y \xrightarrow{s} \mathcal{F}\}$ where $\mathcal{F}$ is the structure sheaf of a subscheme in DT theory or a pure 1-dimensional sheaf in PT theory. The formulas to be discussed will remain the same in either situation. Note that $[\mathcal{E}] = [\mathcal{O}_Y] - [\mathcal{F}]$ since in K-theory it doesn't matter what the map in the complex actually is. As we deform, the $\mathcal{O}_Y$ term remains fixed, so

$$\begin{aligned} T_{\text{vir}} &= \chi(\mathcal{O}_Y,\mathcal{O}_Y) - \chi(\mathcal{E},\mathcal{E}) \\ &= \chi(\mathcal{O}_Y,\mathcal{F}) + \chi(\mathcal{F},\mathcal{O}_Y) - \chi(\mathcal{F},\mathcal{F}). \end{aligned} \tag{92}$$

25.6

Now suppose there is a torus T acting on Y, and for simplicity assume it has isolated fixed points. The terms in (92) are Euler characteristics on Y, so they can be computed by localization in the equivariant K-theory of Y. Only fixed points contribute to this calculation, so

$$T_{\text{vir}} = \sum_{\text{fixed pt } p} \mathcal{T}(\mathcal{F}_p, \mathcal{F}_p)$$

for the appropriate function $\mathcal{T}$, where $\mathcal{F}_p$ is the restriction of $\mathcal{F}$ to the affine chart containing p. Concretely, for 3-folds Y, these affine charts are copies of $\mathbb{C}^3$.

Example. In Figure 41 there are two such charts, one around 0 and one around ∞, and e.g. $\mathcal{F}_\infty \in K_{\text{eq}}(\mathbb{C}^3)$. But everything in $K_{\text{eq}}(\mathbb{C}^3)$ is a multiple of the structure sheaf $\mathcal{O}_0$ (or of $\mathcal{O}_{\mathbb{C}^3}$; up to equivariant factors, it doesn't really matter). So to compute $\mathcal{F}_\infty$ it suffices to know weights of all boxes around ∞, and remembering that weights of functions are inverse to weights of coordinates,

$$\mathcal{F}_\infty = \mathcal{O}_0 \cdot \frac{1 + \cdots + t_1^{-1} t_2^{-1}}{1 - q}.$$

If we call $t_3 \coloneqq q^{-1}$ the weight of the third coordinate, the function $\mathcal{T}$ is then

$$\mathcal{T}(F) = F - t_1 t_2 t_3 \overline{F} - (1 - t_1)(1 - t_2)(1 - t_3) F \overline{F}.$$

To check this formula, it suffices by linearity to check that

$$\begin{aligned} \mathcal{T}(\square) &= 1 - t_1 t_2 t_3 - (1 - t_1)(1 - t_2)(1 - t_3) \\ &= t_1 + t_2 + t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3, \end{aligned}$$

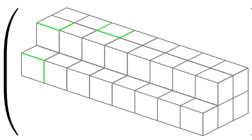
which is the correct virtual tangent space. In general, $\mathcal{T}(F)$ is a big rational function, and only after summing over fixed points does it become a Laurent polynomial. Explicitly, observe that all terms contain a denominator $1 - q^{-1}$.

25.7

A better idea is to think about single out the contributions of "pure edges", meaning that e.g. for Figure 41, instead of thinking of

$$T_{\text{vir}} = (\text{rational function for } 0) + (\text{rational function for } \infty),$$

we instead think

$$T_{\text{vir}} = T_{\text{vir}} \Bigg($$

$$\Bigg) + \text{polynomial}(F_0) + \text{polynomial}(F_\infty).$$

The first term is a "constant" map, in the sense that the section $C \to \widetilde{\mathcal{P}} \times_{\widetilde{\mathsf{G}}} X$ is the "constant" section, and we call it a *pure edge*. The polynomials come from $\mathcal{T}'(\mathcal{F}) \coloneqq \mathcal{T}(\mathcal{F}) - \mathcal{T}(\text{pure edge})$. Concretely, for example,

$$\mathcal{F}_{\text{at } 0} = \mathcal{F}_{\text{pure edge}} + (\text{finitely many boxes}),$$

where the second term corresponds to the colored boxes in Figure 41.

25.8

Suppose now that there is a twist by a cocharacter $\sigma(q)$. Then boxes at 0 are just shifted by $\sigma(q)$. However, the edge character changes non-trivially. Let $\lambda \in \text{Hilb}(\mathbb{C}^2, n)^\sigma$ be the partition of the edge; in particular, σ acts on $T_\lambda \text{Hilb}$. Then by localization,

$$T_{\text{vir}}(\text{pure edge}) = \frac{T_\lambda \text{Hilb}\big|_\sigma}{1 - q^{-1}} + \frac{T_\lambda \text{Hilb}}{1 - q}. \tag{93}$$

Each term is a geometric series, but overall most terms cancel and this is a Laurent polynomial. Recall that T_{vir} makes contributions to the virtual structure sheaf $\mathcal{O}_{\text{vir}}$ as

$$T_{\text{vir}} = \sum a_i \quad \rightsquigarrow \quad \mathcal{O}_{\text{vir}} = \prod \frac{1}{1 - a_i^{-1}}.$$

(We discuss the symmetrized $\widehat{\mathcal{O}}_{\text{vir}}$ later.) So in particular

$$T_{\text{vir}} = \sum \frac{a_i}{1 - q^{-1}} \quad \rightsquigarrow \quad \mathcal{O}_{\text{vir}} = \prod \Gamma_q(a_i^{-1}), \quad \Gamma_q(x) \coloneqq \prod_{i \geq 0} \frac{1}{1 - q^i x}.$$

We conclude that

$$\text{weight of the twisted edge} = \prod_{w \in \text{weight } T_\lambda X} \frac{\sigma \cdot \Gamma_q(w^{-1})}{\Gamma_q(q w^{-1})}$$

where the denominator comes from the other term in (93). In conclusion, a twist by σ affects equivariant weights of the pure edge by a factor

$$\prod_{w \in T_\lambda X} \frac{\sigma \cdot \Gamma_q(w^{-1})}{\Gamma_q(w^{-1})}.$$

This means it is a good idea to multiply vertices by the factor $\prod_w \Gamma_q(w^{-1})$, which is kind of like the "Iritani class" counting maps from $\mathbb{C} \to X$. Such a factor is completely analogous to the double gamma functions in instanton counting, which are "perturbative prefactors" corresponding to maps $\mathbb{C}^2 \to X$.

25.9

Finally, Figure 41b visibly has fewer boxes than Figure 41a in the pure edge part. This means the operator $z^{\deg \mathcal{F}} = z^{\#\text{ boxes}}$ changes. There are infinitely many boxes; here the "number of boxes" is

$$\deg \mathcal{F} \det \mathcal{F}_0 / \det \mathcal{F}_\infty,$$

defined with respect to the line bundle $\mathcal{O}(1)$ on Hilb. For a pure edge, this is

$$\deg \mathcal{F} = \langle \sigma, \mathrm{weight}(\mathcal{O}(1)\big|_\lambda) \rangle$$

where the pairing is of cocharacters and characters. The cocharacter in our running example is $(t_1, t_2) = (q, 1)$, and $\mathcal{O}(1)$ is always the product of all weights in λ, and from this we get the (visible) difference of 4 boxes.

We conclude that it is good to also multiply vertices by the factor

$$\exp\left(\frac{\ln z \cdot \ln \mathrm{weight}(\mathcal{O}(1))}{\ln q}\right), \tag{94}$$

or, more generally, $\exp\left(\sum_i \ln z \ln \mathcal{L}_i / \ln q\right)$ if $\dim \mathrm{Pic} > 1$. This expression has the same behavior under q-shifts of variables as the Poincaré line bundle $\mathcal{U}$ on $\mathrm{Ell}_{\mathrm{eq}}(X) \times (\mathrm{Pic}\, X \otimes_{\mathbb{Z}} E)$. Actually it suffices to put any expression that has the same q-shift behavior, e.g.

$$\frac{\vartheta(z)\vartheta(\mathcal{O}(1))}{\vartheta(z\mathcal{O}(1))} \tag{95}$$

would also work. In Gromov–Witten literature, it is more common to see expressions of the form (94) rather than (95).

Lecture 26. The geometric meaning of our q-difference equations

Geometric meaning of the operator in the q-difference equation in equivariant variables, q-difference equations in Kähler variables, quantum Knizhnik–Zamolodchikov equations.

26.1

Let $\mathsf{T} \subset \mathrm{Aut}(X)$ be the maximal torus, with coordinate denoted t. For a one-parameter subgroup $\sigma\colon \mathbb{C}^\times \to \mathsf{T}$, we are comparing a twisted quasimap invariant, e.g. $\overset{\sigma}{(\!\!-\!\!\!-\!\!\circ}$, with an untwisted one, e.g. $(\!\!-\!\!\!-\!\!\circ$. Last time we discussed that in fact they are equal up to some modifications.

$$\overset{\sigma}{(\!\!-\!\!\!-\!\!\circ} \quad = \quad \overset{\sigma}{(\!\!-\!\!\!-\!\!\circ}\Big|_{t\mapsto \sigma(q)t} \cdot \frac{\sigma\cdot\Gamma_q}{\Gamma_q} z^{\deg}. \tag{96}$$

Recall that the q-Gamma terms come from the difference in the pure edge, and deg here is the degree of the "constant" map.

26.2

In general, a one-parameter subgroup $\sigma\colon \mathbb{C}^\times \to \mathrm{Aut}(Y)$ gives a twisted Y-bundle over $\mathbb{P}^1$ using σ as a clutching function, and σ-fixed points in Y give constant section. For example, if $Y = \mathbb{P}^1$ then the resulting bundle $\widetilde{Y}$ is a Hirzebruch surface. The two σ-fixed points in Y induce sections s_0 and s_∞ whose normal bundles are $\mathcal{O}(k)$ and $\mathcal{O}(-k)$, so that

$$[s_0] - [s_\infty] = k[\text{fiber } \mathbb{P}^1].$$

One way to think about this relation in homology is that degrees of curves are measured by pairing them with line bundles, i.e. by viewing curve classes as elements $[\text{curve}] \in \mathrm{Pic}^\vee$. There is a short exact sequence

$$0 \to \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(\widetilde{Y}) \to \mathrm{Pic}(Y) \to 0$$

where the first term measures degree in the $\mathbb{P}^1$ direction and the quotient measures degree in the fiber direction. Then

$$\left\langle \mathcal{L}, \begin{array}{c}\text{constant section}\\ y \in Y^\sigma\end{array} \right\rangle = \langle \sigma, \mathcal{L}\big|_y\rangle. \tag{97}$$

In our example, $\mathrm{Pic}(\mathbb{P}^1) = \mathrm{Pic}(Y) = \mathbb{Z}$ and we pick the linearization $\mathcal{O}(1)\big|_0 = q$ and $\mathcal{O}(1)\big|_\infty = 1$. Since $\sigma(z) = z^k = q$, the pairing (97) differs by k between 0 and ∞.

The moral of the story is that even constant sections have non-trivial degrees, and different constant sections may have different degrees.

26.3

Returning to (96), we can equally well degenerate the twisted quasimap to get

$$\overset{\sigma}{(\!\!-\!\!\!-\!\!\circ} \quad = \quad \overset{\sigma}{(\!\!-\!\!\!-\!\!)}\ \mathrm{Glue}^{-1}\ (\!\!-\!\!\!-\!\!\circ. \tag{98}$$

In full, not using our shorthand, let $\Psi := \Psi(z,t,q) :=$ $(\!\!-\!\!\!-\!\!\circ$. The combination of (96) and (98) says that

$$\Psi(z,\sigma(q)t,q)\frac{\sigma\cdot\Gamma_q}{\Gamma_q} z^{\langle\sigma,-\rangle} = S_\sigma(z,t,q)\Psi(z,t,q)$$

for the operator

$$S_\sigma(z,t,q) \coloneqq \overset{\sigma}{(\!\!-\!\!-\!\!-\!\!)}\ \mathrm{Glue}^{-1}\,.$$

This is the q-difference equation in equivariant variables. What is remarkable and difficult to prove is that $S_\sigma(z,t,q)$ is a rational function in all variables. So Ψ solves a q-difference equation with rational coefficients.

26.4

A better way to put it is that Ψ intertwines an interesting q-difference connection defined by S_σ (and other operators) with the abelian connection corresponding to the terms $z^{\deg}(w\cdot\Gamma_q)/\Gamma_q$. This abelian connection is very closely related to the "attractive" line bundles $\mathcal{S} = \Theta(T_X^{1/2})\otimes\mathcal{U}$ in the discussion of elliptic stable envelopes. The relation is that sections of $\mathcal{U} = \Theta(\sum(z_i - 1)(\mathcal{L}_i - 1))$ are of the form

$$\prod \frac{\vartheta(z_i\mathcal{L}_i)}{\vartheta(z_i)\vartheta(\mathcal{L}_i)},$$

and a q-shift in $\mathcal{L}_i$ produces the terms z_i^{-1} in $z^{\deg}$. The q-Gamma functions arise from

$$\Gamma_q(w^{-1})\Gamma_q(\hbar w) \approx \vartheta(w).$$

So far we have not done much real work; any matrix trivially satisfies a q-difference equation where the q-difference operator is given by the ratio of the two sides. The real work lies in computing the operator S_σ.

26.5

Now we discuss the q-difference equation in Kähler variables. It will arise in a very similar manner. Consider an equivariant line bundle $\mathcal{L}$ on $\mathbb{P}^1$, for the action of $\mathbb{C}_q^\times$. Localization says

$$c_1(\mathcal{L}\big|_0) - c_1(\mathcal{L}\big|_\infty) = \deg(\mathcal{L})c_1(T_0\mathbb{P}^1)$$

for the equivariant Chern class c_1. Another way of writing the same thing is the K-theoretic version

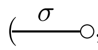
$$\mathcal{L}\big|_0 \otimes \mathcal{L}\big|_\infty^{-1} = (T_0\mathbb{P}^1)^{\deg\mathcal{L}}.$$

If $\mathcal{F}$ is any sheaf on $\mathbb{P}^1$, then $\det\mathcal{F}$ is a line bundle and $\deg\mathcal{F} = \deg\det\mathcal{F}$, and

$$q^{\deg\mathcal{F}} = \frac{\det\mathcal{F}\big|_0}{\det\mathcal{F}\big|_\infty}.$$

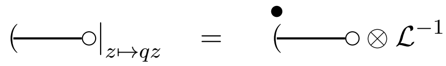
This is the same formula as (97) for the degree of the constant map. Consequently, if $\mathcal{M}$ is a moduli of maps from $\mathbb{P}^1$, then

$$\begin{aligned}&\chi\left(\mathcal{M}, z^{\deg}\widehat{\mathcal{O}}_{\mathrm{vir}}\otimes(\text{insertions})\right)\Big|_{z\mapsto qz}\\ &= \chi\left(\mathcal{M}, z^{\deg}\widehat{\mathcal{O}}_{\mathrm{vir}}\otimes(\text{insertions})\otimes\det(\mathcal{F}\big|_0)\otimes\det(\mathcal{F}\big|_\infty)^{-1}\right).\end{aligned}$$

Now if $\mathcal{M}$ is a moduli of maps like , then the contribution at ∞ is pulled back via the evaluation map, namely

$$\mathcal{F}\big|_{\infty} = \mathcal{F} \otimes \mathcal{O}_{\infty}.$$

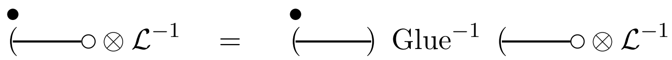
In other words,



where $\mathcal{L}$ is the line bundle on X corresponding to the variable z, and the dot $\bullet$ indicates an insertion $\det(\mathcal{F} \otimes \mathcal{O}_0)$.

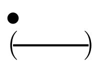
26.6

We can now play the same game as for the q-difference equation in equivariant variables. By degeneration,



Equivalently,

$$\Psi(q^{\mathcal{L}} z, t, q) \otimes \mathcal{L} = M_{\mathcal{L}} \Psi(z, t, q)$$

where $M_{\mathcal{L}} :=$  Glue^{-1} is the q-difference operator in Kähler variables. Its main feature is that it commutes with the q-difference connection in equivariant variables, i.e. both are satisfied by the same function Ψ. It is very rare to have commuting q-difference connections, and it means that $M_{\mathcal{L}}$ is a solution to a q-difference equation. Either of $M_{\mathcal{L}}$ or S_σ will uniquely determine the other. In the current logic, in fact $M_{\mathcal{L}}$ is determined starting from S_σ.

26.7

The overarching goal is to describe the flat q-difference connection, formed from $M_{\mathcal{L}}$ and S_σ, in terms of quantum groups. The plan is to start with particular q-difference equations for certain special equivariant variables.

Recall that $K(T^* \operatorname{Gr}(k,n))$ is the weight-k subspace in $\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$ as $U_\hbar(\widehat{\mathfrak{gl}}_2)$-modules, where the $\mathbb{C}^2(a_i)$ are evaluation representations, and

$$\operatorname{diag}(a_1, \ldots, a_n) \in \mathrm{GL}(n) = \mathrm{GL}(W)$$

acts on $T^* \operatorname{Gr}(k,n)$ by changes of framing (in the language of Nakajima quiver varieties). On this tensor product of evaluation representations, there is a *quantum Knizhnik–Zamolodchikov* (*qKZ*) connection which moves the a_i around. Clearly, it must be the connection we want. This is indeed true.

More generally, let Q be an arbitrary quiver, and $\mathcal{M}_Q(\mathbf{v}, \mathbf{w})$ be its associated Nakajima quiver variety. Then $K(\mathcal{M}_Q(\mathbf{v}, \mathbf{w}))$ is the (integral form of the) weight-$\mathbf{v}$ subspace in

$$\bigotimes_i \bigotimes_{j=1}^{w_i} \begin{pmatrix} i\text{-th fundamental} \\ \text{evaluation rep} \end{pmatrix} (a_{ij})$$

where a_{ij} are coordinates in $\mathrm{GL}(W) = \prod_i \mathrm{GL}(W_i)$.

Theorem. *The q-difference equation in the equivariant variables a_{ij} is exactly qKZ.*

26.8

What is qKZ? There are two ingredients:

- a collection of unitary R-matrices, meaning R-matrices R which satisfy $R_{21}(u^{-1})R_{12}(u) = 1$ along with Yang–Baxter;
- an operator Z such that $[Z \otimes Z, R] = 0$.

For example, take Z to be an element in the maximal torus of the quantum group. For general quantum groups, only the elements in the maximal torus stay group-like, i.e. $Z \otimes Z = \Delta Z = \Delta^{\text{op}} Z$, and since R takes Δ to Δ^{op} it follows automatically that R commutes with ΔZ.

From the R-matrices we get a representation of the type-GL(ℓ) affine Weyl group acting in $V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_\ell(a_\ell)$. Transpositions are given by the operator

$$\check{R} \coloneqq (12) \circ R.$$

Unitarity says $(\check{R})^\vee = 1$, and Yang–Baxter implies the Coxeter relation. This gives an action of the symmetric group $S(\ell)$. We think of the $V_i(a_i)$ as quantum spaces at sites of a 1d lattice. To get the *affine* Weyl group W_{aff}, we make the lattice periodic like in Figure 42, like a periodic spin chain. In fact we should make it *quasi*-periodic using the operator Z, and since $[Z \otimes Z, R] = 0$ the result is still a representation of W_{aff}. In the analogy with spin chains, in e.g. the XXZ spin chain we can always insert operators like $Z = \text{diag}(1, z)$.

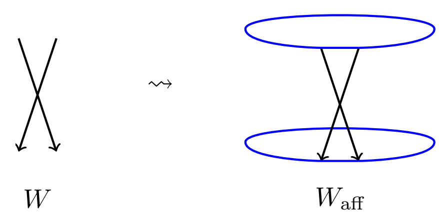


Figure 42: R-matrices on a periodic lattice yield the affine Weyl group.

It is better to think about the extended affine Weyl group $\widetilde{W}_{\text{aff}} \coloneqq S(\ell) \ltimes \mathbb{Z}^\ell$ where $\mathbb{Z}^\ell$ is a lattice of the commuting operators given by taking the ℓ-th strand and moving it all the way around the circle.

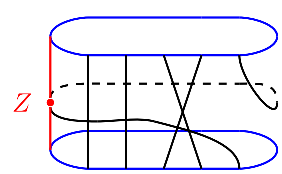


Figure 43: Lattice translations in the extended affine Weyl group.

26.9

To produce commuting q-difference operators, we modify the operator Z and define

$$Z_{\text{new}} V_i(a_i) \coloneqq [Z_{\text{old}} V_i](q a_i).$$

This Z_{new} still commutes with the R-matrix, since the R-matrix

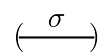
$$R_{V_1(a_1), V_2(a_2)} = R_{V_1, V_2}(a_1/a_2)$$

depends only on the ratios between the a_i.

We may as well treat only the case of a tensor product of two representations, because in any tensor product $V_1(a_1) \otimes \cdots$ we can set $V_2(a_2) \coloneqq \cdots$. Explicitly, the commuting q-difference operators are

$$\Psi(q a_1, a_2) = (z \otimes 1) R_{12}(a_1/a_2) \Psi(a_1, a_2). \tag{99}$$

This is qKZ. The maximal torus of our quantum groups $U_\hbar(\widehat{\mathfrak{g}})$ (modulo central elements) is $\operatorname{Pic}(X) \otimes \mathbb{C}^\times$, which is the home of the Kähler variables z.

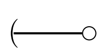
How can something like this be the q-shift operator  Glue^{-1}, where we set $\sigma \coloneqq (q, 1, \ldots)$? This quasimap count (neglecting the glue operator) contains a summation over all degrees with $z^{\deg}$, so in particular it is a series in z, whereas (99) is just a monomial! It would be easier to explain if it were the case that actually (99) were independent of z; then the statement would be that all quantum corrections vanish. The actual explanation is that only "constant" maps contribute, if everything is normalized correctly, but they are constant maps to a non-trivial geometry.

Lecture 27. Singularities of q-shift operators and minuscule cocharacters

Singularities of the difference equations in equivariant variables, minuscule cocharacters and their geometric meaning, quantum Knizhnik–Zamolodchikov equations for shifts by minuscule cocharacters.

27.1

Let's recap. We have an operator

 $\in \operatorname{End}(K_{\text{eq}}(X)_{\text{loc}})$

which depends on equivariant variables in $\mathsf{T} \subset \operatorname{Aut}(X)$ and Kähler variables in $\mathsf{Z} \coloneqq \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times$. This operator is the fundamental solution for a flat q-difference connection in both sets of variables. Specifically, let

$$1 \to \mathsf{A} \to \mathsf{T} \xrightarrow{\hbar} \mathbb{C}^\times \to 1$$

where $\hbar$ is the weight of the symplectic form ω_X. Then the shifts in equivariant variables are only for variables in A, and we do not attempt to shift $\hbar \mapsto q\hbar$. Such a q-difference equation in $\hbar$ is possible and very complicated, e.g. like intertwining operators for $t \mapsto tq$ in Macdonald–Cherednik theory, or like $U_\hbar \mapsto U_{q\hbar}$, and it is unclear what the application would be.

27.2

We begin in our discussion of qKZ by asking where its singularities are. These singularities are q-periodic, so it is good to ask this question mod q and work on $\mathsf{A}/q^{\cdots} = \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})$. The q-periodicity is because for a q-difference equation in one variable

$$\Psi(qa) = S(a,q)\Psi(a),$$

singularities are points where $S(a,q) \notin \mathrm{GL}(n)$, and solutions $\Psi(a)$ will therefore have singularities at $a, qa, q^2a, \ldots$. More generally, the singular locus is a periodic arrangement of hyperplanes, i.e. shifts of

$$\mathrm{Ell}_{\mathsf{A}'}(\mathrm{pt}) \subset \mathrm{Ell}_{\mathsf{A}}(\mathrm{pt}) \quad \mathrm{codim}_{\mathsf{A}}\, \mathsf{A}' = 1.$$

We have seen such arrangements before, given by points $a \in \mathsf{A}$ where the fixed locus $X^a \neq X^{\mathsf{A}}$. Another description comes from looking at the normal bundle for the inclusion $X^{\mathsf{A}} \subset X$. Let $\{w_i\}$ be the A-weights in $N_{X/X^{\mathsf{A}}}$. Then the hyperplane arrangement, viewed in $\mathrm{Lie}\,\mathsf{A}$ instead of $\mathrm{Ell}_{\mathsf{A}}(\mathrm{pt})$, consists of hyperplanes ξ such that $\langle \xi, w_i\rangle \in \mathbb{Z}$, since then $a = e^{2\pi i\xi}$ acts trivially on some normal direction.

To be more precise, weights in $N_{X/X^{\mathsf{A}}}$ appear in pairs w and $1/w_i\hbar$. So hyperplanes come in pairs as well: for every hyperplane $wq^n = 1$ there is another one $w\hbar q^n = 1$. It is good to think $\hbar \approx 1$, since at exactly $\hbar = 1$ all poles and zeros cancel, and the pair of hyperplanes annihilate each other. Hence, while none of our theory of stable envelopes, difference equations, etc. interacts directly with $\hbar$, it is nonetheless very important that $\hbar$ is there in order to get nontrivial objects. The same consideration is true at $\hbar = q^n$. For example, recall that the vertex function for $\mathbb{P}^1$ is

$$\sum_d z^d \frac{(\hbar)_d(\hbar a)_d}{(q)_d(qa)_d},$$

which has no zeros or poles at $\hbar = q$, and more generally at $\hbar = q^n$ it becomes some rational function.

The weights $\{w_i\}$ of $N_{X/X^{\mathsf{A}}}$ which define the hyperplane arrangement are called *equivariant roots*. Analogously, on the Kähler side, i.e. for q-difference equations in Kähler variables, there will be *Kähler roots*. These will live in characters of Z, which is just $H_2(X,\mathbb{Z})$. The two sets of roots are swapped by 3d mirror symmetry.

27.3

Among all difference equations in equivariant variables, we want to find qKZ. By previous discussion, it is enough to do this for two variables, where qKZ looks like (99). The qKZ operator $(z \otimes 1)R_{12}(a_1/a_2)$ arose geometrically as

$$S_\sigma := \left(\xrightarrow{\;\sigma\;}\right)\ \mathrm{Glue}^{-1}$$

counted with $z^{\deg}$ as usual, with $\sigma := (q, 1)$. In the result, however, z appears only as an overall factor, as a *monomial*; this is very surprising. More importantly, there is no q at all! In classical treatments of the KZ equation, e.g. [FR92], q arises as $q \approx \hbar^{\text{level}}$. But for us, q is not related to $\hbar$. This means that in the q-difference equation, Ψ only has singularities at $q^n b = 1$ for a particular weight b. Typically this is not the case, since if some equivariant root w has pairing $\langle w, \sigma\rangle = m$, meaning that $w(\sigma(b)) = b^m$, then there are singularities at $q^n b^m = 1$ which involve m-th roots of q. Hence it must be the case that

$$|\langle w, \sigma\rangle| \leq 1.$$

Such a condition on cocharacters σ is very well-known in Lie theory: these σ are called *minuscule*.

27.4

For us, it is easier to establish a different, equivalent condition on σ. As an example, think of $X = T^*\mathbb{P}^1$, where σ must act with weights as in Figure 44. In the projection $X \to X_0$, some of these weights may become trivial, from the collapsed $\mathbb{P}^1$, but some other weights will remain. The condition on σ is therefore that X_0 embeds into a vector space where σ acts with weights $\pm 1, 0$. Specifically, functions on this vector space generate $H^0(X_0, \mathcal{O}_{X_0}) = H^0(X, \mathcal{O}_X)$. Hence we arrive at a equivalent, geometric condition.

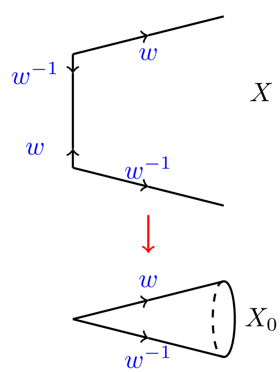


Figure 44: Weights of any cocharacter σ acting on $T^*\mathbb{P}^1$ and its affinization.

Definition. The cocharacter σ is *minuscule* if $\mathbb{C}[X]$ is generated by functions of weight $-1, 0, 1$ with respect to σ.

The fundamental theorem of invariant theory usually directly gives us generators for $\mathbb{C}[X]$, so this geometric condition is much easier to check. As an exercise, show that cocharacters of the form

$$\sigma(b) := \mathrm{diag}(b, b, \ldots, b, 1, 1, \ldots, 1) \in \mathrm{GL}(W)$$

are minuscule. (These are exactly the minuscule cocharacters for $\mathrm{GL}(n)$.)

27.5

This is not yet a proof, but at least for such σ the operator S_σ has a chance to be qKZ. The difficult and more mysterious part is to show that only "constant" quasimaps contribute to S_σ, so that the result is monomial in z. All other contributions must somehow vanish. The way to prove such vanishing is by rigidity, which recall involves two steps: a properness argument, and a bound on weights. Properness tells us the answer is a Laurent polynomial, and boundedness tells us the Newton polygon of that Laurent polynomial contains no lattice points. The minuscule condition on σ is also important for properness.

27.6

Example. We focus on a (somewhat silly) example for clarity. Let $X = \mathbb{C}^2$ be the simplest possible symplectic manifold, and let

$$\sigma(b) \coloneqq \operatorname{diag}(b^k, b^{-k}) \subset \operatorname{Aut}(X, \omega)$$

for $k > 0$. A σ-twisted quasimap to X is the bundle

$$\widetilde{X} \coloneqq \mathcal{O}(k) \oplus \mathcal{O}(-k) \to \mathbb{P}^1.$$

Sections of $\widetilde{X}$ come only from $\mathcal{O}(k)$, but the space of sections is a $(k+1)$-dimensional vector space, which is clearly bad for properness. But thinking of such counts as forming an operator, taking a matrix element of the operator means to impose some conditions on the sections at 0 and ∞, e.g. that they vanish at both 0 and ∞. Then the space of sections has dimension

$$\dim H^0(\mathcal{O}(k - [0] - [\infty])) = k - 1.$$

Hence if $k = 1$ exactly, i.e. the minuscule condition, then there are no non-zero sections, and the space of sections is now proper.

This example is not great when we start counting weights in the second part of the rigidity argument, to show boundedness. For example

$$\mathcal{O}_{\text{pt}} = (1 - b^{-1})(1 - b)$$

and its Newton polygon is $[-1, 1]$, which clearly contains lattice points. The solution is to impose *different* conditions at 0 and ∞; instead of using a single point, we use the repelling directions Attr_-. The resulting space of sections is still proper, since there are no sections in the Attr_- direction anyway where normal bundles have negative degree.

27.7

For general X, the map $X \to X_0$ is proper and by hypothesis there is an embedding (also proper) of X_0 into a vector space of weights $0, \pm 1$. Hence σ-twisted quasimaps to X have a proper evaluation map to sections of some bundle

$$\mathcal{O}(1)^{\cdots} \oplus \mathcal{O}(-1)^{\cdots} \oplus \mathcal{O}^{\cdots}.$$

From our silly example we know the minuscule condition implies properness here. Hence we get properness for σ-twisted quasimaps. In general, the Attr_- conditions at 0 and ∞ in the silly example should be replaced by stable envelopes.

Theorem. *The (composite) operator*

$$\begin{array}{ccc} K_{\mathrm{eq}}(X^\sigma) & & K_{\mathrm{eq}}(X^\sigma) \\ \downarrow{\scriptstyle \mathrm{Stab}_-} & & \uparrow{\scriptstyle (\mathrm{Stab}_-)^{transpose}} \\ K_{\mathrm{eq}}(X) & \xrightarrow[S_\sigma]{} & K_{\mathrm{eq}}(X) \end{array}$$

is proper, i.e. defined in non-localized K-theory, and vanishes for non-constant quasimaps.

The K-theoretic stable envelope depends on a *slope* parameter which comes from the Kähler variables in the elliptic setting. The slopes in the theorem must be chosen with care.

Proof. We have essentially proved properness. To show vanishing it remains to bound the weights, see [Oko17] for details. □

Recall that $\mathrm{Stab}_-^{\mathrm{transpose}} = (\mathrm{Stab}_+)^{-1}$, and the R-matrix is $\mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$. Hence for constant quasimaps, we get exactly qKZ.

27.8

What about the q-difference equations for all other non-minuscule shifts? For minuscule shifts, the picture is that there is one wall-crossing (modulo q) from $X^\sigma \to X \to X^\sigma$, and therefore the shift operator is $R_\sigma = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$. For general σ, there is therefore a natural conjecture that

$$S_\sigma = \overrightarrow{\prod} (\text{R-matrices for } X^a)$$

where the product is over all root hyperplanes crossed by σ. These X^a are also Nakajima quiver varieties. Good progress has been made toward this conjecture in [KS20].

Lecture 28. The dynamical groupoid of q-difference operators

q-difference equations in Kähler variables, Dubrovin connection for Nakajima varieties, dynamical groupoids, slope R-matrices in equivariant K-theory, Khoroshkin–Tolstoy factorization of R-matrices, slope subalgebras in quantum loop groups, fusion operators J for slope subalgebras, dynamical groupoid associated to slope subalgebras and quantum q-difference equiations for Nakajima varieties.

28.1

Today we discuss the q-difference equation in Kähler variables. The object

$$(\!\!\!-\!\!\!-\!\!\!-\!\!\circ \in \operatorname{End} K_{\mathrm{eq}}(X)_{\mathrm{loc}}[[z]]$$

is a fundamental solution. In fact the solution is *rational* in z and works for all birational models X_{flop} of X, e.g. for different choices of GIT stability, by performing different series expansions in z. The $[[z]]$ should be viewed as the semigroup ring of $H_2(X, \mathbb{Z})_{\mathrm{effective}}$.

28.2

Let $\mathcal{M}(v, w)$ be a Nakajima quiver variety, and abbreviate $\mathcal{M}(w) \coloneqq \bigsqcup_v \mathcal{M}(v, w)$. Then the K-theory of $\mathcal{M}(w)$ is a module over a quantum group, and each $\mathcal{M}(v, w)$ gives a weight space. Recall that $\mathcal{M}(w) \times \mathcal{M}(w')$ is a component of the fixed locus of $\mathcal{M}(w + w')$, for the action of some $\mathbb{C}^\times$ whose coordinate we call a. There is therefore a stable envelope map

$$K(\mathcal{M}(w)) \otimes K(\mathcal{M}(w')) \xrightarrow{\text{Stab}} K(\mathcal{M}(w + w')),$$

making the collection $\{K(\mathcal{M}(w))\}$ into a tensor category. We know that $(\!\!\!-\!\!\!-\!\!\!-\!\!\circ$ is a solution of q-difference equations in both z and a, and in particular the q-difference equation in a is just qKZ. The claim is that this information determines the q-difference equation in z up to a scalar multiple. The general principle is that (almost) nothing commutes with R-matrices, meaning that our tensor category has very few tensor automorphisms.

28.3

The scalar multiple for the q-difference equation in z is actually very interesting, even in cohomology. In cohomology qKZ becomes the quantum differential equation, also known as the Dubrovin connection. This connection lives on a trivial $H^*_{\mathrm{eq}}(X)$-bundle over $H^2(X) = \operatorname{Pic}(X) \otimes \mathbb{C} = \operatorname{Lie}(\mathsf{Z})$. If λ denotes the coordinate on the base, then for a curve class z^α

$$\frac{d}{d\lambda} z^\alpha = (\lambda, \alpha) z^\alpha$$

The Dubrovin connection is then $d/d\lambda - (\lambda \star -)$ where

$$(\lambda \star -) = (\lambda \cup -) + \hbar \sum_{\substack{\text{effective} \\ \text{roots } \alpha}} (\lambda, \alpha) \frac{z^\alpha}{1 - z^\alpha} C_\alpha + (\text{constant}) \tag{100}$$

is quantum multiplication by λ. Note that the term $z^\alpha/(1 - z^\alpha)$ is the contribution of multiple covers. The term C_α is some Steinberg correspondence, which can be written in terms of the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$$

from the classical r-matrix. Recall that elements of $\mathfrak{h}$ act by linear functions of v and w, corresponding to classical multiplication by elements of $\operatorname{Pic}(X) \otimes \mathbb{C}$ and central elements respectively. A root α is effective iff $\theta \cdot \alpha > 0$, e.g. if $\theta = (1, 1, \ldots, 1)$ is the stability condition then this really is the condition for a root to be positive. The Steinberg correspondence C_α is the Casimir element in $\mathfrak{g}_{-\alpha}\mathfrak{g}_\alpha$. The cup product $(\lambda \cup -)$ is the Yangian deformation of $\lambda t \in \mathfrak{h}[t]$.

In cohomology, the constant in (100) is determined by the condition that

$$\lambda \star 1 = \lambda,$$

i.e. quantum corrections kill $1 \in H^0(X)$. (In terms of the quantum group, 1 is a "Whittaker" vector.) For example, for the moduli of rank-r instantons, the constant vanishes if $r > 1$.

In comparison, in K-theory:

- sums become products;
- roots of $\mathfrak{g}$ are promoted to roots of $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{Q}[t^\pm]$;
- the situation is best phrased in terms of the *dynamical groupoid*.

28.4

Consider a periodic arrangement of rational hyperplanes, given by $\langle \beta, - \rangle + n = 0$ in e.g. $\mathrm{Lie}(\mathsf{A})$. Equivalently this is an arrangement of translates of codimension-1 subvarieties in an abelian variety, given by $a^\beta q^n = 1$ in e.g. A or $\mathrm{Ell}_\mathsf{A}(\mathrm{pt})$. For every wall w, we would like an operator $B_w \in \mathrm{End}(\text{vector space})$ associated to w such that the following hold.

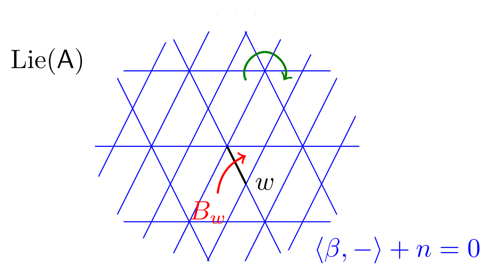


Figure 45: A periodic hyperplane arrangement in $\mathrm{Lie}(\mathsf{A})$. The green arrow indicates one automorphism of the arrangement, and one wall w with its operator B_w is drawn.

- $\{B_w\}_w$ satisfies the braid relations $\prod_{\text{loop}} B_w = 1$, where the product is over walls crossed by any closed loop. An example is the Yang–Baxter equation with spectral parameter.
- The automorphism group Aut of the arrangement, which by periodicity always includes translation by the lattice, acts on $\{B_w\}$ by

$$B_w(\tau^{-1} \cdot a) = O(\tau) B_{\tau \cdot w}(a) O(\tau)^{-1}$$

for an automorphism τ. An example is the arrangement in Figure 46a, which has $\mathrm{Aut} = S_3 \rtimes \mathbb{Z}^2$. This is slightly bigger than the affine Weyl group $W_{\mathrm{aff}}(\widehat{\mathfrak{gl}}_3)$, by the outer automorphism of the Dynkin diagram which rotates the whole diagram. Previously we already saw an action of Aut on q-difference operators, e.g. the automorphism in Figure 46a corresponds to Figure 46b. The operator Z corresponds to conjugation by $O(\tau)$.

(a) The element of Aut

(b) The element of W_{aff}

Figure 46: An automorphism of the arrangement as an affine Weyl group element.

28.5

Such a structure gives rise to a q-difference connection in the following way. The subgroup of translations in Aut really acts by q-shifts, since if τ_λ is translation by a lattice element λ then

$$B_w(\tau_\lambda^{-1} a) = B_w(q^{-\lambda} a).$$

Hence pick a path which goes to infinity from any given alcove (in a sufficiently generic way), and consider the formal product

$$\Psi \coloneqq \overrightarrow{\prod} B_w \coloneqq B_{w_1} B_{w_2} B_{w_3} \cdots$$

of operators at each wall crossed by the path. By the braid relations, this product is independent of the choice of path. Completely formally, it satisfies a q-difference equation of the form

$$\tau_\lambda \Psi O(\lambda) = \underbrace{O(\lambda) B_{w_1} \cdots B_{w_k}}_{M_\lambda} \Psi \tag{101}$$

where $B_{w_1} \cdots B_{w_k}$ is a path which connects the original alcove to its translate under λ.

28.6

Last time we discussed the conjecture that, for the equivariant torus A, there is a hyperplane arrangement $\{a \in \mathsf{A} : X^a \neq X^{\mathsf{A}}\}$ and the general q-difference equations in equivariant variables is a product of R-matrices for X^a (decorated by z like in qKZ and with an appropriate

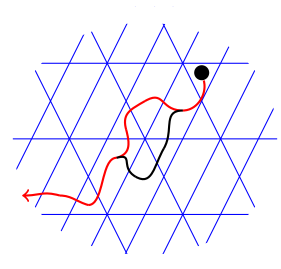


Figure 47: Two paths to infinity from the same alcove, differing by a single braid relation.

shift by q). Today we saw an analogous picture for the torus $\mathsf{Z} := \mathrm{Pic}(X) \otimes \mathbb{C}^\times$, where the hyperplane arrangement consists of slopes where stable envelopes jump. Recall that stable envelopes in elliptic cohomology are sections of $\Theta(T^{1/2}) \otimes (\text{deg zero})$ on $\mathrm{Ell}_{\mathrm{eq}}(X)$, which is a union of abelian varieties $\mathsf{T}/q^{\mathrm{cochar}(\mathsf{T})}$. In the limit $q \to 0$, these abelian varieties become a periodic gluing of toric varieties and assemble into $\bigcup_t \operatorname{Spec} K_{\mathrm{eq}}(X^t)$. Stable envelopes in K-theory are therefore polynomials with Newton polygon the same as the Newton polygon of

$$\wedge^\bullet(T^{1/2}) + (\text{shift})$$

where the shift is by something in $\mathrm{Pic}(X) \otimes \mathbb{R}$, called the *slope*, and comes from the degree-zero line bundle on $\mathrm{Ell}_{\mathrm{eq}}(X)$. The lattice points contained in the Newton polygon are locally constant with respect to changes of slope, and places in $\mathrm{Pic}(X) \otimes \mathbb{R}$ where they change are the walls.

Definition. Given two alcoves $s, s' \in \mathrm{Pic}(X) \otimes \mathbb{R}$ separated by a single wall, define the *wall R-matrix*

$$\mathrm{Stab}_{s'}^{-1} \circ \mathrm{Stab}_s .$$

It satisfies the Yang–Baxter equation, and the spectral variable $a \in \mathsf{A}$ only appears in a trivial way as a monomial of a given degree. Hence its define a quantum group $U_\hbar(\mathfrak{g}_w)$ which is *not* a quantum loop group.

The old R-matrix, corresponding to changing attracting directions, e.g. $\mathfrak{C} \to -\mathfrak{C}$, has a Khoroshkin–Tolstoy factorization

$$R_s^{-\mathfrak{C}\leftarrow\mathfrak{C}} = \prod_{\substack{\text{path from} \\ -\infty \text{ to } s}} R_{s''\leftarrow s'}^{-\mathfrak{C}} \cdot R_{-\infty\leftarrow\infty}^{-\mathfrak{C}\leftarrow\mathfrak{C}} \cdot \prod_{\substack{\text{path from} \\ s \text{ to } \infty}} R_{s''\leftarrow s'}^{\mathfrak{C}}. \tag{102}$$

The middle operator $R_{-\infty\leftarrow\infty}^{-\mathfrak{C}\leftarrow\mathfrak{C}}$ is an explicit product of q-Gamma functions. The choice of the overall path from $-\infty$ to s to ∞ is like the choice of factorization of the longest element of the Weyl group W, e.g. as in Lusztig's work on quantum groups.

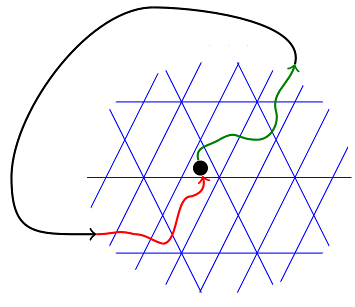


Figure 48: Paths to infinity and back, forming the Khoroshkin–Tolstoy factorization of the R-matrix.

28.7

The factorization implies that each *wall sub-algebra* $U_\hbar(\mathfrak{g}_w)$ embeds into $U_\hbar(\widehat{\mathfrak{g}})$.

Example. A familiar example may be $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$, where $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$. Here roots are just $\{\beta\} = \mathbb{Z}$, and hyperplanes in $\mathrm{Pic}(X) \otimes \mathbb{R}$ are therefore $\{x : \beta x + n \in \mathbb{Z}\} = \mathbb{Q}$. All but finitely many wall R-matrices act trivially in $K_{\mathrm{eq}}(\mathrm{Hilb}(\mathbb{C}^2, n))$ for a fixed n; explicitly, the only non-trivial walls are at a/b for $|b| \le n$. The wall sub-algebras are all isomorphic to $U_\hbar(\widehat{\mathfrak{gl}}_1)$, and some are shown in Figure 49. The infinite slope R-matrix, in the KT factorization, corresponds to the vertical (commutative) sub-algebra.

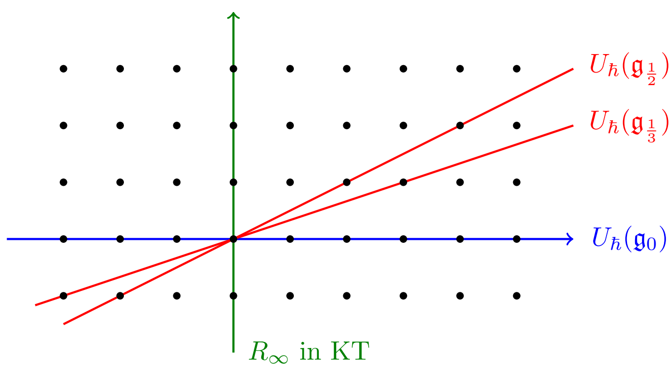


Figure 49: Some wall sub-algebras of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$.

28.8

We return to the discussion of the operators B_w. The wall sub-algebras $U_\hbar(\mathfrak{g}_w)$ always have $\mathfrak{g}_w$ of rank 1. Also, qKZ for $U_\hbar(\mathfrak{g}_w)$ is not a q-difference equation, since the q-shift is on the loop rotation variable, which is not present here. Hence qKZ becomes a linear equation

$$(z \otimes 1)RJ = J(z \otimes 1)R(0)$$

for an operator J. Explicitly, $R(0) = \hbar^{\cdots}$ is the diagonal part of the R-matrix. There is a universal formula for $B_w(z)$ in terms of J called *dynamical reflection*, of the form

$$B_w(z) = m\left((1 \otimes S)J_{21}^{-1}\right). \tag{103}$$

Here "universal" refers to how it only involves multiplication and antipode in the Hopf algebra. In particular we can check it in the case of $\mathfrak{sl}_2$, where we should get the explicit formulas for the *dynamical Weyl group* of Etingof–Varchenko [EV02].

Theorem ([OS16])**.** *These $B_w(z)$ form a dynamical groupoid, and give the q-difference equation for Nakajima quiver varieties.*

Proof. The main idea is that, given two slopes s and s' separated by a single wall, the wall operator B_w is essentially an intertwiner for the qKZ operators in s and s'. Moving in a closed loop, like in the Khoroshkin–Tolstoy factorization (102), we obtain an operator commuting with qKZ. But nothing commutes with qKZ. This verifies the braid relations. □

28.9

Example. What we need for $\mathrm{Hilb}(\mathbb{C}^2, n)$ is an explicit formula for J in the case $\widehat{\mathfrak{gl}}_1 = \bigoplus_{n>0}$ Heisenberg/centers. It suffices to write J for a single Heisenberg, with presentation

$$H \coloneqq \hbar^{xd/dx}, \quad K \coloneqq \hbar, \quad E \coloneqq x, \quad F \coloneqq -d/dx$$

where $\Delta(K) \coloneqq K \otimes K$ is group-like and

$$\Delta(E) \coloneqq E \otimes 1 + K^{-1} \otimes E, \quad \Delta(F) \coloneqq F \otimes K + 1 \otimes F.$$

Then

$$J = \exp\left(-(\hbar - \hbar^{-1})\frac{z}{1-z} F \otimes E\right). \tag{104}$$

28.10

Remark. The hyperplane arrangement in Z has to do with singularities of both the quantum difference equation and the quantum differential equation. For $\mathrm{Hilb}(\mathbb{C}^2, n)$, the singularities are

$$\bigcup_{b \le n} \sqrt[b]{1} \subset \mathbb{C}^\times = \mathsf{Z}.$$

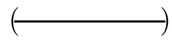
One can ask: what is the monodromy around one of these singularities? By general principles, if we view a differential equation in z as a degeneration of a difference equation in z, then the monodromy of the differential equation is a less severe degeneration and does not involve z. For each wall operator $B_w(z)$, the monodromy is given by the operator $B_w(\infty)$; this can be proved by the same sort of argument as in the theorem, using that nothing commutes with qKZ.

Lecture 29. Rigidity results for the glue operator and capped vertex

Tube = Glue, *the glue matrix in terms of the dynamical groupoid, varieties* X' *associated to strata in* $\mathrm{Pic}(X) \otimes \mathbb{R}$*, conjectural formula for the wall operators in terms of these* X'*, what it says for rank-*r *framed sheaves on* $\mathbb{C}^2$*, capped vertex with descendants, large framing vanishing, Smirnov's formula for the capped vertex with descendants, the fusion operator* J *again.*

29.1

Recall that we have two operators

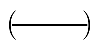
Tube := 

Glue := 

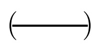
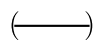
Today we will prove that Tube = Glue, and in fact we will prove it in two ways.

29.2

Proof 1. Consider the operator

$P :=$  Glue^{-1}.

By degenerating the Tube operator, we get

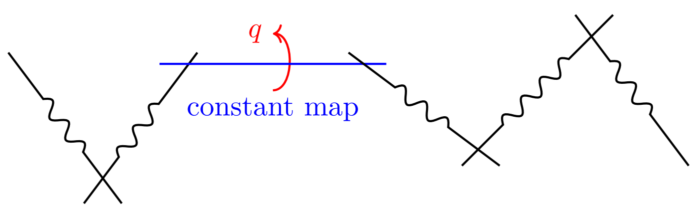
$P =$  Glue^{-1}  $\mathrm{Glue}^{-1} = P^2.$

So P is in fact a projector. Since $P\big|_{z=0} = 1$, it follows that $P = 1$. □

Remark. In cohomology, the Glue operator is trivial, so this also proves that Tube is trivial.

29.3

Proof 2. Proceed by localization with respect to the $\mathbb{C}_q^\times$ acting on Tube. Since $\mathbb{C}_q^\times$ acts only on the non-rubber part, fixed points are of the form



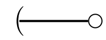
and q does not act in the bubbles. In this picture, q only acts on the normal bundle to the fixed locus, and this normal bundle has to do with smoothing the nodes connecting the rubber to the non-rubber parts. Such a smoothing has a tangent weight $q\psi$, where ψ is the line bundle of tangent spaces (at a marked point) for the moduli of curves corresponding to the rubber part. These tangent weights contribute factors

$$\frac{1}{1-q^{-1}\psi^{-1}} \to \begin{cases} 1 & q\to\infty \\ 0 & q\to 0 \end{cases}$$

to localization. The same is true for smoothing at ∞, but with a substitution $q \leftrightarrow q^{-1}$. Hence, in total, Tube is some Laurent polynomial in q (due to properness) such that

$$\text{Tube} \to \begin{cases} \text{Glue (from 0)} & q\to\infty \\ \text{Glue (from } \infty) & q\to 0. \end{cases} \qquad \square$$

29.4

The argument in proof 2 can also be applied to our fundamental solution $\Psi :=$  as well. It follows that

$$\Psi \to \begin{cases} \text{Glue (from 0)} & q\to\infty \\ 1 & q\to 0. \end{cases}$$

Plugging this into the quantum difference equation,

$$\begin{aligned} M_{\mathcal{L}} &\to \mathcal{L}, & q&\to 0 \\ M_{\mathcal{L}}(q^{-\mathcal{L}}z,q) &\to \text{Glue}\,\mathcal{L}, & q&\to\infty \end{aligned}$$

for $\mathcal{L}$ ample [Oko17, Corollary 8.1.19]. But the q-difference operator $M_{\mathcal{L}}$ we know from last time, and therefore we learn what Glue is. In terms of the dynamical groupoid, Glue is therefore the operator moving from the anti-ample to the ample alcove, across $0 \in \text{Pic}(X)\otimes\mathbb{R}$.

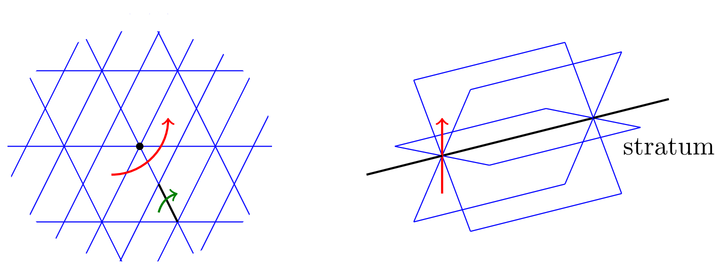


Figure 50: The glue operator for X, in red, and (conjecturally) a glue operator for a related X', in green.

Thinking of walls as reflections in the affine Weyl group, and a path $-\infty \to \infty$ as the longest element in the affine Weyl group, the Glue operator is the "longest element in the finite Weyl group". Here of course there is no "finite Weyl group"; it is just a groupoid.

29.5

In general, in our groupoid, a special role is played by the transformations of the kind in Figure 50. In the equivariant lattice, this is like the R-matrix which moves from an attracting chamber $\mathfrak{C}$ to the opposite chamber $\mathfrak{C}^{\text{opp}}$. The geometric meaning of these operators should be the following (research project): they are glue matrices for a different variety X', obtained from X by a procedure sort of like deformation, as follows. Our hyperplane arrangement consists of subgroups in a torus $\text{Pic}(X) \otimes \mathbb{C}^\times$, so the hyperplanes passing through the origin have a geometric interpretation as the discriminant in

$$\text{Pic}(X) \otimes \mathbb{C} = \begin{pmatrix} \text{deformation space of} \\ (X, \omega) \text{ as a symplectic manifold} \end{pmatrix},$$

i.e. the hyperplanes are where X_0 is singular or X is not affine. In each stratum there is a space X_0' which is the deepest singularity. For example, if $X_0 \subset \mathfrak{g}0$ is the nilpotent cone, then deformations are other conjugacy classes in $\mathfrak{g}$ and the discriminant consists of roots. But what about hyperplanes which do not pass through $0 \in \text{Pic}(X)$? There are many answers, but the easiest may be to study deformations of the multiplicative analogue of X. To really get a new example, it is important to go outside of Lie theory, because we really want a fractional hyperplane.

29.6

Example. The usual symplectic deformation of the moduli of rank-r instantons is

$$\mathcal{M}(r) \rightsquigarrow \{XY = YX + (\text{const}) + (\text{rank } r)\}$$

where $(\text{const}) \in \text{Pic}(\mathcal{M}(r)) \otimes \mathbb{C}$ is the deformation parameter. Instead, take the multiplicative deformation (part of a general theory of multiplicative Nakajima varieties)

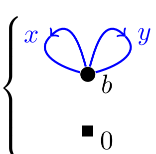
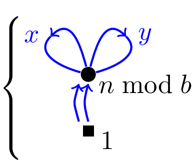
$$\{XY = zYX + (\text{rank } r)\}$$

where $z \in \text{Pic}(\mathcal{M}(r)) \otimes \mathbb{C}^\times$. When z is a primitive b-th root of unity, then the algebra with relations $XY = zYX$ has a b-dimensional irreducible representation $\underline{b}$ where

$$\text{spectrum}(X) = \text{spectrum}(Y) = \mu_b$$

consists of all b-th roots of unity. We will study it in terms of slices in the moduli of representations, as previously discussed.

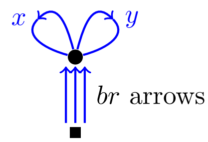
To study representations of algebras, it is easier to use the Crawley-Bovey notation where a rank-r framing node is replaced by r maps from a rank-1 ordinary node. In the setting of $\underline{b}$, the overall representation therefore decomposes into representations

$\underline{b} = \{$  $\}^{\oplus\cdots}$, remainder $:= \{$  $\}$.

Compute that

$$\chi(\underline{b}, \underline{b}) = 0, \quad \chi(\underline{b}, \text{remainder}) = 2rb.$$

We don't care about $\chi(\text{remainder}, \text{remainder})$, because these are coordinates along the stratum where we take the slice. The slice therefore corresponds to the quiver



In summary, if $X = \mathcal{M}(r)$ and $z = \sqrt[b]{1}$, then $X' = \mathcal{M}(br)$. This makes sense by 3d mirror symmetry, because $X^\vee = \text{Hilb}(\mathcal{A}_{r-1}, \text{points})$ and $z = \text{diag}(z, z^{-1})$ is an equivariant variable acting on $X^\vee$, and

$$(X')^\vee = (X^\vee)^z$$

has the same quiver (but with a different stability condition) as $\text{Hilb}(\mathcal{A}_{br-1}, \text{points})$. Additionally, putting together (101), (103) and (104), one can obtain an explicit formula

$$M_{\mathcal{O}(1)} = \mathcal{O}(1) \overleftarrow{\prod_{\substack{s=a/b\in\mathbb{Q} \\ -1\leq s<0}}} :\!\exp\left(\sum_{k\geq 0} \frac{c_k \hbar^{-kbr/2}}{1 - z^{-kb} q^{ka} \hbar^{-kbr/2}} \alpha_{-(kb,ka)} \alpha_{(kb,ka)}\right)\!: .$$

In this formula, the scalar br enters exactly as expected, in the sense that e.g. the operator of crossing the wall at 0 is Glue for $\mathcal{M}(r)$ while the operator crossing the wall at a/b is Glue for $\mathcal{M}(br)$ (with a mild change of parameters).

29.7

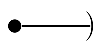
This sort of behavior should hold in general. The 3d mirror dual story is that, at walls for the equivariant variables, we have R-matrices $R(X^a)$ for the fixed locus $X^a \subset X$ of the wall.

Put differently, Nakajima quiver varieties X have natural multiplicative analogues X_{mult}, and there is usually a map

$$\text{Pic}(X) \otimes \mathbb{C}^\times \to \text{Def}(X_{\text{mult}})$$

to the deformation space of X_{mult}. For a point $z \in \text{Pic}(X) \otimes \mathbb{C}^\times$, we set X' to be the deepest singularity of the corresponding deformation of X_{mult}.

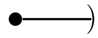
29.8

Consider the object  . The evaluation maps ev_0 and ev_∞ land in the stack $\mathfrak{X}$ and X respectively, and so in this object the stable and unstable loci in $\mathfrak{X}$ really interact. Both evaluation maps are proper and therefore have no denominators in localization, and so there is not much room for complexity. There is, however, a lot of room for rigidity.

Example (Large framing vanishing)**.** Let $X = \mathcal{M}(v, w)$ be a Nakajima quiver variety. The stack is $\mathfrak{X}$ is a quotient by $\mathrm{GL}(V) \coloneqq \prod_i \mathrm{GL}(V_i)$. Hence

$$\operatorname{Spec} K_{\mathrm{eq}}(\mathfrak{X}) \subset \operatorname{Spec} \operatorname{Rep} \mathrm{GL}(V) \times \cdots$$

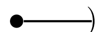
where $\operatorname{Rep} \mathrm{GL}(V)$ is explicitly just the ring of symmetric Laurent polynomials in groups of v_i variables. This ring is independent of w, and we can choose an element λ in it. For any given λ, we can choose the framing w so large that

λ  $= \lambda \otimes \widehat{\mathcal{O}}^{\mathrm{vir}}_{\text{const maps}}$

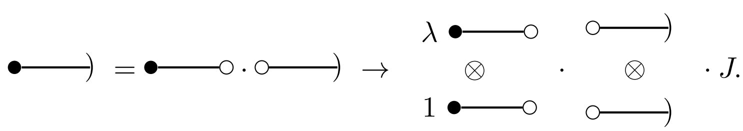
consists of purely classical contributions, with no quantum corrections [Oko17, Theorem 7.5.23]. This is really a rigidity result, not a dimensional vanishing since dimension is independent of degree. Picking the correct normalization, which involves some choice of polarization, the term $\widehat{\mathcal{O}}^{\mathrm{vir}}_{\text{const maps}} = \hbar^{\cdots}$ is just some multiple of $\hbar$.

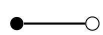
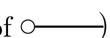
29.9

For simplicity, assume λ is a polynomial on $\mathrm{GL}(V)$. Take $W = W_1 \oplus W_2$, where W_1 is the framing we want and W_2 is very large so that we have vanishing. Let $a \in \mathbb{C}^\times$ act on W_2 by scaling, and consider $\mathcal{M}(w) = \bigsqcup_v \mathcal{M}(v, w)$.

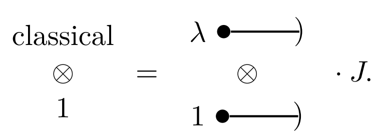
- By large framing vanishing, λ  is purely classical.
- By localization, in the $a \to 0$ limit,

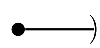
  $$\text{classical} \to \text{classical} \otimes 1$$

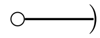
  It is fairly clear that the bare vertex with descendants  is just a tensor product of two bare vertices in the limit, but the limit of  and the presence of the slope-0 fusion operator J is a theorem of A. Smirnov.

Putting this together, we get the formula

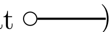


This is Smirnov's formula for the object λ  . Actually Smirnov has a better (unpublished) version of this formula which is more explicit.

29.10

Where does this J come from? Recall that  is a fundamental solution of $\Psi(q^{\mathcal{L}}z)\mathcal{L} = M_{\mathcal{L}}\Psi(z)$, and one can ask what happens when $a \to 0$. From geometry, $M_{\mathcal{L}}$ becomes upper-triangular, and up to gauge equivalence it is just

$$M_{\mathcal{L}} \to \begin{pmatrix} M_{\mathcal{L}} & \\ & M_{\mathcal{L}} \end{pmatrix}.$$

The gauge equivalence is essentially J. Recall also that  solves a q-difference equation in a, namely qKZ, given by

$$R(a) = \overrightarrow{\prod} R_{\text{slope}}(\cdots)$$

where the variable a enters monomially in $\cdots$. As $a \to 0$, this becomes the slope-0 R-matrix, and J is the universal solution for qKZ for this R-matrix.

Lecture 30. The stratification of the unstable locus

Capped vertex with descendants as a correspondence between X and the ambient quotient stack, stable and unstable loci in GIT, stratification of the unstable locus, inductive construction of stable envelopes for quotient stacks.

30.1

We continue to discuss the capped vertex with descendents, which involves a moduli space of quasimaps with two evaluation maps: ev_0 valued in $\mathfrak{X}$, and ev_∞ valued in X. Push-pull using this object is how X and the stack $\mathfrak{X}$ talk to each other.

30.2

The stack $\mathfrak{X}$ is a quotient stack, and admits an embedding into a stack of the form $[V/\mathsf{G}]$ where V is a vector space. For Nakajima quiver varieties, $\mathsf{G} = \prod \mathrm{GL}(V_i)$. The K-theory of the stack is very simple:

$$K_{\text{eq}}([V/\mathsf{G}]) = K_{\text{eq}\times\mathsf{G}}(\text{pt}) = K_{\text{eq}}(\text{pt})[x_{i,j}^{\pm}]^W$$

where the $x_{i,j}$ are coordinates in the maximal torus of G.

30.3

On the other hand, X is the quotient of an open *stable* locus $Y_{\text{sst}} \subset Y := \mu^{-1}(0)$. Further, Y is a closed subset in some ambient space Z (which for us is a vector space) which comes equipped with an ample G-equivariant line bundle $\mathcal{L}$. We now briefly discuss stability. There are two notions of stability, one algebraic and the other geometric. In this setting,

$$X = \operatorname{Proj} \bigoplus_{n>0} H^0(Y, \mathcal{L}^n)^{\mathsf{G}}.$$

So we can say a point $y \in Y$ is stable iff there exists a function $f \in H^0(Y, \mathcal{L}^n)^{\mathsf{G}}$ such that $f(y) \neq 0$. Geometrically, what does this mean? Take a G-equivariant embedding $Z \hookrightarrow \mathbb{P}(V)$ for a vector space V such that $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$. Let $\widehat{Y}$ be the affine cone of Y, so that

$$X = \operatorname{Proj} \mathbb{C}[\widehat{Y}]^{\mathsf{G}}.$$

In general, if a reductive group acts on an affine variety, then G-invariants separate closed orbits on $\widehat{Y}$. So two orbits in $\widehat{Y}$ intersect iff they contain $0 \in \widehat{Y}$ in the orbit, and therefore $y \in Y \subset \widehat{Y}$ is unstable iff $0 \in \overline{\mathsf{G}y}$.

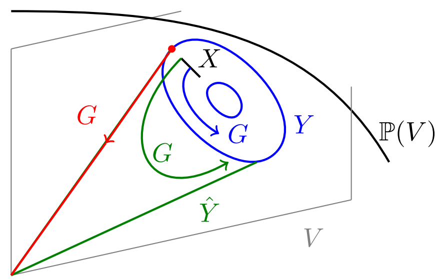


Figure 51: Various G-orbits in the affine cone $\widehat{Y}$. The red one contains 0 and therefore corresponds to an unstable point in Y.

The Hilbert–Mumford criterion says

$$0 \in \overline{\mathsf{G}y} \iff 0 \in \overline{\mathsf{T}y}$$

for a maximal torus $\mathsf{T} \subset \mathsf{G}$. Written out, it means there exists a one-parameter subgroup $\sigma \colon \mathbb{C}^\times \to G$ such that

$$\sigma(t) \cdot y \to 0 \quad \text{as } t \to 0.$$

But there is a very easy way to tell if $0 \in \overline{\mathsf{T}y}$: draw the Newton polygon of y, namely the convex hull

$$\operatorname{Conv}(\text{weights appearing in } T_y Y),$$

and see if it contains the zero weight. If it does not, there is a hyperplane σ which separates zero from the Newton polygon, showing that $0 \in \overline{\mathsf{T}y}$. It is very natural to talk about the Newton polygon since $\overline{\mathsf{T}y}$ is a toric variety.

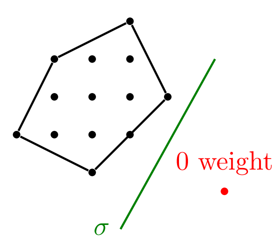


Figure 52: A hyperplane σ separating the Newton polytope from the zero weight.

30.4

In practice, e.g. for Nakajima quiver varieties, the setup is slightly different and the ambient space is actually affine instead of sitting in a projective space. This is equivalent because the affine variety is the complement of a very ample divisor in projective space, so one reduces to the other very easily. But the statement of stability is slightly different. A general $\mathcal{L}$ on an affine Y is of the form

$$\mathcal{L} = \mathcal{O}_Y \otimes (\text{character } \chi \text{ of } \mathsf{G}),$$

and if we follow through the discussion earlier, the Newton polytope is replaced by a *cone* spanned by the T-weights that appear in y, and if $-\chi$ lives in this cone then y is stable.

Example. Embed

$$T^* \operatorname{Gr}(k,n) \hookrightarrow [\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^k) \oplus \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^n) / \operatorname{GL}(k)],$$

and let (A, B) be coordinates on the pre-quotient. Let $\mathsf{T} = \operatorname{diag}(t_1, \dots, t_k) \subset \operatorname{GL}(k)$ be the maximal torus. Then the Newton polygon of weights in A remembers the non-zero rows in A, i.e. it is the convex hull of t_i for all non-zero rows i. Let $\chi \coloneqq \det^{-1}$, i.e. $-\chi = (1, 1, \dots, 1)$ in the weight lattice. Then

$$\begin{aligned} (A,B) \text{ are stable for the } \mathsf{T}\text{-action} &\iff \text{all rows of } A \text{ are non-zero} \\ (A,B) \text{ are stable for the } \mathsf{G}\text{-action} &\iff \operatorname{rank} A = k, \end{aligned}$$

where the latter condition $\operatorname{rank} A = k$ means that, however conjugated, all rows of A are still non-zero.

30.5

The topic of today and next lecture is that the capped vertex with descendents is (like) a stable envelope, in the following sense. Recall that stable envelopes solve an extension problem for cohomology classes. We can pose a very similar problem. Since the action of G on Y_{sst} is free,

$$K_{\text{eq}}(X) = K_{\text{eq} \times \mathsf{G}}(Y^{\text{sst}}).$$

One can in principle consider non-free actions and take orbifold K-theory (or cohomology), and this is perhaps interesting. We can ask about the extension problem from $K_{\text{eq}\times\mathsf{G}}(Y^{\text{sst}})$ to $K_{\text{eq}\times\mathsf{G}}(Y)$. But because Y is in general singular, it is better to ask about an extension

$$K_{\text{eq}\times\mathsf{G}}(Y^{\text{sst}}) \to K_{\text{eq}\times\mathsf{G}}(Z) \text{ supported on } Y$$

to the smooth ambient space Z; smoothness is rather crucial for the construction of stable envelopes. Of course, the same question may be posed in elliptic cohomology.

30.6

Like for stable envelopes, this extension problem may be solved inductively using a stratification of the unstable locus. Recall that for stable envelopes, we built a stratification

$$X \supset X_1 \supset X_2 \supset \cdots$$

by ordering the components $X^{\mathsf{A}} = \bigsqcup F_i$ and setting

$$X_1 \setminus X_2 \coloneqq \mathrm{Attr}(F_1), \quad X_2 \setminus X_3 \coloneqq \mathrm{Attr}(F_2), \quad , \ldots .$$

In fact this is a special case of the stratification of the unstable locus, which itself is a classical perspective (see e.g. [VP89]). In our more general GIT setting on Z, let $\mathrm{supp}(z)$ denote the Newton polytope for a point $z \in Z$ (as in e.g. Figure 52) for each point. The stable locus $Z \setminus Z_1$ is where $0 \in \mathrm{supp}(z)$, and then the unstable locus can clearly be stratified by how far away 0 is from $\mathrm{supp}(z)$. To quantify this measure of instability we must choose an invariant metric on $\mathrm{Lie}(\mathsf{G})$, which then induces a metric on (co)weights. The shortest path from 0 to $\mathrm{supp}(z)$ specifies a 1-parameter subgroup $\sigma^\vee$ which brings z to 0 the fastest.

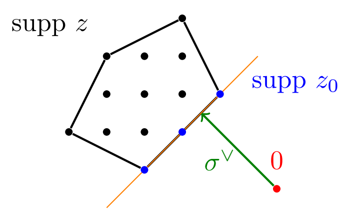


Figure 53: The maximally destabilizing 1-parameter subgroup $\sigma^\vee$, with the closest face $\mathrm{supp}(z_0)$ indicated in blue.

30.7

The point z is *reduced* if $\mathrm{supp}(z)$ cannot be made any further from 0 by the G-action. Let $z_0 \in \mathbb{P}(V)$ be the projection of $z \in \mathbb{P}(V)$ onto the weight spaces on the closest face of $\mathrm{supp}(z)$ to 0. This point z_0 has its own polytope $\mathrm{supp}(z_0)$, and the projection of 0 gives a new point $\overline{0} \in \mathrm{supp}(z_0)$.

Theorem. *z is reduced if z_0 is stable under the action of G^σ.*

Example. Continue with $T^* \mathrm{Gr}(k, n)$ from Example 30.4. Let $A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)$ be such that only rows 1 and 3 are non-zero. So it is an unstable point, and its Newton polytope lies in the plane spanned by t_1 and t_3. The vector $-\chi = (1, 1, \ldots, 1)$ projects to the new vector $(1, 1)$, and we are now asking for the stability of the projection of A to only its non-zero rows, under the $\mathrm{GL}(2)$ action. This projection is clearly stable iff its rank is exactly 2.

Proof idea. Consider $\mathsf{G}^\sigma \subset \mathsf{G}$ and its positive/negative parabolics $P_\pm \subset \mathsf{G}$ consisting of ≥ 0 and ≤ 0 σ-weights respectively. By hypothesis, P_+ preserves the fact that $0 \in \mathrm{supp}(gz_0)$. On the other hand, $\mathsf{G} = P_+ W P_+$. $\square$

30.8

Taken all together, this gives the following stratification. First, for unstable points z, order their paths $\sigma_i^\vee$ by their length. By construction, $\mathrm{supp}(z_0)$ lies on the hyperplane orthogonal to $\sigma_i^\vee$. So $z_0 \in \mathbb{P}(V)$ is fixed by σ_i, meaning that if $w(\sigma_i)$ is the weight of σ_i then

$$z_0 \in V_{w(\sigma_i)} \subset V$$

lies in a specific weight space of V. Then set

$$F_i := V_{w(\sigma_i)} \cap (\text{stable locus for } \mathsf{G}^{\sigma_i}).$$

For a fixed torus $\mathsf{T} \subset \mathsf{G}$, the i-th stratum is then $\mathrm{Attr}_{\sigma_i}(F_i)$, as before.

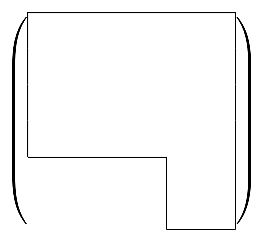
Theorem ([VP89, Theorem 5.6])**.** *As we vary the torus T, the strata become*

$$\mathsf{G} \times_{P_{+,i}} \mathrm{Attr}_{\sigma_i}(F_i)$$

where $P_{+,i}$ is the positive parabolic corresponding to σ_i.

To be precise, $\mathrm{Lie}(P_{+,i}) = \mathfrak{g}_{\geq 0,\sigma_i}$ consists of those weight spaces $\mathfrak{g}_\alpha$ such that $\langle \sigma_i, \alpha\rangle \geq 0$. For example, in $\mathrm{GL}(n)$,

$$\sigma(t) = \begin{pmatrix} t &&&&& \\ & t &&&& \\ && t &&& \\ &&& t && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \quad \rightsquigarrow \quad P_+ =$$



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30.9

As a corollary, if Z is smooth then these strata are smooth with normal bundle

$$N := N_{Z/F_i,<0} \ominus \mathfrak{g}_{<0,\sigma_i}$$

where the second term is the tangent space to $\mathsf{G}/P_{+,i}$. The induction strategy therefore proceeds using the cofibration sequence

$$Y := Z \setminus \text{stratum} \to Z \to \mathrm{Thom}(N) \to \cdots.$$

However, recall that after passing to elliptic cohomology, we must twist the whole sequence $\cdots \to \Theta^i(-N) \to \mathcal{O}^i_{\mathrm{Ell}(Z)} \to \mathcal{O}^i_{\mathrm{Ell}(Y)} \to \cdots$ by an "attractive" line bundle $\mathcal{S}$ so that $\Theta^i(-N) \otimes \mathcal{S}$ has degree 0 in σ and therefore has no cohomology. In fact the definition of attractive was some condition on degree. The definition of attractive in the new situation requires us to look at both

$$\mathcal{S}_0 := \Theta(T^{1/2}Z) \otimes (\text{degree } 0)$$
$$\mathcal{S} := \ker(\mathcal{O}_{\mathrm{Ell}(Z)} \to \mathcal{O}_{\mathrm{Ell}(Z\setminus Y)}),$$

and while the former is a line bundle, the latter is not. So a somewhat new argument is needed. Note that the latter sheaf is supported on Y, as desired.

30.10

For Nakajima varieties, we already know the answer to the extension problem, i.e. the elliptic stable envelopes on the stack. Namely, our formulas for elliptic stable envelopes were written in terms of Chern roots of universal bundles, and therefore make sense on the stack as well. So for Nakajima varieties, our discussion introduces no new generality.

Example. For $T^* \operatorname{Gr}$, stable envelopes have the form

$$\operatorname{Symm} \prod_{i<j} \frac{1}{\vartheta(x_i/x_j)\vartheta(\hbar x_i/x_j)} \prod \vartheta(\cdots)$$

on X, and so also on the pre-quotient Y_{sst}. The pushforward to Z comes from the inclusion with normal bundle given by $\{\mu = 0\}$, and so corresponds to multiplication by $\prod_{i,j} \vartheta(\hbar x_i/x_j)$. On Z_{sst}, the result is

$$\operatorname{Symm} \prod_{i<j} \frac{\vartheta(\hbar x_j/x_i)}{\vartheta(x_i/x_j)} \prod \vartheta(\cdots).$$

Upon symmetrization, the first-order poles at $x_i = x_j$ disappear, and hence this is a regular function. So one may wonder how it can extend in some other interesting way to all of Z. In fact it already *is* the correct extension, with no further change.

Lecture 31. Non-abelian stable envelopes

Nonabelian stable envelopes for algebraic symplectic reductions, the relation between abelian and nonabelian stable envelopes, K-theoretic stable envelopes and equivalence between descendant and relative insertions, integral solutions of the quantum difference equations, Bethe equations, K-theoretic stable envelopes as the off-shell Bethe eigenfunction.

31.1

To summarize the setup, recall that an algebraic symplectic reduction is of the form

$$X \coloneqq Z \mathbin{/\!\!/\!\!/\!\!/} \mathsf{G} \coloneqq \mu^{-1}(0) \mathbin{/\!\!/} \mathsf{G} \coloneqq \mu^{-1}(0)_{\mathrm{st}}/\mathsf{G}.$$

We assume that there are no strictly semistable points in $\mu^{-1}(0)$. In addition, for Nakajima quiver varieties it turns out there are even no finite stabilizers. The problem is to take a class α on X and extend it to $Y \coloneqq \mu^{-1}(0)$:

$$\begin{array}{ccccc} & & Y = \mu^{-1}(0) & & \\ & \swarrow & & \nwarrow & \\ Z & & & & Y_{\mathrm{st}} \xrightarrow{/\mathsf{G}} X \\ & \nwarrow & & \swarrow & \\ & & Z_{\mathrm{st}} & & \end{array}.$$

The extension $Y_{\mathrm{st}} \to Y$ is problematic since Y is not smooth. So it is better to first pushforward to Z_{st} and do the extension $Z_{\mathrm{st}} \to Z$ such that the result is supported on Y.

31.2

The extension $Z_{\mathrm{st}} \to Z$ is in fact a non-abelian stable envelope. This is like the abelian stable envelope in many ways.

31.2.1

Property 1. It solves an interpolation problem.

The inclusion of an open $Z_{\mathrm{st}} \hookrightarrow Z$ induces a map in cohomology, e.g.

$$\mathrm{Ell}_{\mathrm{eq}}(Z_{\mathrm{st}}) \to \mathrm{Ell}_{\mathrm{eq}}(Z),$$

which is a *closed* embedding. Hence the extension problem is asking for a section of a line bundle which takes given values at given points, and this is exactly what interpolation means.

The result must also vanish on the open $Z \setminus Y \hookrightarrow Z$, meaning on the image of the induced closed embedding $\mathrm{Ell}_{\mathrm{eq}}(Z \setminus Y) \to \mathrm{Ell}_{\mathrm{eq}}(Z)$. Such vanishing conditions translate into what are more commonly known as "wheel conditions".

Example. For $\mathrm{Hilb}(\mathbb{C}^2, n)$, we have $Z = \mathrm{Hom}(V, V) \otimes (t_1 + t_2) \oplus V \oplus V^* t_1 t_2$. Let the coordinates on the maximal torus on $\mathrm{GL}(n)$ be $x_1, x_2, \ldots, x_n$. Check that the vanishing condition here is

$$\{x_i, x_j, x_k\} = \{c, ct_1, ct_1t_2\} \text{ or } \{c, ct_2, ct_1t_2\}$$

for any weight c. This is the original "wheel condition" in the literature.

31.2.2

Property 2. It may be solved inductively, using the stratification of the unstable locus discussed previously.

31.2.3

Property 3. It takes values in

$$\mathcal{S}_0 = \Theta(T^{1/2} Z) \otimes (\text{degree } 0),$$

or, more specifically, $\mathcal{S} \coloneqq \ker(\text{restriction to } Z/Y)$, i.e. some sheaf which already encodes the wheel conditions.

The main source for the degree-0 twist is G-equivariant line bundles on Z, especially $\mathcal{L} = \mathcal{O}_Z \otimes (\text{character } \chi \text{ of } \mathsf{G})$. For these $\mathcal{L}$, let $\mathcal{U}(\mathcal{L}, z)$ be the line bundle whose sections have the form

$$\frac{\vartheta(\chi(x)z)}{\vartheta(\chi(x))\vartheta(z)}$$

where z is a dual Kähler variable.

31.2.4

Property 4. If $\mathsf{G} = \prod \mathrm{GL}(V_i)$, then these non-abelian stable envelopes reduce to the abelian ones.

31.3

Here are the main applications of non-abelian stable envelopes.

1. The vertex with descendents ●——○, viewed as a map $K(X) \to K(\text{stack})$, is the elliptic stable envelope up to some Γ_q factors.

2. The capped vertex ●——) is related to the K-theoretic stable envelope for the stack.

The degeneration from elliptic to K-theoretic is the limit $q \to 1$, and the behavior of $\ln z / \ln q$ determines the K-theoretic slope.

31.4

Let $\alpha \in K_{\text{eq}}(X)$, and let $\text{Stab}(\alpha) \in K_{\text{eq}\times\mathsf{G}}(Z)$ be the K-theoretic abelian stable envelope. It is supported on $\{\mu = 0\}$.

Theorem ([AO17])**.** *Let $\Delta_\hbar$ be the weight of μ. Then*

$$\frac{\text{Stab}(\alpha)}{\Delta_\hbar} ●——) = \alpha\ (——)\ .$$

Example. For $\mathsf{G} = \text{GL}(n)$ the factor is $\Delta_\hbar = \prod_{i,j}(1 - \hbar x_i/x_j)$. The vertex ●——) studies quasimaps $f\colon \mathbb{P}^1 \to Z$ such that $\mu(f(t)) = 0$, and so $\text{ev}_0(f) = f(0)$ already satisfies the moment map constraint. To divide by $\Delta_\hbar$ means to cancel this very same factor of $\Delta_\hbar$ arising from the localization formula.

Remark. This goes the opposite direction compared to Smirnov's formula, which tells us the specific relative insertion necessary to produce a given descendent insertion. Note that there is no z and no q in the K-theoretic $\text{Stab}(\alpha)$, whereas in Smirnov's formula the relative insertion must involve z and q.

31.5

Corollary. *In any source curve, any relative insertion α can be replaced by a descendent insertion* $\text{Stab}(\alpha)/\Delta_\hbar$.

Proof. Bubble off the point where we want to change the insertion, and apply the theorem to the $\mathbb{P}^1$ bubble. □

Corollary.

$$\alpha\ (——○\beta = \frac{\text{Stab}(\alpha)}{\Delta_\hbar} ●——○\beta.$$

On the lhs, (——○ is the fundamental solution of our q-difference equations. On the other hand, the rhs ●——○ is some Mellin–Barnes integral

$$\int_{\gamma_\beta} \prod \frac{dx_i}{x_i} \frac{\text{Stab}(\alpha)(x)}{\prod(1 - \hbar x_i/x_j)} \prod \frac{\Gamma_q(\cdots)}{\Gamma_q(\cdots)}.$$

This is because, acting on $\bullet\!\!-\!\!\!-\!\!\!-\!\!\circ$ by the scaling $\mathbb{C}_q^\times$, fixed loci are themselves GIT quotients by G. If $\mathcal{F}$ is a G-equivariant sheaf on $\mathfrak{Z}$ then

$$\chi(\mathfrak{Z}/\mathsf{G}, \mathcal{F}^{\mathsf{G}}) = \chi(\mathfrak{Z}, \mathcal{F})^{\mathsf{G}} = \int_{\substack{\text{max compact} \\ \text{torus in } \mathsf{G}}} (\text{Weyl integration formula}).$$

Similarly, for a GIT quotient,

$$\chi(\mathfrak{Z} /\!\!/_\chi \mathsf{G}, \mathcal{F}^{\mathsf{G}}) = \int_{\substack{\text{contour} \\ \text{depending on } \chi}} \cdots$$

with the same integrand. This is because $\chi(\mathfrak{Z} /\!\!/_\chi \mathsf{G}, \mathcal{F}^{\mathsf{G}})$ is the value at $m = 0$ of

$$\chi(\mathfrak{Z}/\mathsf{G}, (\mathcal{F} \otimes \chi^m)^{\mathsf{G}}) \qquad m \gg 0,$$

which is an analytic function of m. Analytic continuation in m means we perform steepest descent, which involves changing the contour; see [AFO18, Appendix] for details. The conclusion is that

$$\left\langle \alpha \,\middle|\, \begin{array}{c} \text{fundamental solution of} \\ q\text{-difference equations} \end{array} \,\middle|\, \beta \right\rangle = \int_{\text{cycle}(\beta)} \mathrm{Stab}(\alpha) \frac{\Gamma_q(\cdots)}{\Gamma_q(\cdots)} \prod \frac{dx_i}{2\pi i x_i}.$$

31.6

In this formula, we can take the $q \to 1$ limit. The K-theoretic $\mathrm{Stab}(\alpha)$ has no q, but the Γ_q factors will blow up as

$$\frac{\Gamma_q(\cdots)}{\Gamma_q(\cdots)} \sim \exp\left(\frac{W(x, \ldots)}{\ln q}\right).$$

Saddle points of the integral, i.e. where $\partial W/\partial x_i = 0$, are therefore the only contributions in the limit. On the other hand, for a q-difference equation $\Psi(qz) = M(z,q)\Psi(z)$, sending $q \to 1$ produces

$$\Psi \sim \exp\left(\frac{1}{\ln q} \int \lambda_i\right) \psi_i$$

where $\{\lambda_i\}$ and $\{\psi_i\}$ are the eigenvalues and eigenvectors respectively of $M(z,q)$. Comparing the two,

$$\frac{\partial}{\partial x_i} W = 0$$

must be the Bethe equations for the spectrum (cf. Nekrasov–Shatashvili), and therefore $\mathrm{Stab}(\alpha)$ must be the off-shell Bethe eigenfunction, i.e. the Bethe ansatz for eigenvectors. The variable z is a parameter, similar to the quasi-periodic boundary condition in spin chains.

31.7

The $z = 0$ limit is strange from the spin chain perspective, but for us it reduces the quantum integrable system to classical multiplication in $K_{\text{eq}}(X)$, and the spectrum becomes nothing more than $\operatorname{Spec} K_{\text{eq}}(X)$. For example, given $p \in X^{\mathsf{T}}$, the structure sheaf $\mathcal{O}_p$ is always an eigenvector since

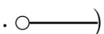
$$\mathcal{F} \otimes \mathcal{O}_p = \mathcal{F}\big|_p \otimes \mathcal{O}_p.$$

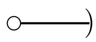
Irreducible components of $\operatorname{Spec} K_{\text{eq}}(X)$ correspond to these fixed points. If fixed points are not isolated, then the components may be non-reduced, and the spectrum has non-trivial Jordan blocks. Deforming away from $z = 0$, we get the Bethe equations, and

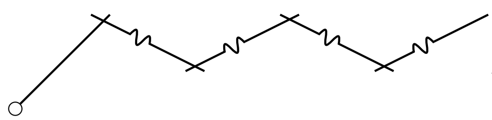
$$\operatorname{Stab}(\cdots)\Big|_{\text{Bethe eq.}} = \text{eigenvectors}.$$

31.8

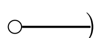
Proof of Theorem 31.4. The input to the theorem is a K-theoretic $\operatorname{Stab}(\alpha)$, which involves a slope and a polarization, both of which must be fixed correctly. Recall that the rhs is the glue operator, and one of our proofs for this was by showing that it is independent of q using a rigidity argument. We will do the same here. By localization, the lhs becomes

$\dfrac{\operatorname{Stab}(\alpha)}{\Delta_\hbar}$  $\cdot$ (edge term) $\cdot$  .

The last term  in reality means contributions of the form



At the first node there is a contribution $1/(1 - q\psi_\infty)$. Hence, by the same argument as in the Glue = Tube theorem,

 $\to \begin{cases} \text{Glue} & q \to 0 \\ 1 & q \to \infty. \end{cases}$

It therefore suffices to show that

$\dfrac{\operatorname{Stab}(\alpha)}{\Delta_\hbar}$  $\to \begin{cases} \alpha & q \to 0 \\ \text{bounded} & q \to \infty. \end{cases}$

This is a typical rigidity proof, involving bounding weights that appear in stable envelopes. $\square$

31.9

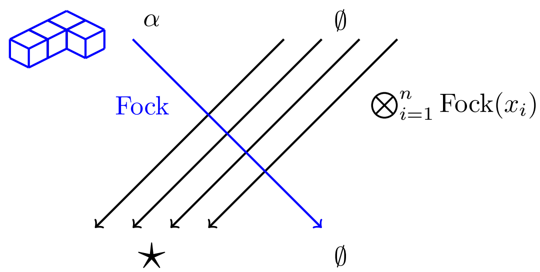
Example. For $X = \operatorname{Hilb}(\mathbb{C}^2, n)$, the non-abelian stable envelope is a map

$$\alpha \xrightarrow{\text{Stab}} (\text{symmetric functions of } x_1, x_2, \ldots, x_n).$$

If Fock is the Fock representation of the quantum group, e.g. $\mathsf{Fock} = \mathrm{Ell}_{\mathrm{eq}}(\mathrm{Hilb})$, then the claim is that Stab is a matrix element of the R-matrix on

$$\mathsf{Fock} \otimes \bigotimes_i \mathsf{Fock}(x_i).$$

In other words, $(x_1, \ldots, x_n)$ plays the role of $(a_1, \ldots, a_n) \in \mathrm{GL}(W)$. Specifically, the matrix element of interest is of the form



where $\star$ is the class consisting of a single box in each $\mathsf{Fock}(x_i)$ representation (at the origin $x = y = 0$), e.g. if we write $\alpha = \{1, t_1, t_2, t_2^2\}$ in terms of the weights of its boxes then $\star = \{x_1, x_2, x_3, x_4\}$.

Lecture 32. Vertices with descendents as non-abelian stable envelopes

Nonabelian stable envelopes for $\mathsf{G} = \prod \mathrm{GL}(V_i)$, algebraic Bethe Ansatz, vertex with descendents and nonabelian stable envelopes, maps from the formal disk and q-Gamma functions, integral solutions of the quantum difference equations and the monodromy of the vertex functions.

32.1

For a general Nakajima quiver variety, there are special points where all maps from the framing W to the nodes V are *isomorphisms*, and all other maps are zero (to satisfy $\mu = 0$). Then the framing variables $(x_1, \ldots, x_n) \in \mathrm{GL}(W)$ become elements of $\mathrm{GL}(V)$, via these isomorphisms. We take the class $\star$ of these points and consider, as we did last time, the R-matrix elements shown in Figure 54.

Example. For $X = T^* \mathrm{Gr}(k, n)$, which is a quotient by $\mathrm{GL}(k)$, we want a symmetric function in $x_1, \ldots, x_k$. The auxiliary representation $\bigotimes_{i=1}^k \mathbb{C}^2(x_i)$ is the cohomology of $\bigsqcup_{\ell=0}^k T^* \mathrm{Gr}(\ell, k)$. The point $\star$ is the component $T^* \mathrm{Gr}(k, k) = \mathrm{pt}$. In terms of spin chains,

$$\emptyset = \{\downarrow\downarrow \cdots \downarrow\}$$

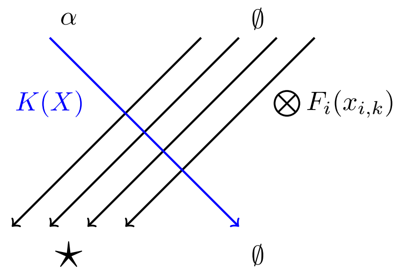
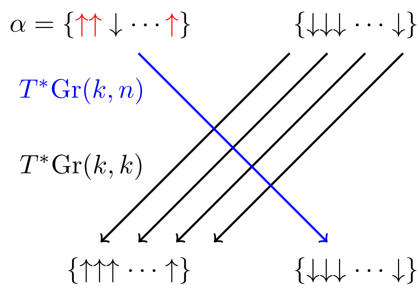


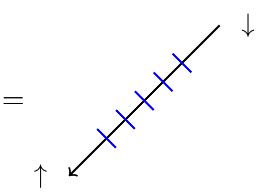
Figure 54: K-theoretic nonabelian stable envelope as an abelian one, by taking R-matrix elements. The $F_i(x_{i,k})$ for $i \in I$ are fundamental representations, with $k = 1, \ldots, \dim V_i$.

corresponds to all spins down, and since α is a class on $T^* \operatorname{Gr}(k, n)$ we can view it as consisting of k spins up (out of n). We can draw the matrix element in this language:



where α has k spins up (in red) out of n. Reading this matrix element backward, we recognize the very classical formula

$$B(x_1)B(x_2)\cdots B(x_k)\,|\emptyset\rangle\,, \quad B(x) =$$



in the algebraic Bethe ansatz for $Y(\mathfrak{gl}_2)$, $U_\hbar(\widehat{\mathfrak{gl}}_2)$, etc.

32.2

Let $X \coloneqq Z /\!\!/\!\!/\!\!/ \operatorname{GL}(V)$ be an algebraic symplectic reduction by a single $\mathsf{G} \coloneqq \operatorname{GL}(V)$ (as opposed to $\prod \operatorname{GL}(V_i)$), for simplicity. Let

$$X \xleftarrow{/\mathsf{G}} Y_{\mathrm{st}} \hookrightarrow Y \coloneqq \mu^{-1}(0) \hookrightarrow Z$$

be the usual setup. Introduce the space

$$\widetilde{Z} \coloneqq Z \times T^* \operatorname{Hom}(V', V)$$

where V' is in fact the framing W, but with $\dim V' = \dim V$ in general and $V' \cong V$ on the locus we care about. Write the new maps as

$$V' \underset{A}{\overset{B}{\leftrightarrows}} V \ .$$

Then on $\widetilde{Z}$, there are enlarged versions of everything:

$$\begin{aligned} \widetilde{Y} &\coloneqq \widetilde{\mu}^{-1}(0), \quad \widetilde{\mu} \coloneqq \mu + AB \\ \widetilde{X} &\coloneqq \widetilde{Y}_{\mathrm{st}} / \operatorname{GL}(V). \end{aligned}$$

Note that any point where $V' \xrightarrow{\sim} V$ is an isomorphism (or with the arrow the other way around) is stable. On $T^* \operatorname{Hom}(V', V)$ there is an action of $\operatorname{GL}(V')$, inside which are scalars $u \cdot 1_{V'}$. Then

$$\left(\widetilde{Z}/\mathsf{G}\right)^u \supset (Z/\mathsf{G} \times \{0\}) \sqcup (Z^u/\mathsf{G} \times \{V' \xrightarrow{\sim} V,\, V \xrightarrow{0} V'\}).$$

Here Z^u means the fixed locus under $u \cdot 1_V \in \operatorname{GL}(V)$. Hence it makes sense to take stable envelopes on $\widetilde{X}$ for $u' \in \operatorname{GL}(V')$. They are $\operatorname{GL}(V')$-equivariant objects on the quotient by $\operatorname{GL}(V)$, but equally well, they are $\operatorname{GL}(V)$-equivariant objects on the quotient by $\operatorname{GL}(V')$, i.e. on

$$\widetilde{Y}_{\mathrm{iso}} / \operatorname{GL}(V')$$

where $\widetilde{Y}_{\mathrm{iso}}$ is the locus where $V' \xrightarrow{\sim} V$ is an isomorphism. The action of $\operatorname{GL}(V')$ is therefore free on $\widetilde{Y}_{\mathrm{iso}}$ and the quotient is well-behaved.

32.3

Remember that non-abelian stable envelopes are about extending a class from Y_{st} to Z. We can embed

$$Z \hookrightarrow \widetilde{Y}_{\mathrm{iso}} / \operatorname{GL}(V')$$

as follows. In addition to the coordinates on Y, the extra coordinate on $T^* \operatorname{Hom}(V' \to V)_{\mathrm{iso}} / \operatorname{GL}(V')$ is $AB \in \operatorname{Hom}(V, V)$. Since $\widetilde{Y} = \{\mu + AB = 0\}$, the desired embedding is therefore given by $AB = -\mu$. Put differently, in $\widetilde{Y}$ we have one extra coordinate and one extra equation, and we can exactly satisfy the equation by choosing the coordinate appropriately.

It is easy to check that, after this whole process, the resulting stable envelope on $\widetilde{Y}_{\mathrm{iso}} / \operatorname{GL}(V')$ is a section of the correct bundle. So it remains to check that it is supported on Z. This follows because it is supported on $\widetilde{\mu}^{-1}(0)$, by construction, but on the other hand AB is invariant with respect to u and vanishes on $X \subset \widetilde{X}$, so $AB = 0$ on the whole u-attracting manifold.

See [AO17] for more details about this argument.

32.4

Finally, we prove a relationship between the vertex with descendents and the non-abelian stable envelope:

$$\overset{K(\text{stack})}{\bullet}\!\!\longrightarrow\!\!\!\!-\!\!\!\!-\!\!\overset{K(X)}{\circ} \;\approx \mathrm{ch}\begin{pmatrix}\text{nonabelian elliptic stable} \\ \text{envelope for } E = \mathbb{C}^\times/q^{\mathbb{Z}}\end{pmatrix}.$$

This requires some setup. There is a map $\mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}} = E$ which induces a map of groups

$$\operatorname{Spec} K_{\mathrm{eq}}(X) \to \mathrm{Ell}_{\mathrm{eq}}(X).$$

Induced maps of sheaves go the other way, and we denote it as the "Chern character"

$$\mathrm{ch}\colon \text{(sheaves on } \mathrm{Ell}_{\mathrm{eq}}(X)) \to \text{(sheaves on } \operatorname{Spec} K_{\mathrm{eq}}(X)).$$

This is analogous to the usual Chern character, which is induced by the map of curves $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$.

The vertex with descendents is like a q-hypergeometric function, which looks nothing like an elliptic function. To have a meaningful comparison, namely an actual equality instead of $\approx$, we need:

1. suitable Γ_q-factors;
2. to work with matrix elements $(\alpha, \bullet\!\!-\!\!-\!\!-\!\!\circ)$, which are functions of *all* variables.

32.5

Theorem. *Let* $X = Z /\!\!/\!\!/\!\!/ \mathsf{G}$*, and suppose* $1 \to \mathsf{G} \to \widetilde{\mathsf{G}} \to \mathsf{G}_{\mathrm{Aut}} \to 1$ *with* $\mathsf{G}_{\mathrm{Aut}}$ *acting on* X*. Then*

$$\begin{array}{ccc} K_{\mathsf{G}_{\mathrm{Aut}}}(X)_{z,mero} & \xrightarrow{\ \mathrm{ch}(\mathrm{Stab}^{\mathrm{Ell}}\ \textit{for}\ \mathsf{G})\ } & K_{\widetilde{\mathsf{G}}}(X)_{z,mero} \\ \Big\downarrow{\scriptstyle \Gamma} & & \Big\downarrow{\scriptstyle \Gamma' \otimes \hbar^{-\frac{\dim X}{4}}} \\ K_{\mathsf{G}_{\mathrm{Aut}}}(X)_{z,mero} & \xrightarrow[\ \textit{vertex with descendents}\ ]{} & K_{\widetilde{\mathsf{G}}}(X)_{z,mero}. \end{array} \tag{105}$$

The vertical arrows Γ and $\Gamma' \otimes \cdots$ are q-analogues of the Iritani map. The subscripts $K_{\mathsf{G}}(X)_{z,\mathrm{mero}}$ mean to take meromorphic functions on

$$\operatorname{Spec} K_{\mathsf{G}}(X) \times \{|q| < 1\} \times \mathrm{Pic}(X) \otimes \mathbb{C}^\times$$

which are analytic in $z \in \mathrm{Pic}(X) \otimes \mathbb{C}^\times$ in a neighborhood of 0. Here 0 is some point on the boundary of a (partial) toric compactification of $\mathrm{Pic}(X) \otimes \mathbb{C}^\times$, and corresponds to the choice of stability condition.

32.6

In particular, as a consequence of the theorem,

$$\begin{aligned}\overset{\alpha}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\beta}{\circ} &= \left(\alpha, \operatorname{Stab}(\beta)\frac{\Gamma}{\Gamma' \otimes \cdots}\right)_{\text{stack}} \\ &= \frac{1}{|W|}\int_{|x_i|=1} \prod \frac{dx_j}{2\pi i x_j} \alpha(x) \operatorname{Stab}(\beta)(x) \frac{\Gamma}{\Gamma' \otimes \cdots}.\end{aligned}$$

Last time we showed that

$$\frac{\operatorname{Stab}(\alpha)}{\Delta_\hbar} \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ \cdot \beta = \alpha\ (\!\!-\!\!\!-\!\!\!-\!\!\circ \cdot \beta,$$

and recall that $(\!\!-\!\!\!-\!\!\!-\!\!\circ$ is the fundamental solution to the q-difference equations.

32.7

Corollary.

$$\left\langle \alpha \,\middle|\, \begin{matrix}\textit{fund.}\\ \textit{solution}\end{matrix} \,\middle|\, \beta \right\rangle = \frac{1}{|W|}\int_{|x_i|=1} \prod \frac{dx_j}{2\pi i x_j} f_\alpha(x) g_\beta(x) \frac{\Gamma}{\Gamma' \otimes \cdots} \tag{106}$$

where:

- f_α *is the K-theoretic stable envelope of* α*;*
- g_β *is the elliptic stable envelope of* β*;*
- *the* $\Delta_\hbar$ *factor has been absorbed into the gamma functions.*

The difference between this and the formulas from last time is that the $g_\beta(x)$ factor was not present, and instead we integrated over some contour, depending on stability conditions, different from $\{|x_i| = 1\}$. Importantly, note that the Γ factors and the choice of contour are interchangeable in the world of q-difference equations; the former is elliptic and therefore behaves like a constant with respect to q-shifts, and changing the integrand by a constant is like changing the contour. It is much better to insert a function g_β instead, which takes the place of integrating over some complicated contour.

A beautiful consequence of this integral representation is that it immediately gives the analytic continuation and monodromy of the vertex functions. This is because the integrand in (106) is already meromorphic. Put differently, the vertex with descendents in (105) is a series in z, and the other way of traversing the commutative square is its analytic continuation.

32.8

The Γ_q factors have to do with maps from $\mathbb{A}^1$ (or the formal disk) to X. Namely, consider $\operatorname{Maps}(\mathbb{A}^1 \to \mathbb{A}^1)$ and let q and w act on the source and target respectively by scaling. It is some infinite-dimensional space, with coordinates which are polynomails $1, t, t^2, t^3, \ldots$. These coordinates have weights $w, w/q, w/q^2, \ldots$, and therefore

$$\mathcal{O}_{\mathrm{Maps}(\mathbb{A}^1 \to \mathbb{A}^1)} = \frac{1}{(1-w^{-1})(1-qw^{-1})(1-q^2w^{-1})\cdots} = \Gamma_q(w^{-1}).$$

This is the origin of q-Gamma factors in K-theory, and is much more direct than the original story by Iritani in cohomology. More generally, if S is a smooth stack and $\mathrm{ev}_0 \colon \mathrm{Maps}(\mathbb{A}^1 \to S) \to S$ is evaluation at $0 \in \mathbb{A}^1$,

$$\mathrm{ev}_{0,*} \mathcal{O}_{\mathrm{Maps}(\mathbb{A}^1 \to S)} = \Gamma_q(qT^\vee S)$$

where we extend $\Gamma_q(-)$ as a multiplicative genus.

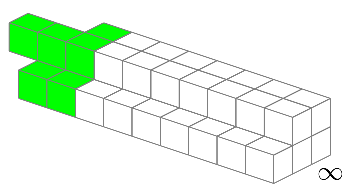
In the diagram (105),

$$\begin{aligned} \Gamma' &:= \Gamma_q(qT^\vee X - q\mathfrak{g}^\vee + q\mathfrak{g}^*) \\ \Gamma &:= \Gamma_q(qT^\vee X - q\mathfrak{g}^\vee + \mathfrak{g}^*). \end{aligned}$$

The $-q\mathfrak{g}^\vee$ factor comes from the quotient by G, while $\mathfrak{g}^*$ is as in the target of the moment map, with a weight $\hbar$. We add $\mathfrak{g}^*$ because it is much easier to work with p-fields.

32.9

Proof idea. We look at the main ideas in the example of $\mathrm{Hilb}(\mathbb{C}^2, 5)$, where the quasimap moduli is exactly the PT moduli. Specifically, consider the point



and its contribution to the matrix element.

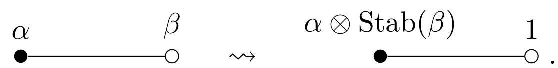
1. The contribution of the (unsymmetrized) virtual structure sheaf $\mathcal{O}_{\mathrm{vir}}$, at this point, has the form
$$\frac{\mathrm{ev}_\infty(\Gamma_q \text{ expression})}{\mathrm{ev}_0(\Gamma_q \text{ expression})}.$$
This is because $\mathcal{O}_{\mathrm{vir}}$ comes from the virtual tangent space
$$T_{\mathrm{vir}} = H^0(\cdots) - H^1(\cdots) = \frac{\text{term at } 0}{1-q^{-1}} - \frac{\text{term at } \infty \cdot q^{-1}}{1-q^{-1}},$$
and these terms produce the Γ_q expressions. Hence the vertical arrows in (105) take care of $\mathcal{O}_{\mathrm{vir}}$.

2. Let γ be an elliptic stable envelope of some class. Then

$$\frac{\mathrm{ev}_0^*(\gamma)}{\mathrm{ev}_\infty^*(\gamma)} = \frac{\text{section of some line bundle } (\dots, q^3 t_1^{-1} t_2^{-1}, \dots)}{\text{section of some line bundle } (\dots, t_1^{-1} t_2^{-1}, \dots)}$$
$$= z^{-\text{degree}} \cdot (\text{constant in } z),$$

and in fact this constant turns $\mathcal{O}_{\text{vir}}$ into the symmetrized $\widehat{\mathcal{O}}_{\text{vir}} \otimes \hbar^{-\frac{\dim}{4}}$.

3. Using (1) and (2), (and Theorem 31.4), we can move the insertion at ∞ to 0:



It remains to forget the non-singularity condition at ∞, i.e. to extend the integration over

$$\mathrm{QMaps}(X) \subset_{\text{stable}} \mathrm{Maps}(\mathbb{A}^1, \text{stack})$$

from the stable locus to the whole space, since integration over $\mathrm{Maps}(\mathbb{A}^1, \text{stack})$ is what gives the Γ_q factors. This is the hardest step and can be found in [Oko20, Section 3.4].

□

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