# Enumerative geometry meets number theory

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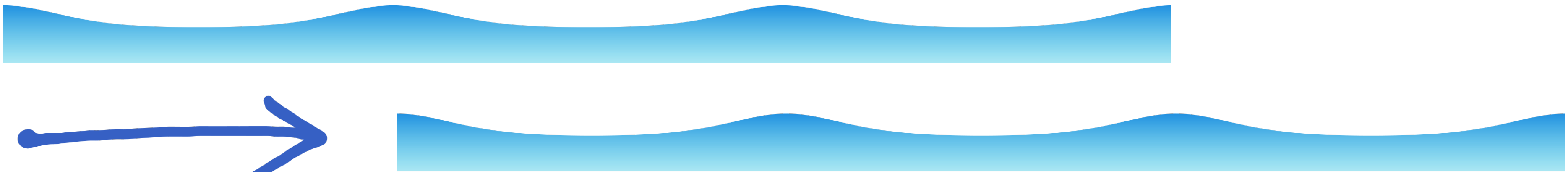
based on a joint work with D. Kazhdan

# Part 1. Periodic functions



periodicity means that a translational symmetry in space/time is broken to subgroup which looks like the original group at large distances

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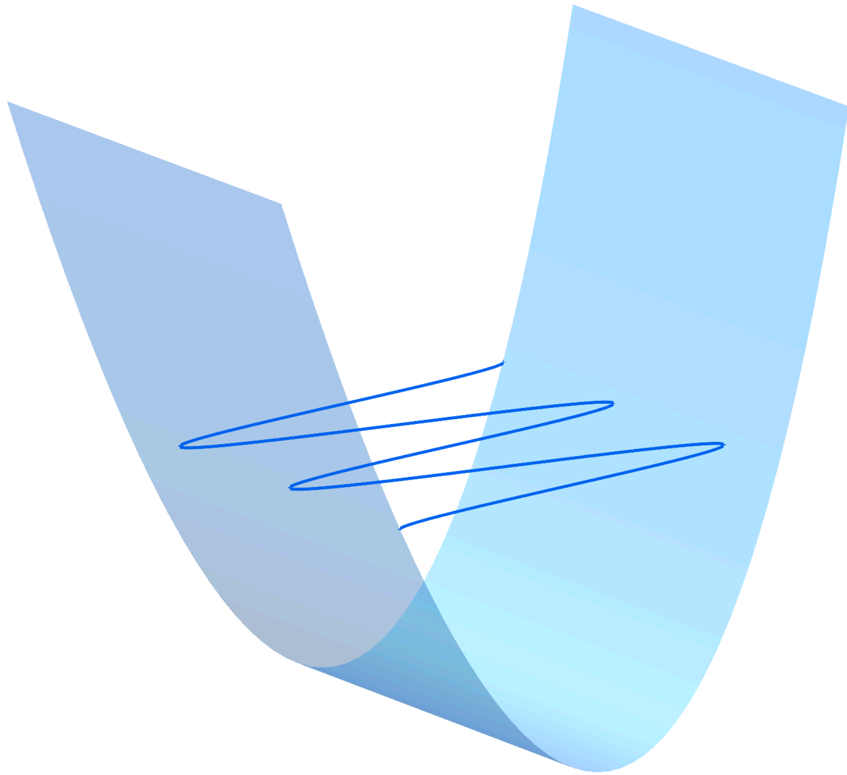
*e.g.* translation by $1$ water wavelength on the ocean scale, or translation by $4.5{\cdot}10^{-7}m$ ($\approx$ the wavelength of this color) on the human scale

Periodic functions *exist* in *abundance around* us starting with the classical mechanics, where *e.g.*

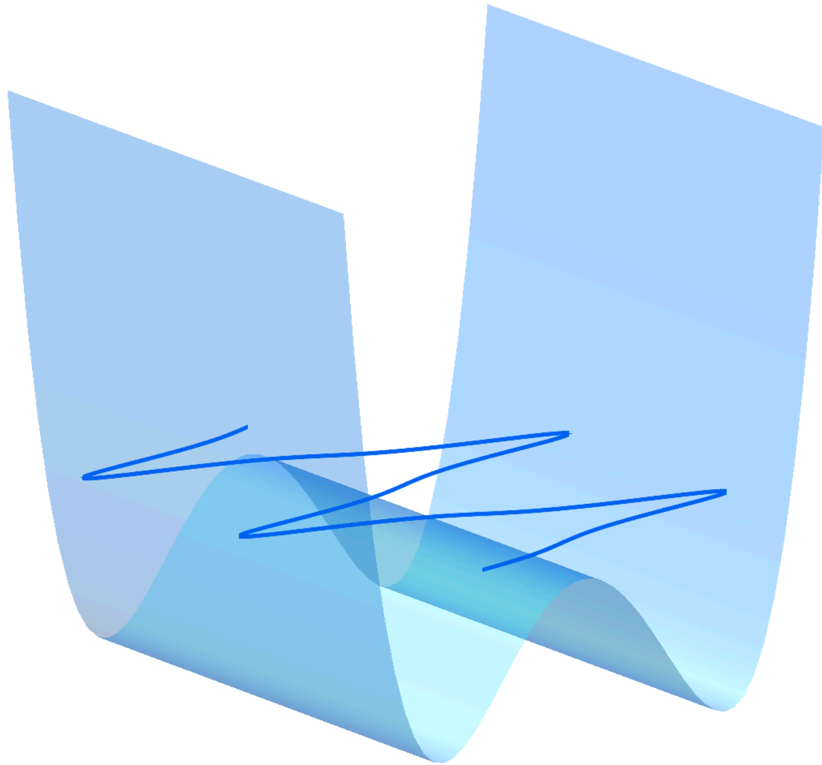
$$x = \cos(\omega t)$$

describes the motion in *a* 1-dimensional harmonic well.



We rely on periodic phenomena to keep time *and* measure distances

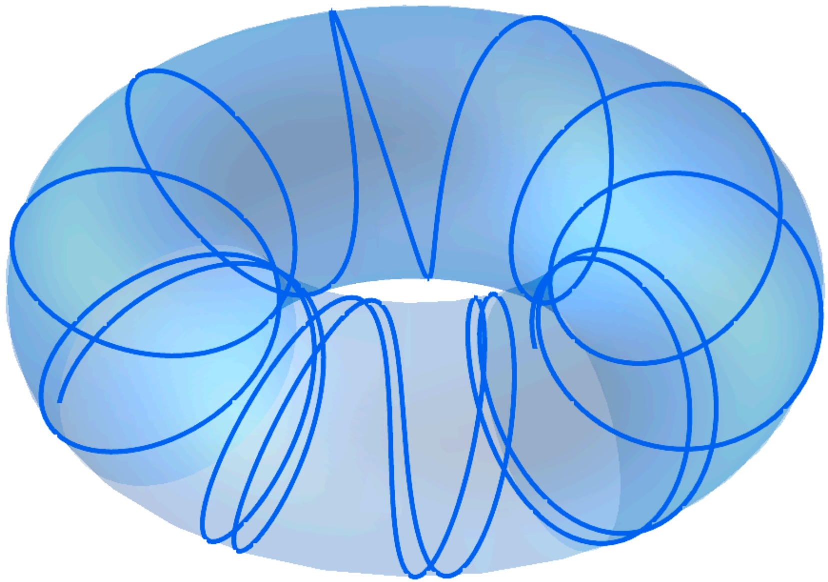
In fact, the classical motion in any 1-dimensional well is not just periodic but, being integrable, *ex*-hibits *additional* periodicity for complex time.



Fun to see it explicitly for the physical pendulum, that is, for solutions of $\ddot{\phi}(t) = -\sin\phi(t)$, which is not *exactly* in *a* well and *can* swing around.

In integrable dynamical systems, trajectories lie on tori, and thus demonstrate a periodic or quasiperiodic behaviour, depending on whether the line of the physical time closes in the given torus.

This also works for complex time and complex tori.



Tori may degenerate by having one or more periods go to infinity, this will be important later.

A famous example are integrable equations describing nonlinear waves.

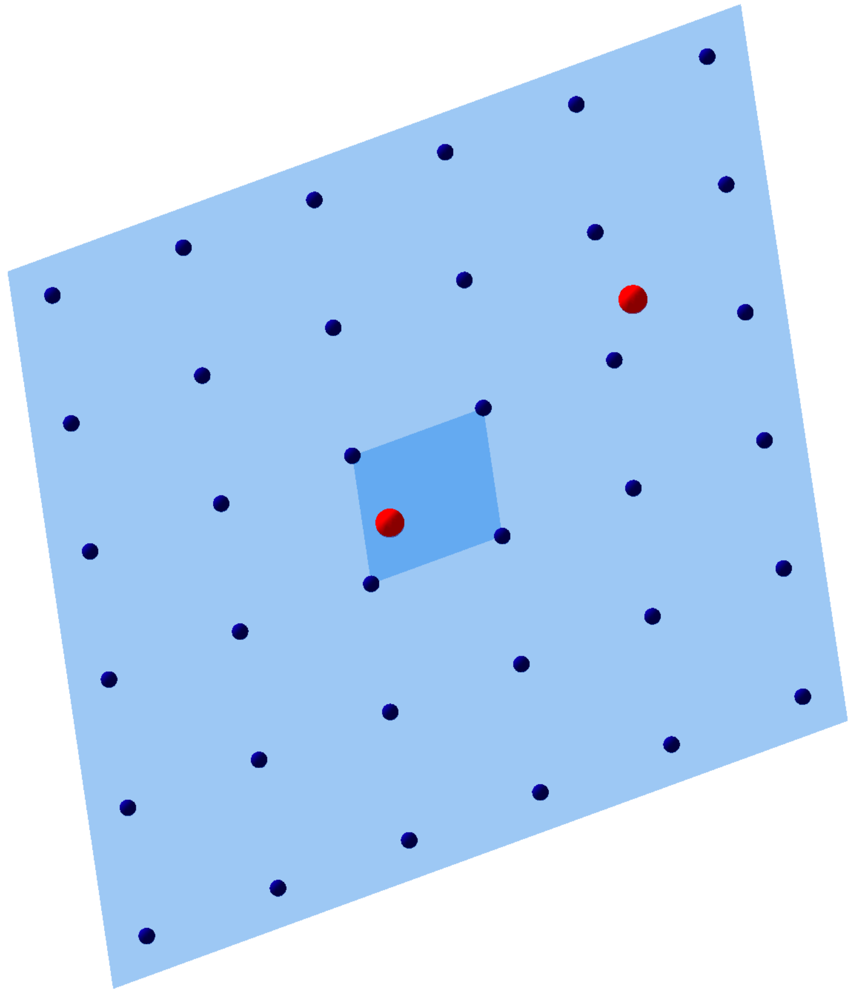
They exhibit a hypnotizing variety of periodic and quasiperiodic patterns, including solitary waves when the periods become infinite.

cnoidal wave for the Korteweg-deVries equation, from Wikimedia, ©Kraaiennest

A torus is a quotient $V/\Gamma$ of a vector space $V$ by a lattice $\Gamma$ of full rank.

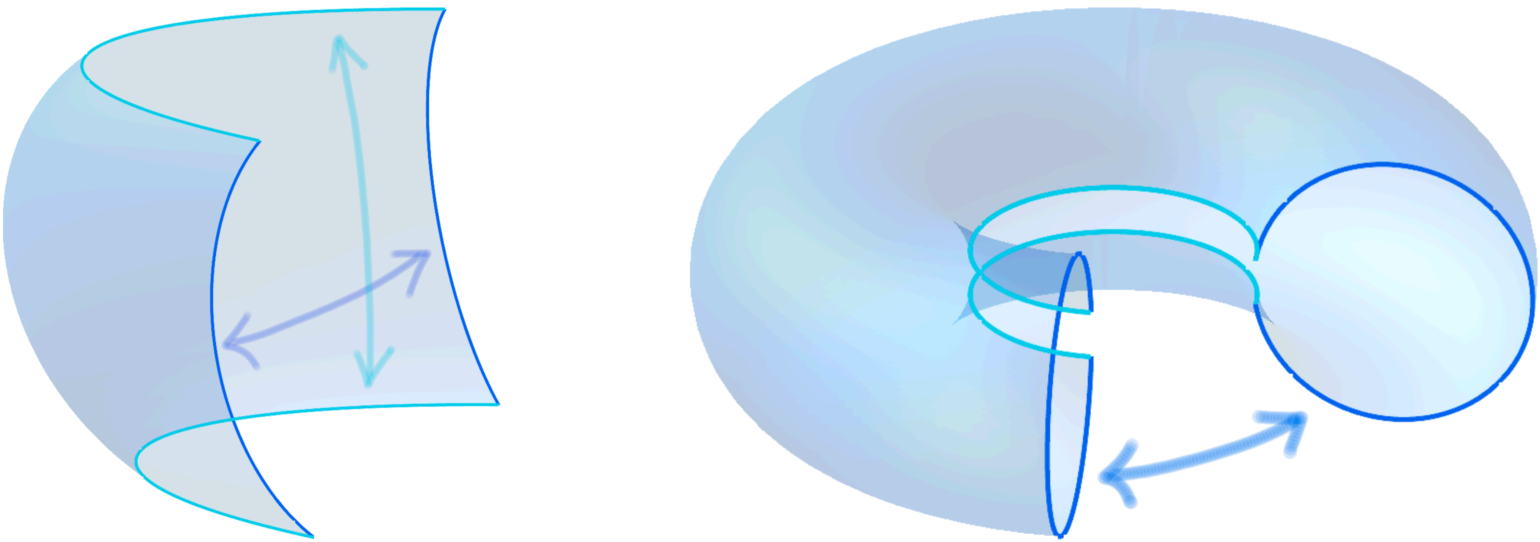
This means we identify two points if their difference lies in the lattice of periods.

Every point has a representative in the fundamental parallelepiped, but neither the representative nor the fundamental parallelepiped are unique.



For a complex torus, the rank of the lattice is twice the complex dimension

We can also represent a torus $V/\Gamma$ by a fundamental parallelepiped with opposite sides glued. This procedure is familiar to anyone who *ever* imposed periodic boundary condition on a system in a box.



lattices $\Gamma \subset V$ and tori $V/\Gamma$ are the same thing

$\Gamma$-periodic functions of $v \in V$ are functions on the corresponding torus $V/\Gamma$

An exponential $e^{2\pi i(\lambda,v)}$ is periodic if and only if $\lambda$ lies in the dual lattice $\Gamma^*$. They form a Fourier basis in periodic functions

In particular, the spectrum of the Laplace operator $\Delta$ on the torus $V/\Gamma$ is $-4\pi^2$ times the lengths squared $\|\lambda\|^2$ of the vectors $\lambda \in \Gamma^*$ in the dual lattice

# Part 2. Automorphic functions

Superficially, just like periodic functions, except we:

- replace a vector space $V$ by a noncommutative group $\mathsf{G}$, for instance the group $SL(n, \mathbb{R})$ of $n \times n$ real matrices with $\det = 1$,
- take the one-sided quotient $\mathsf{G}/\Gamma$ by a lattice $\Gamma \subset \mathsf{G}$, for instance the group $\Gamma = SL(n, \mathbb{Z})$ of integer matrices with $\det = 1$

Noncommutativity has a very profound effect

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By definition, $\Gamma \subset \mathsf{G}$ is a lattice if the volume of $\mathsf{G}/\Gamma$ is finite. This is a quantitative way to say that $\Gamma$ looks like $\mathsf{G}$ at large distances.

Our main interest will be the spectrum of $\Delta$ on $\mathsf{G}/\Gamma$ in a certain very general situation.

But to keep this talk concrete, we will stick to one example

$$\mathsf{G}/\Gamma = SL(n,\mathbb{R})/SL(n,\mathbb{Z})\,.$$

In fact, we are already familiar with the points of this space.

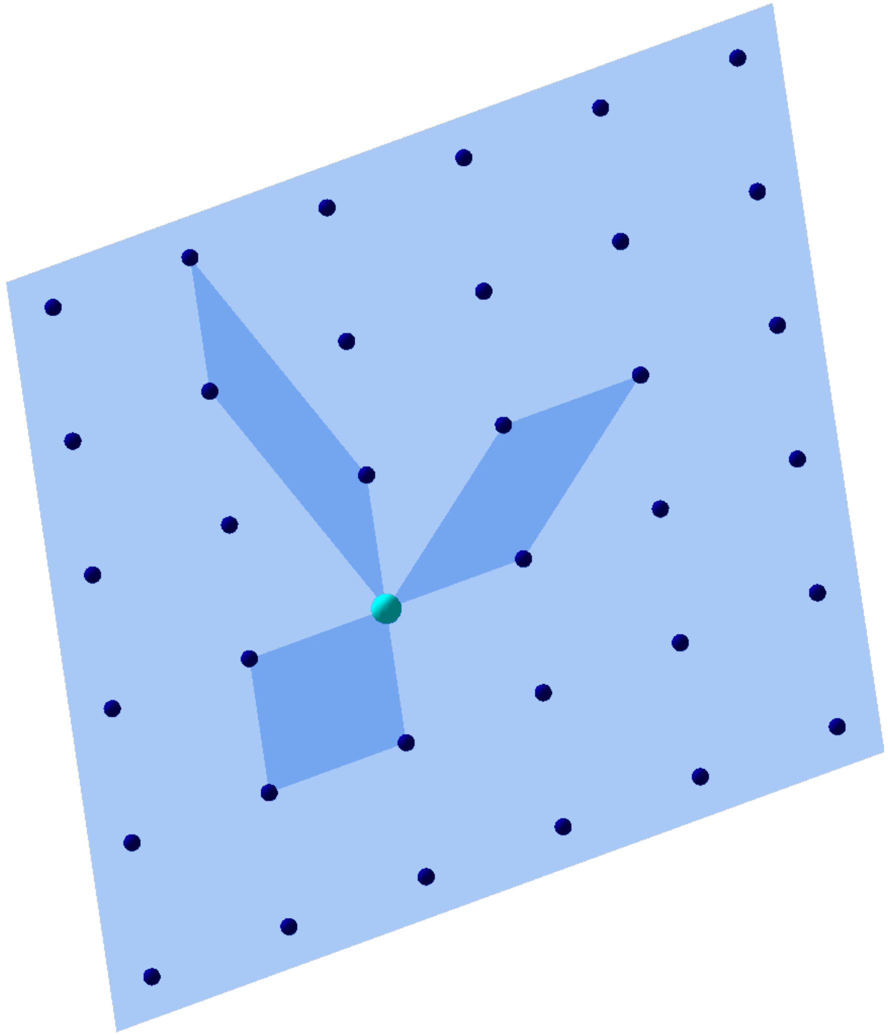
The points of

$$SL(n,\mathbb{R})/SL(n,\mathbb{Z})$$

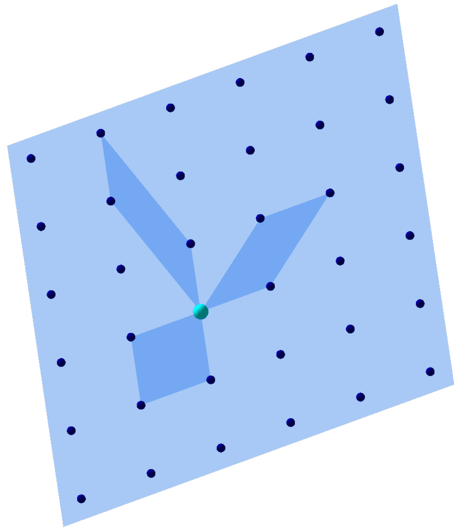
correspond to lattices $\Omega \subset \mathbb{R}^n$ such that $\mathrm{vol}(\mathbb{R}^n/\Omega) = 1$.

A basis $\{\omega_1, \ldots, \omega_n\}$ of such lattice is a matrix with det $= 1$.

Two bases generate the same lattice if they differ by a matrix in $SL(n,\mathbb{Z})$.

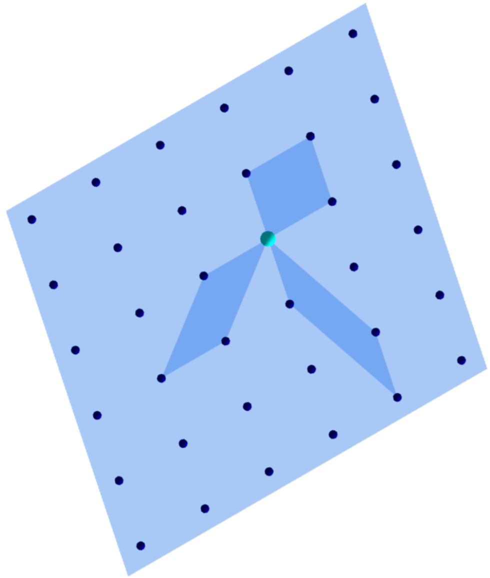


What comes up in many application is the space of lattices $\Omega$ up to isometry, that is, up to rotations of $\mathbb{R}^n$.



This is the double quotient

$$SO(n,\mathbb{R})\backslash SL(n,\mathbb{R})/SL(n,\mathbb{Z})$$



Instead of fixing $\mathrm{vol}(\mathbb{R}^n/\Omega) = 1$, we can consider lattices up to scale.

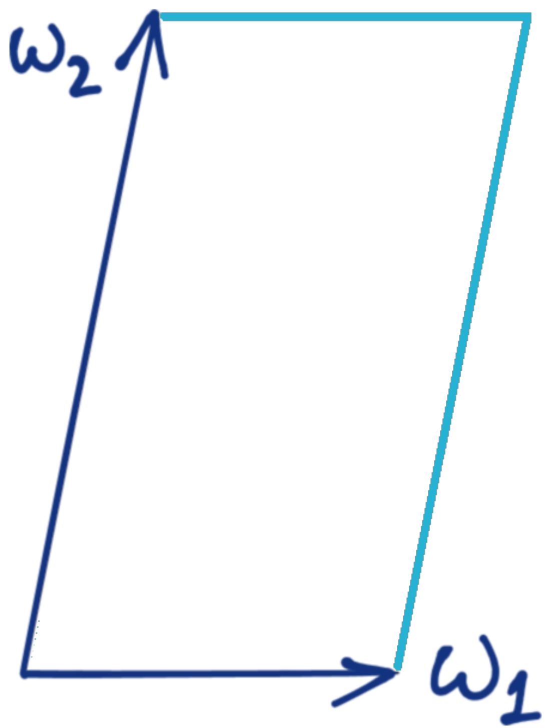
Let's see what this space is like for $n = 2$.

To visualize

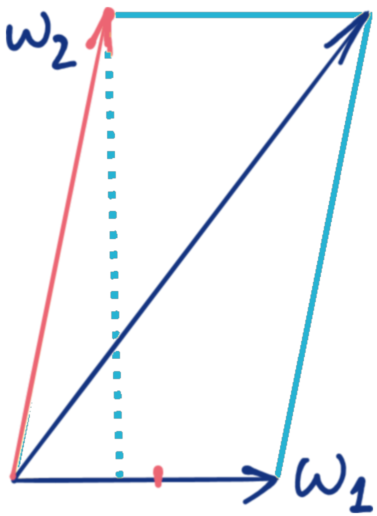
$$SO(2,\mathbb{R})\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$$

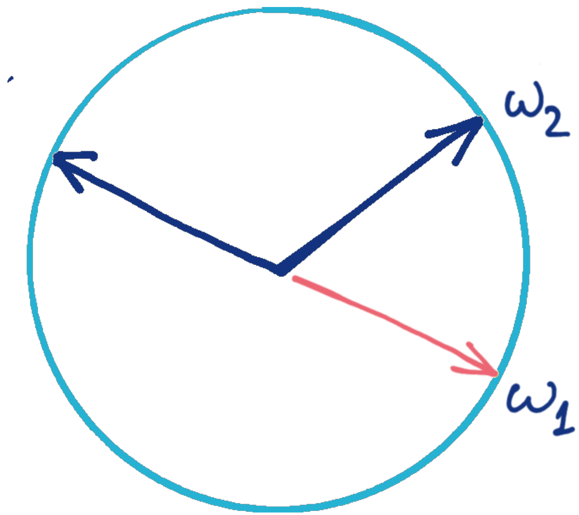
let's pick a basis $\{\omega_1, \omega_2\} \subset \Omega$ such that:

- $\omega_1$ is the shortest vector,
- $\omega_2$ is the shortest vector among those not proportional to $\omega_1$.



It follows that $|(\omega_1, \omega_2)| \leq \frac{1}{2}\|\omega_1\|^2$ and if it happens that $\|\omega_2\| = \|\omega_1\|$ then we can assume that the angle between them is $\leq 90°$.



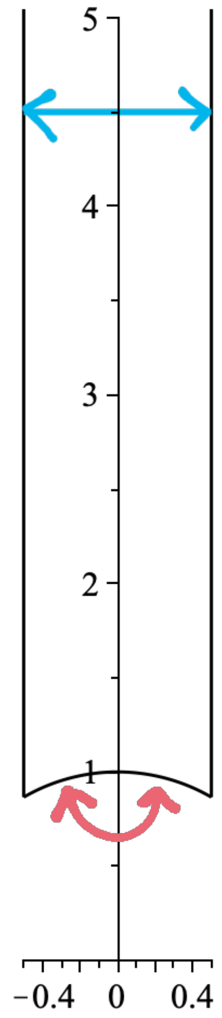
So, if we take

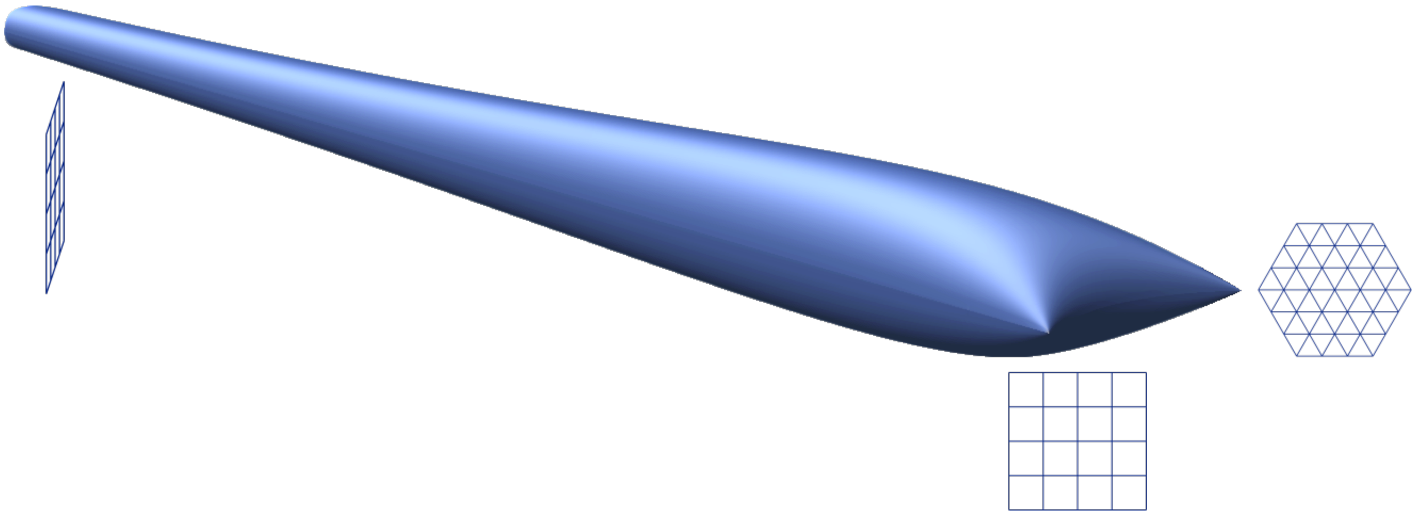
$$\omega_1 = (1, 0)\,,$$

then the possibilities for $\omega_2$ are

$$\omega_2 \in \Big\{(x, y) \,\Big|\, |x| \leq \tfrac{1}{2}\,, \\ x^2 + y^2 \geq 1\Big\}\,,$$

and we glue the points of the boundary of this domain by the reflection in the $y$-axis.





Once we glue the boundaries, we get an object like this. Note it is not compact. At infinity (called the "cusp") of $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ are unimodular lattices $\Omega$ that have a very short vector $\omega_1$.

Whence there will be a mixture of discrete and continuous spectrum for the Laplace operator.

Automorphic functions are eigenfunctions of the Laplace operator (and of the related Hecke operators) on $\mathsf{G}/\Gamma$.

They are incredibly important in mathematics, especially in number theory. Suffices to say that at the core of Wiles' proof of Fermat's last theorem was the following (extremely profound) correspondence

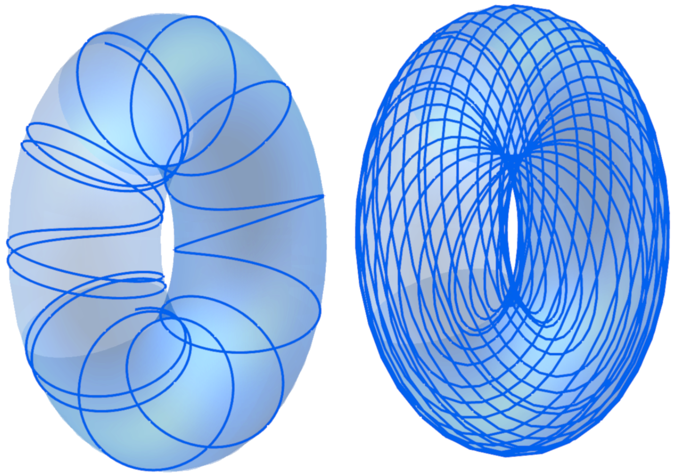
$$y^2 = x(x-B)(x-C) \longrightarrow \text{an automorphic function}$$

$$y^2 = x(x-b^p)(x-c^p) \longrightarrow \text{problem if } a^p + b^p = c^p$$

In physics, classical or quantum dynamics on the space of parameters comes up, for instance, when there is a big separation of temporal/spatial scales.

The fast dynamics can fill out a torus, while the slow *averaged* dynamics is taking place on the space of tori.

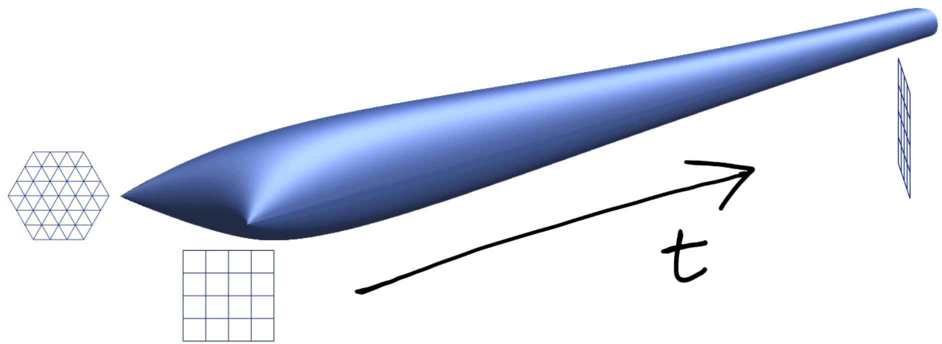
The fast physics may be as simple as a particle bouncing in a potential or as complex as a string *exporing* a geometry set by a given period matrix.



# Part 3. Eisenstein spectrum

Ferdinand Gotthold Max Eisenstein (Berlin, 1823 – Berlin, 1852) lived a very short, tragic, and influential life, very much like Galois and Abel.

Back to the space of lattices, we want to know the Laplace *eigenfunctions* on this *surface* with two conical points *and a* cusp.



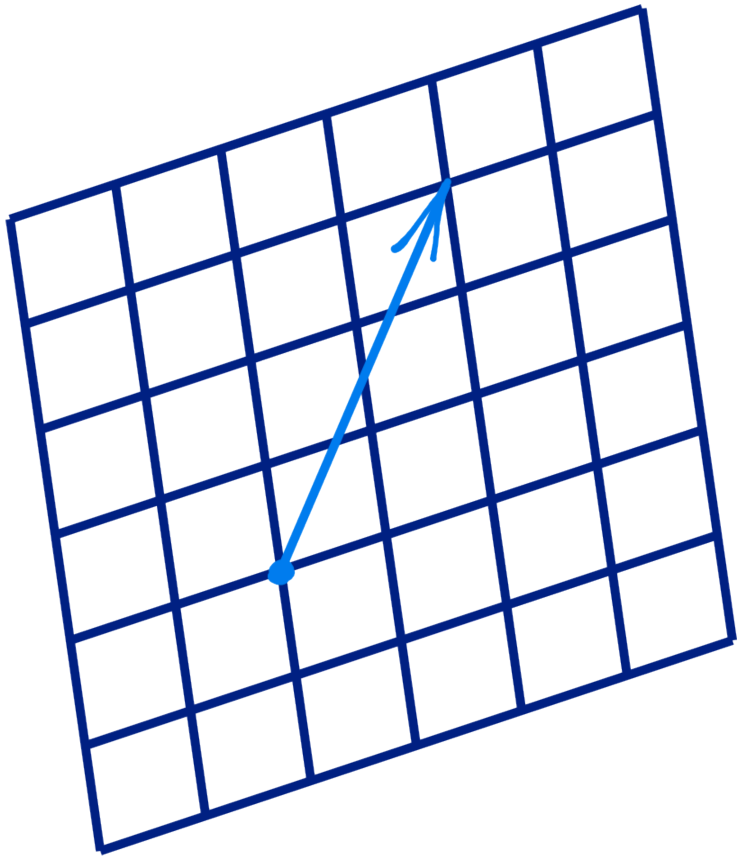

Fall into two *categories*:

(B) Bound states. *Decay* superexponentially with the distance $t$ into the cusp. Important, but very mysterious

(E) Eisenstein spectrum. These come from the infinity *and are* of interest to us. Lies in the span of rather concerete functions $E(\lambda, \Omega)$.

Concretely,

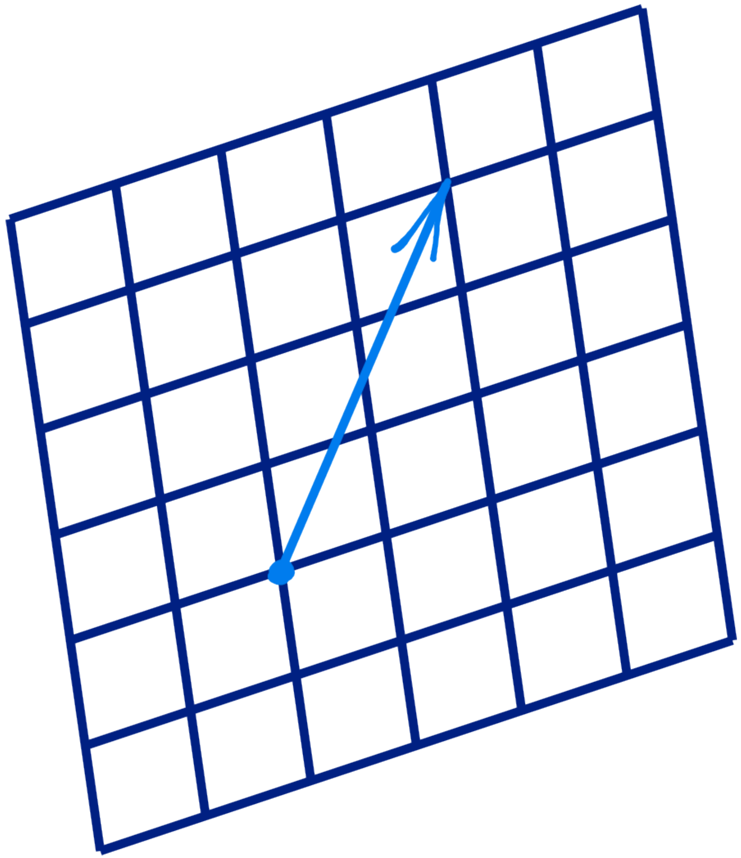
$$E(\lambda, \Omega) = \tfrac{1}{2} \sum_{\text{primitive } \omega \in \Omega} \frac{1}{\|\omega\|^{\lambda+1}}.$$



For the dual torus $R^2/\Omega^*$, this is essentially $\operatorname{tr} \Delta^{-\frac{\lambda+1}{2}}$, so a spectral $\zeta$-function.

Concretely,

$$E(\lambda, \Omega) = \tfrac{1}{2} \sum_{\text{primitive } \omega \in \Omega} \frac{1}{\|\omega\|^{\lambda+1}}.$$



For $t \to \infty$, these behave like

$$E(\lambda, \Omega) = e^{\frac{(1+\lambda)t}{2}} + \frac{\xi(\lambda)}{\xi(-\lambda)} e^{\frac{(1-\lambda)t}{2}} + \dots$$

and since the radius of goes like $R = e^{-t}$ in the cusp, none of these are in $L^2$.

The function $\xi(\lambda)$ in the reflection coefficient is, essentially, the Riemann $\zeta$-function.

A more detailed look at

$$E(\lambda,\Omega) = e^{\frac{(1+\lambda)t}{2}} + \frac{\xi(\lambda)}{\xi(-\lambda)}\, e^{\frac{(1-\lambda)t}{2}} + \dots\,, \qquad R = e^{-t}\,,$$

shows the Eisenstein spectrum has two pieces:

- the functions for $\lambda \in i\mathbb{R}$ are almost in $L^2$ and contribute to the continuous spectrum $\frac{\lambda^2}{4} - \frac{1}{4} \in [-\infty, -\frac{1}{4}]$
- there is a pole of $\xi(\lambda)$ at $\lambda = 1$, and the residue at that pole gives the constant function, which is in $L^2$.

There is the following nonobvious way to state this result:

$$E(\lambda) \text{ is in the spectrum} \Leftrightarrow \begin{bmatrix} \lambda & \\ & -\lambda \end{bmatrix} = \phi\left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\right) + X\,,$$

where

$$\phi : \text{Lie}\, SL(2,\mathbb{C}) \to \text{Lie}\, SL(2,\mathbb{C})$$

preserves the commutation relations and $X = -X^* \in \text{Lie}\, SL(2,\mathbb{C})$ commutes with all matrices in the image of $\phi$. Indeed, the two possibilities are:

$$\begin{array}{lll} \phi(\xi) = \xi & X = 0 & \left[\begin{smallmatrix} \lambda & \\ & -\lambda \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right] \\ \phi(\xi) = 0 & X \sim \left[\begin{smallmatrix} is & \\ & -is \end{smallmatrix}\right] & \left[\begin{smallmatrix} \lambda & \\ & -\lambda \end{smallmatrix}\right] = X \end{array}$$

Several generations ago, R. Langlands conjectured that this is a general pattern for <u>any</u> split reductive group $\mathsf{G}$ over a global field $\mathbb{F}$, the above example corresponding to

$$\mathsf{G} = SL(2)\,, \quad \mathbb{F} = \text{rational numbers } \mathbb{Q}\,.$$

Namely, the Eisenstein spectrum in

$$L^2(\text{maximal compact}\backslash\mathsf{G}/\Gamma)$$

is always parametrized by homomorphisms

$$\phi : SL(2,\mathbb{C}) \to {}^{\mathsf{L}}\mathsf{G}(\mathbb{C})\,,$$

where ${}^{\mathsf{L}}\mathsf{G}$ is the Langlands dual group.

There were many result for in that direction by R. Langlands himself, C. Mœglin and J.-L. Waldspurger, V. Heiermann, M. de Martino and E. Opdam.

In recent joint work with D. Kazhdan, we prove the conjecture for arbitrary $\mathsf{G}$ and $\mathbb{F}$.

Our main tool is a topological interpretation of computation with Eisenstein series. Namely, we translate them into computations with characteristic classes of

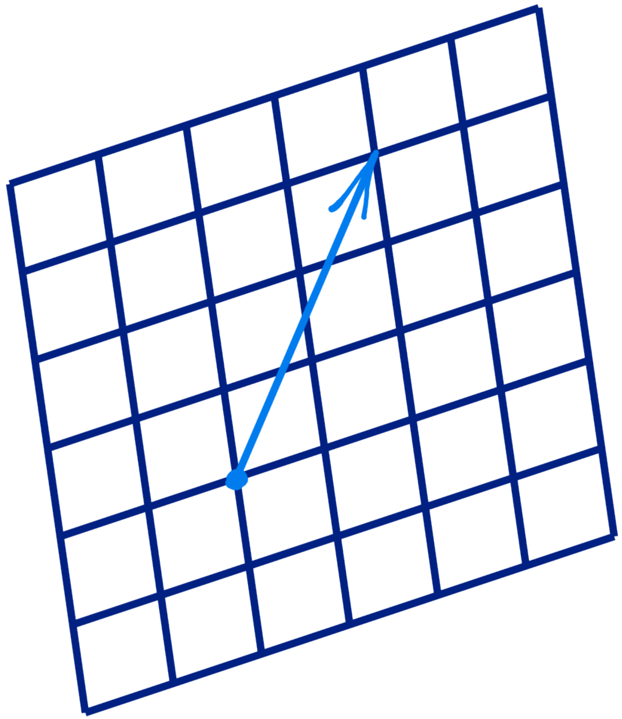
$$^{\mathsf{L}}\mathsf{X} = T^*\left({}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B}\right) .$$

This is not that far from the interests of many high-energy physicists at the IPMU, as I will try to explain.

# Part 4. Enumerative geometry

Recall that

$$E(\lambda, \Omega) = \frac{1}{2} \sum_{\text{primitive } \omega \in \Omega} \frac{1}{\|\omega\|^{\lambda+1}}\,.$$



The sum is effectively over

$$r = \text{slope}(\omega) \in \mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q})\,,$$

e.g. here

$$\omega = 2\omega_1 + 3\omega_2$$

so $r = 3/2$. We have

$$\|\omega\| = \text{height}_\Omega(r)\,.$$

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For a general group G over a number field $\mathbb{F}$, Eisenstein series count heights of $\mathbb{F}$-points in the corresponding flag varieties.

There is a well-known parallel between rational numbers and rational functions (or, more generally, rational functions on an algebraic curve/Riemann surface $C$)

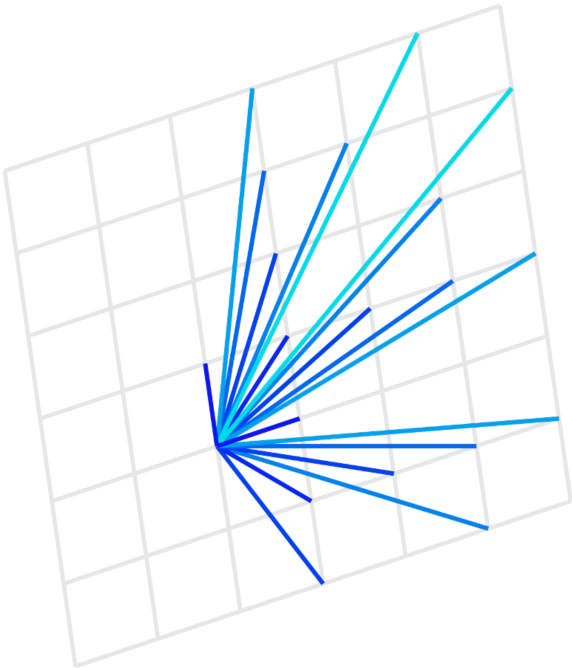
$$r = \pm \frac{2^3 \cdot 7 \cdot 71}{3^2 \cdot 7919} \quad \Leftrightarrow \quad f(x) = c\,\frac{(x-z_1)^3(x-z_2)(x-z_3)}{(x-w_1)^2(x-w_2)}\,,$$

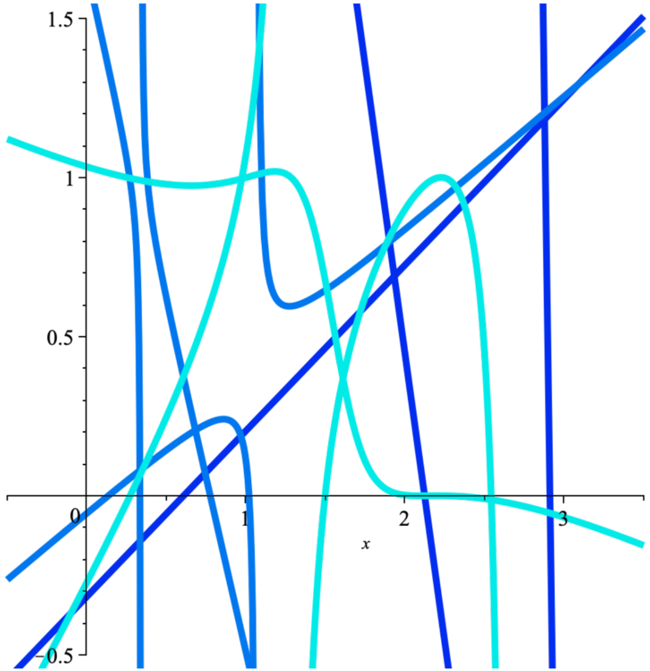
in which the prime factors of $r$ correspond to zeros and poles of $f(x)$.

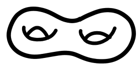
In this parallel,

$$\text{height}(r)^{-\lambda} \quad \Leftrightarrow \quad e^{-\lambda \deg f}$$

Thus counting rational numbers $r$ weighted by $\text{height}(r)^{-\lambda}$ is like counting rational functions $f$ weighted by $e^{-\lambda \deg f}$






For complex $f$, these are the holomorphic maps  $\to \mathbb{CP}^1$.

For many people working in or near susy string and gauge theories, it is a daily routine to count holomorphic maps

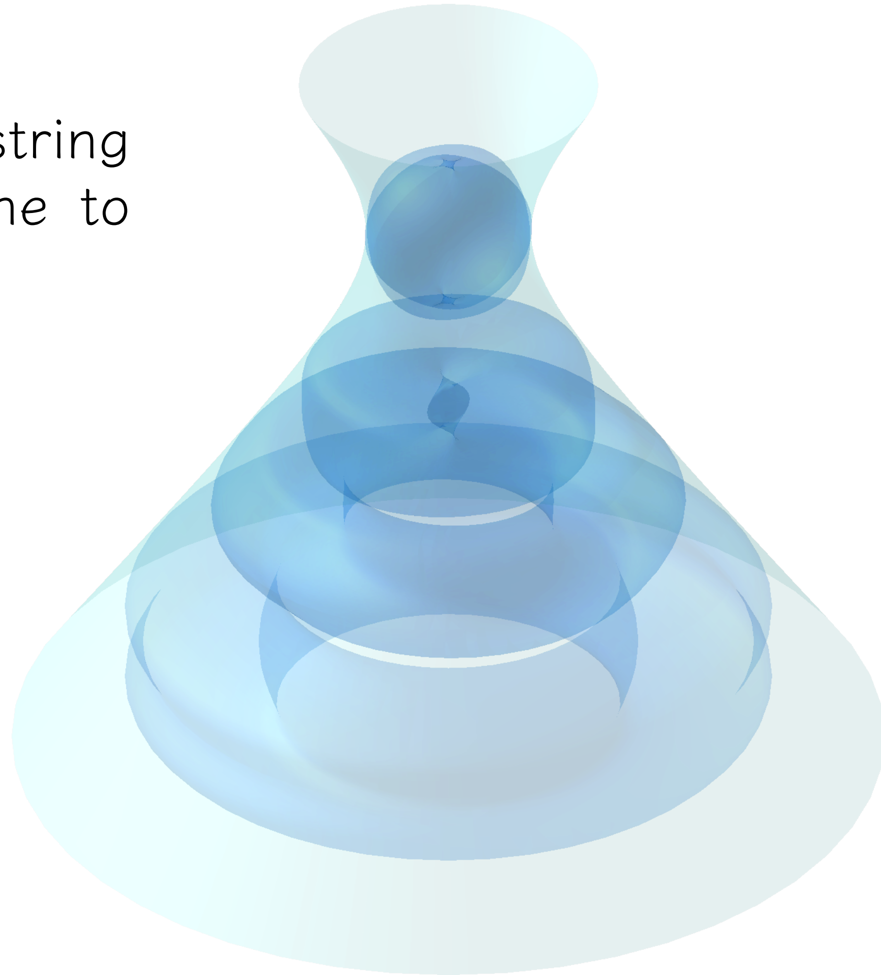
$$\to \mathsf{X}\,,$$

weighted by

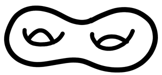
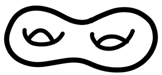
$$e^{-\lambda \deg f} = \exp\left(-\lambda\, \mathrm{Area}(f(\;))\right)$$

The relevant targets for us are:

$$\mathsf{X} = \mathsf{G}/\mathsf{B}\,, \quad \text{or} \quad T^*(\mathsf{G}/\mathsf{B})\,.$$



Very powerful ideas have been developed to do these counts, including those of 3-dimensional mirror symmetry. These, in particular, relate counts

 $\to \mathsf{X} = T^*(\mathsf{G}/\mathsf{B}) \quad \Leftrightarrow \quad$  $\to {}^{\mathsf{L}}\mathsf{X} = T^*({}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B})$

with a very dramatic exchange of parameters.

Many fundamental contributions to this subject came from our IMPU people.

For this talk, the main takeaway from this is the following metaprinciple:

> "All" integrals involving Eisenstein series for $\mathsf{G}$ can be expressed as certain characteristic classes of $T^*({}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B})$.

Characteristic classes can be defined, for instance, as integrals constructed from curvature invariants. Very basic object in many branches of physics, from GR to strings, to phases of matter.

For the spectral decomposition we use integrals of the form

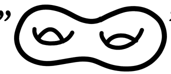
$$\int_{\mathsf{G}/\Gamma} \psi\,(\Delta^k \psi')\,dg = \int_{T^*({}^{\mathsf{L}}\mathsf{B}\backslash{}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B})} C_1(\psi)\,C_2(\Delta)^k\,C_3(\psi')\,C_4(dg)\,,$$

where $\psi$ and $\psi'$ are arbitrary functions in the span of the Eisenstein series and $C_i(\,\cdot\,)$ are certain characteristic classes, of which $C_4$ is the most interesting one.

The exact form of $C_i(\,\cdot\,)$ is not important here, because the spectral decomposition corresponds to the decomposition of the space itself:

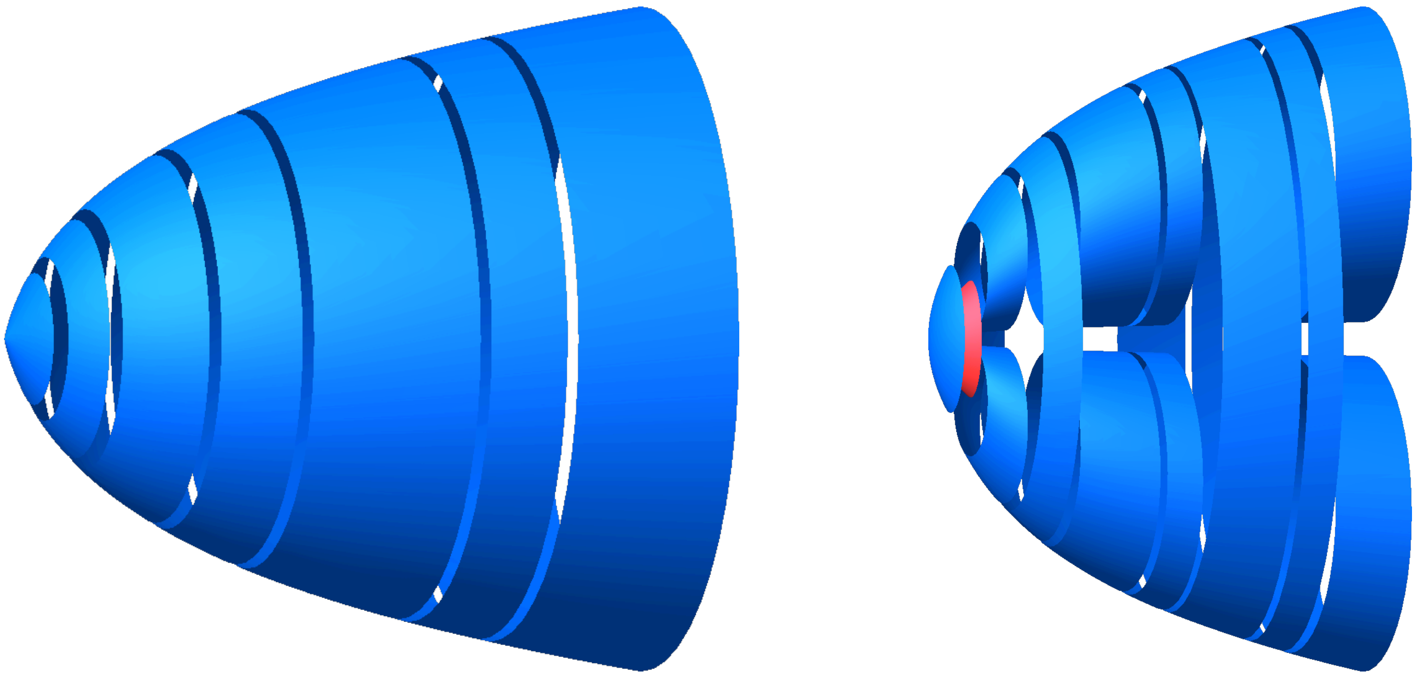
$$T^*({}^{\mathsf{L}}\mathsf{B}\backslash{}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B}) = \bigsqcup_{\text{nilpotent } e\,\in\,{}^{\mathsf{L}}\mathfrak{g}} \mathsf{Fix}_e({}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B})^{\times 2}\Big/\mathsf{Fix}_e{}^{\mathsf{L}}\mathsf{G}$$

But since it is natural to look deeper into the nature of these characteristic classes, one natural interpretation of them is as of counts of maps

$\mathscr{Q} =$ ”  ” $\to {}^{\mathsf{L}}\mathsf{X} = T^*({}^{\mathsf{L}}\mathsf{G}/{}^{\mathsf{L}}\mathsf{B})\,,$

where the source curve $\mathscr{Q}$ is the best complex likeness of a global field $\mathbb{F}$.

Namely $\mathscr{Q}$ has an strip of length $\ln p$ for every nonarchimedian place of $\mathbb{F}$ and disk with cone angle $\pi$ (resp., $2\pi$) for every real/complex place of $\mathbb{F}$. Here what it looks like for $F = \mathbb{Q}$ and $F = \mathbb{Q}(\sqrt{-1})$.



The red strip corresponds to the ramification $(1+\sqrt{-1})^2 = 2\sqrt{-1}$ of $p = 2$ in $\mathbb{Q}(\sqrt{-1})$.