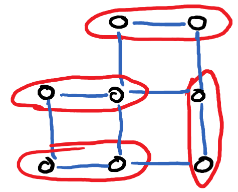
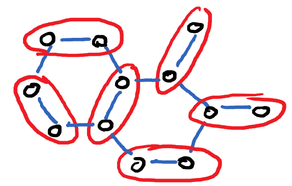
Geometry of dimer models

A. Borodin

based on joint works with T. Berggren and M. Duits

Dimer models study random dimer coverings or perfect matchings on a given graph.





- Introduced in 1937 by Fowler-Rushbrooke for "statistical theory of perfect solutions" in liquid mixtures.

- For planar graphs, "solved" via Pfaffians/dets in early 1960's by Kasteleyn and Temperley-Fisher; connection to the Ising model of ferromagnetism.

- In mid-1980's, related to roughening transitions in equilibrium crystals by Nienhuis-Hilhorst-Blöte.

- Since early 1990's studied by mathematicians.

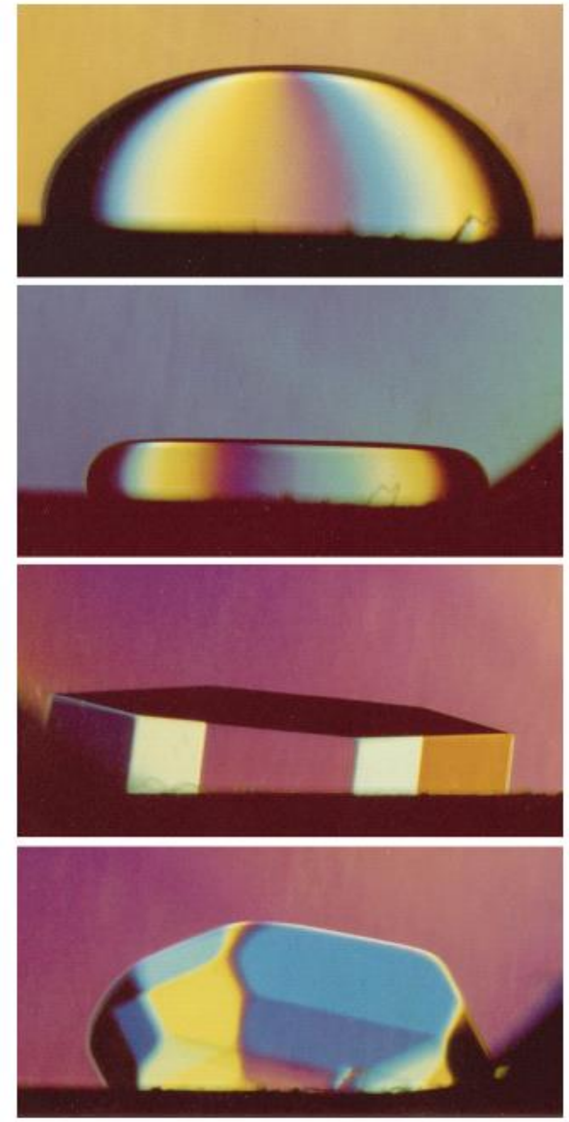
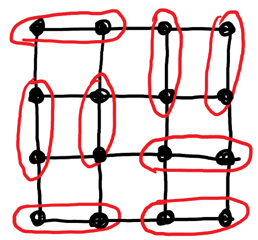
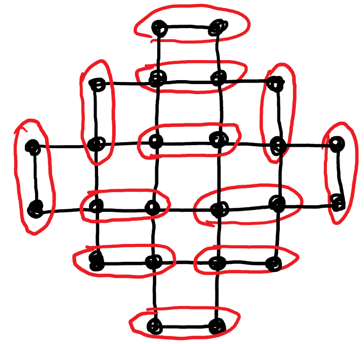


FIG. 2. (Color) Faceting of ^{4}He crystals. As temperature goes down, more and more facets appear at the surface of ^{4}He crystals. From top to bottom, the temperature is successively 1.4, 1, 0.4, and 0.1 K. The size of the facets is larger than on equilibrium shapes, due to the slow growing from the surrounding superfluid (see Fig. 20). The colors are real, obtained by Balibar, Guthmann, and Rolley (1994) with a prism, a lens, and a small mask (see Sec. II.B.1).

This talk — square domains in a square lattice.

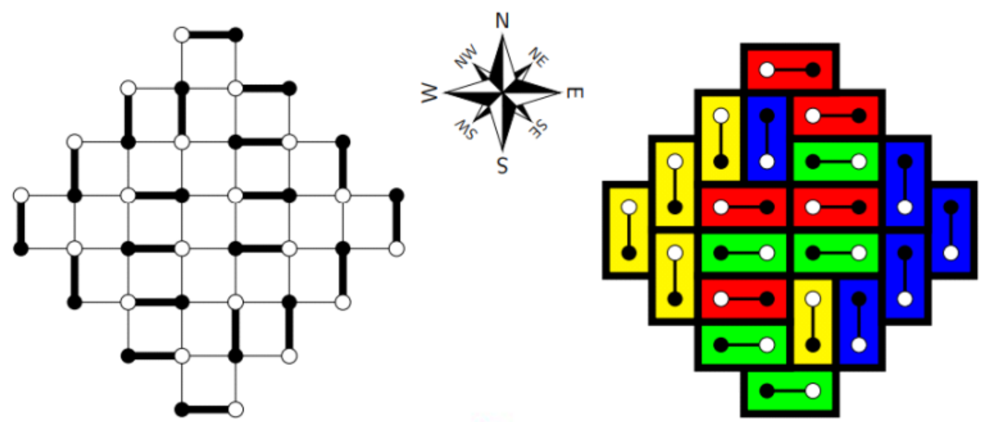
Two obvious possibilities:



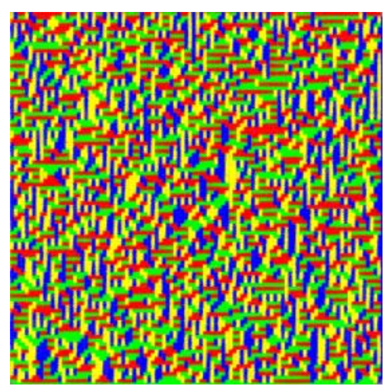


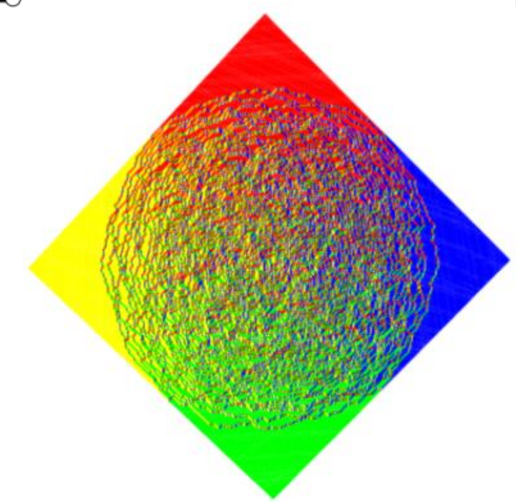
Aztec diamond (after J. Propp)

Useful tools: domino tilings and coloring



Random sample:





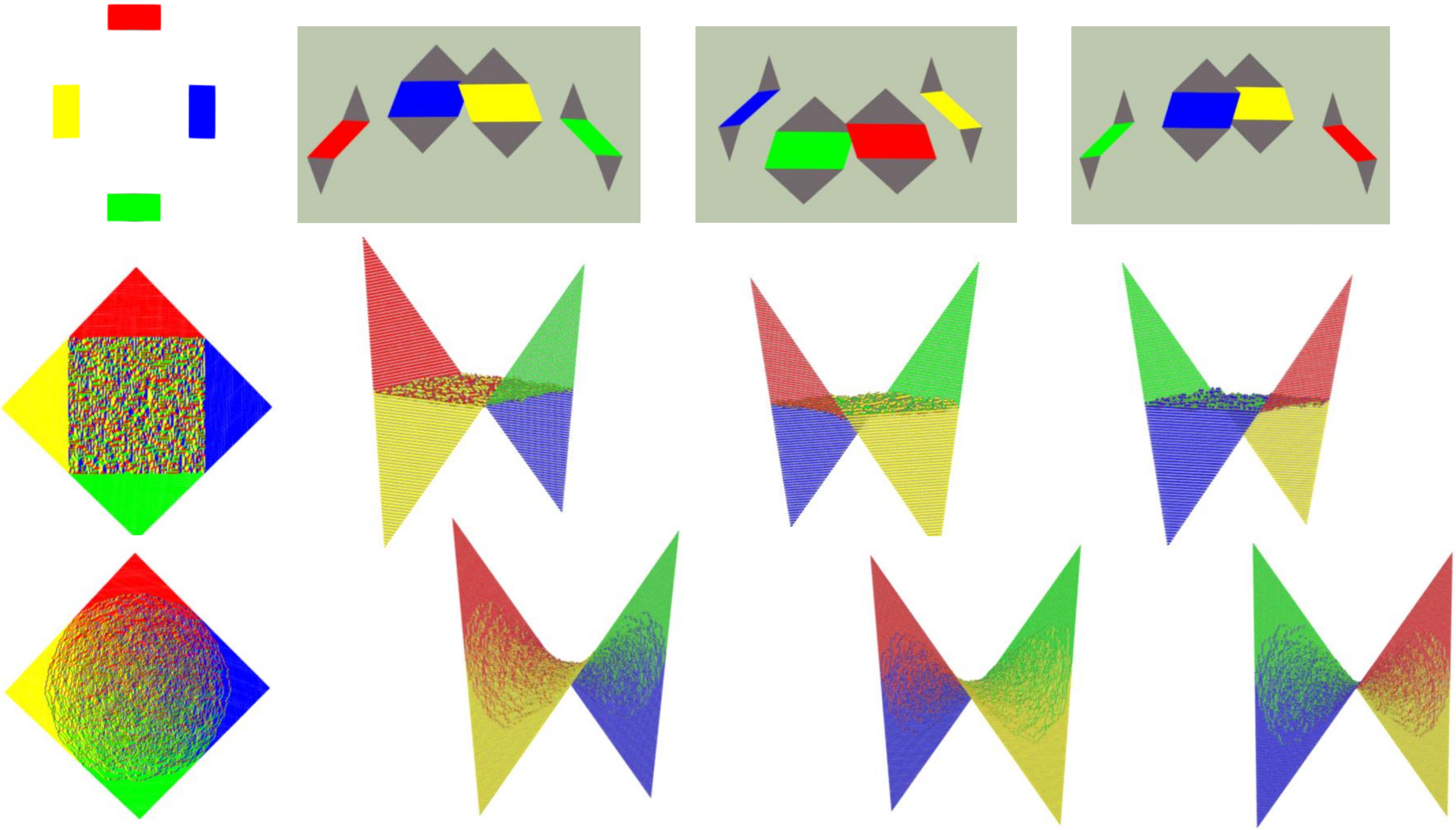
Exact sampling via domino shuffling alg.

Elkies-Kuperberg-Larsen-Propp 1992

Propp, 2001

Another useful tool: **Height function**

Conway-Lagarias, 1990
Thurston, 1990
Levitov, 1990



Random domino tilings of the Aztec diamond is a **very well studied model**. Here are a few key facts:

- Total number of tilings is $2^{n(n+1)/2}$.

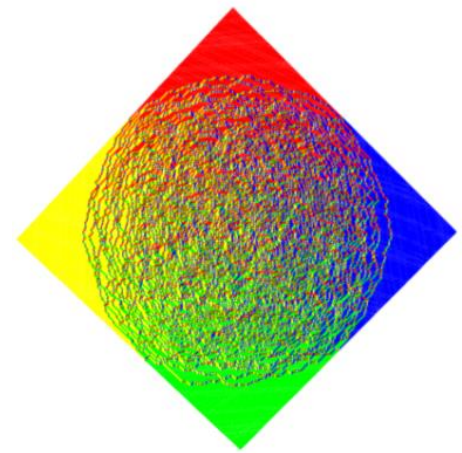
- There is a sampling algorithm, known as shuffling, that involves only independent Bernoulli $\{0,1\}$ trials.

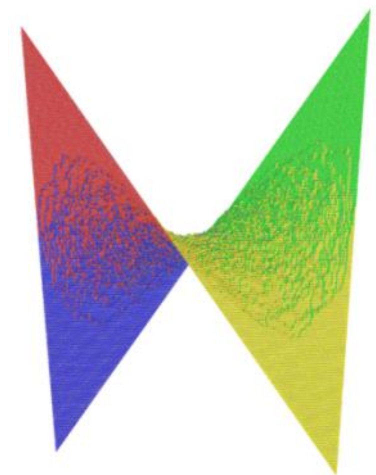
- The frozen boundary is a circle, called Arctic.

- Local fluctuations are described by an explicit 2-param. family of translation invariant Gibbs measures.

- Frozen edge fluctuations are described by the Airy_2 process.

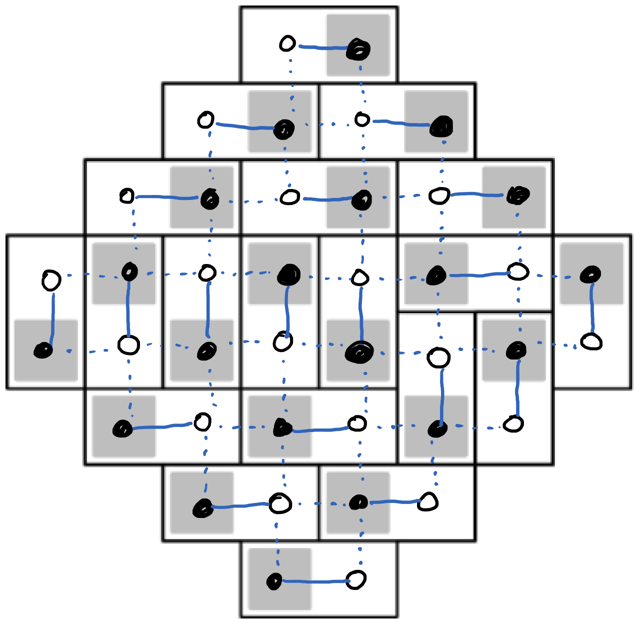
- Global surface fluctuations are given by the 2d GFF.

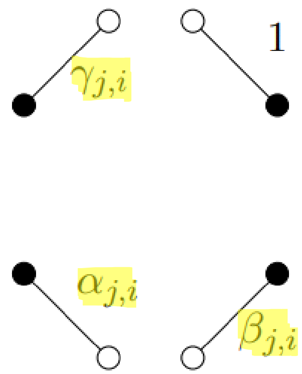


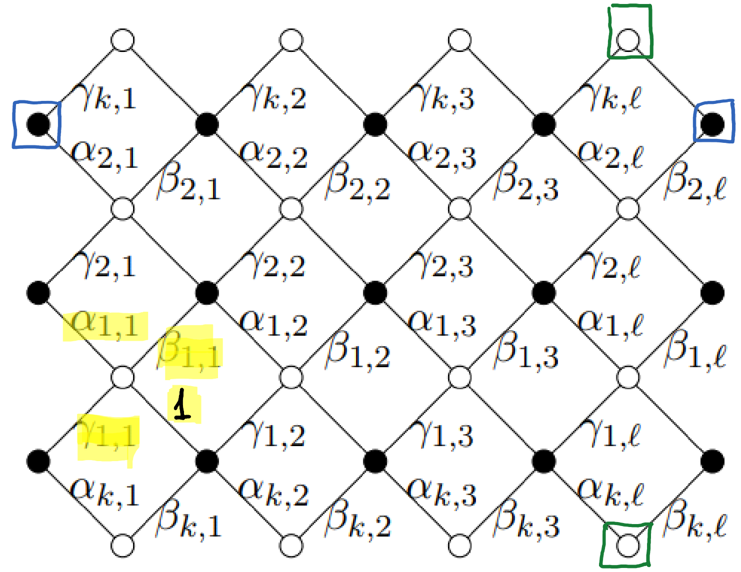


N. Elkies, G. Kuperberg, M. Larsen, J. Propp, 1992.
W. Jockusch, J. Propp, P. Shor, 1995
K. Johansson, 2000.
K. Johansson, 2003
R. Kenyon, A. Okounkov, S. Sheffield, 2003
S. Chhita, K. Johansson, B. Young 2012
A. Bufetov, V. Gorin, 2016
A. Bufetov, A. Knizel, 2016

Periodic edge weights







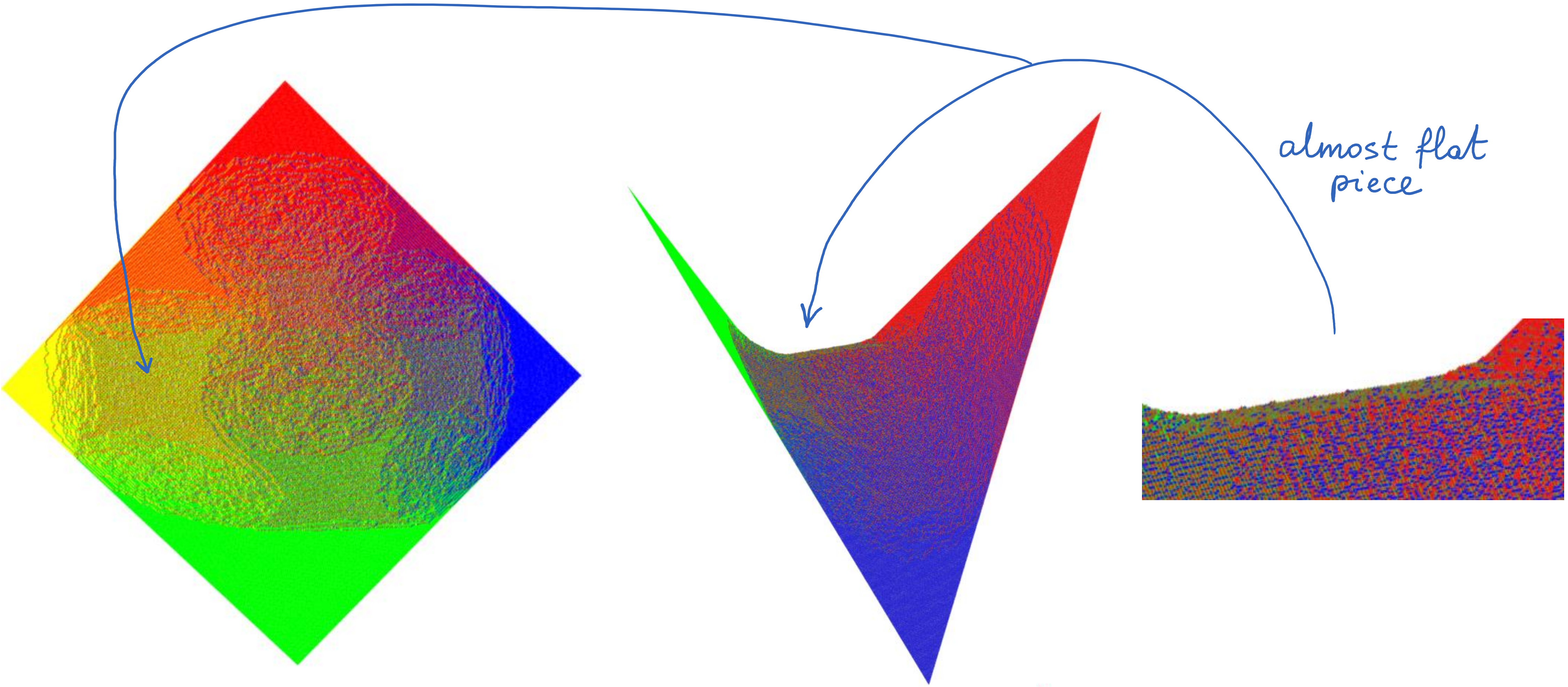
Prob(matching)

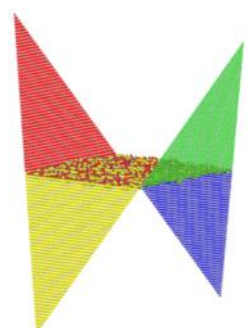
$\sim$ product of edge weights

Fundamental domain (rotated by 45°)

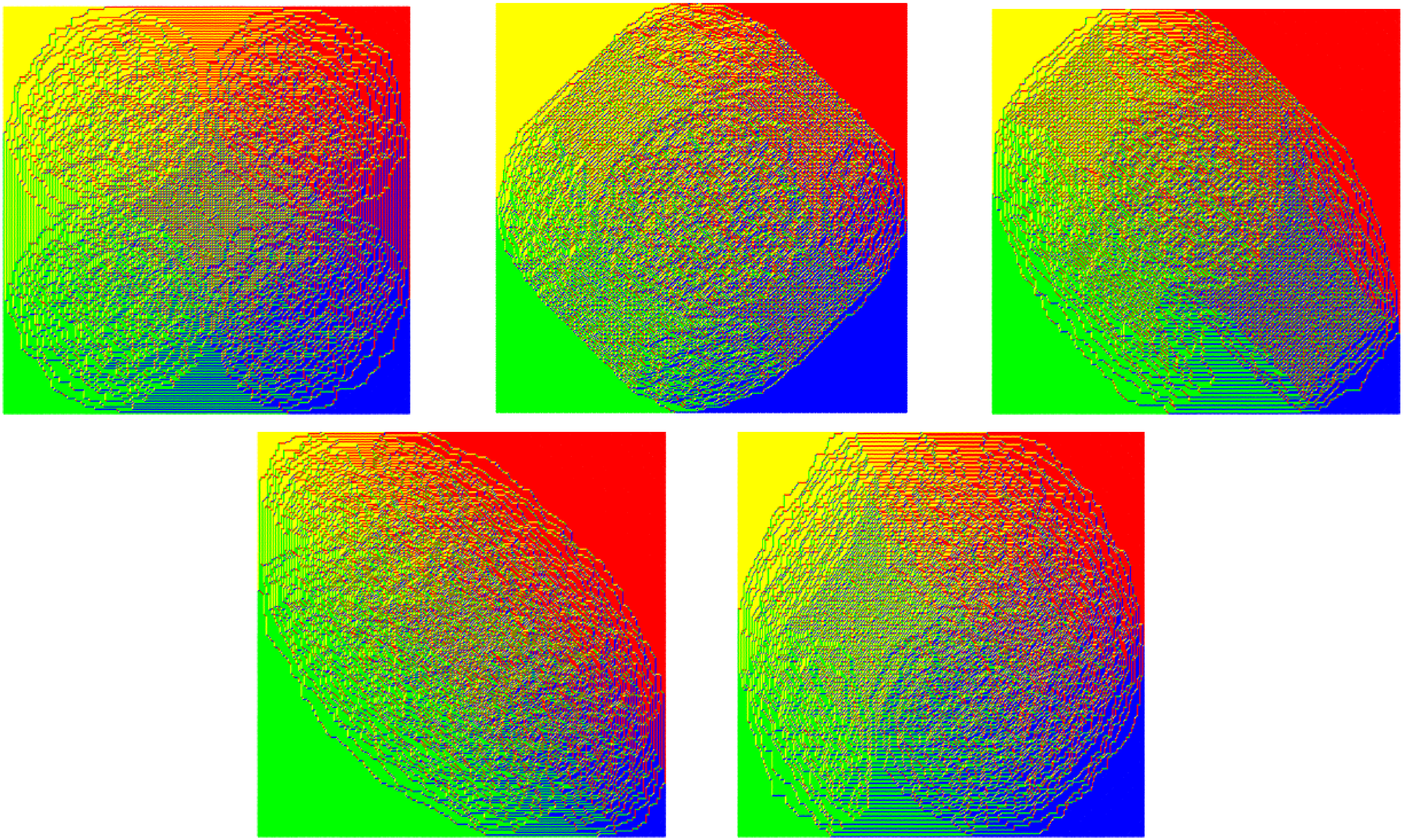
$k=3$, $\ell=4$ are vertical and horizontal periods

A 3×3-periodic example

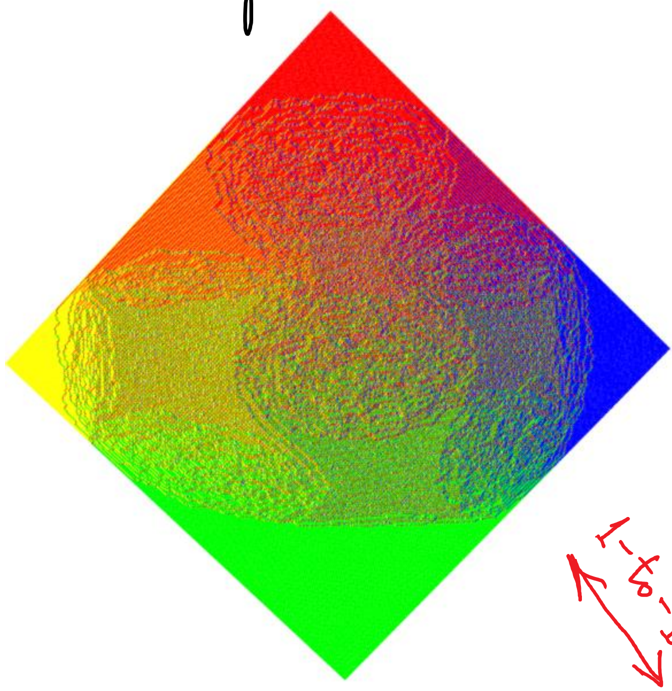


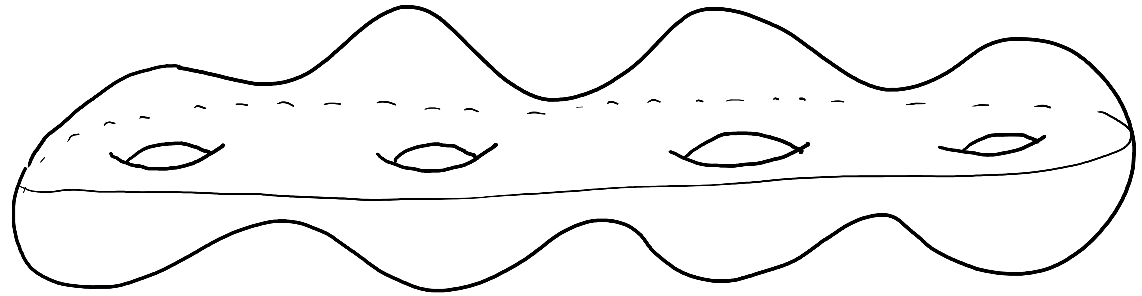
This flat piece is much smoother than  . New phase!

More examples for periodic edge weights



Main message: This is a compact Riemann surface of genus $(k-1)\cdot(\ell-1)$ (or a sphere with $(k-1)(\ell-1)$ handles)...



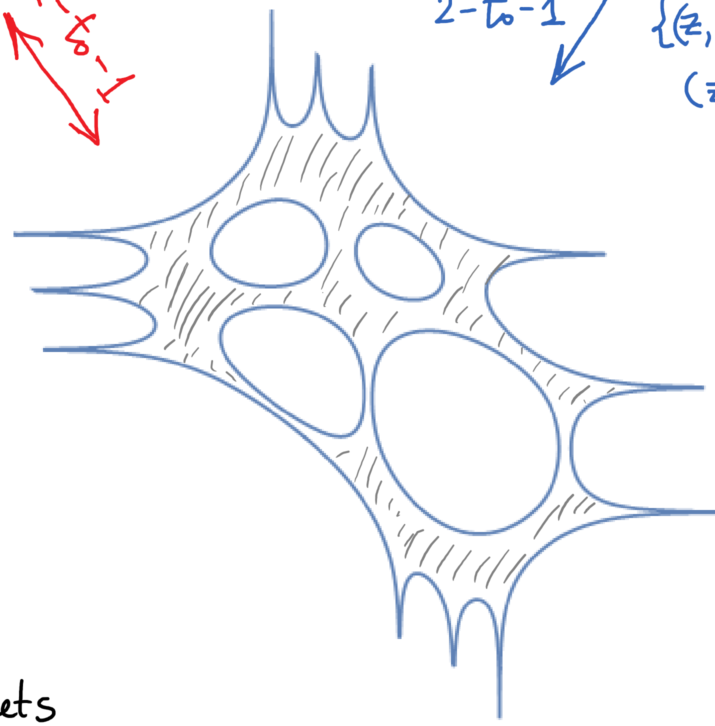


1-to-1

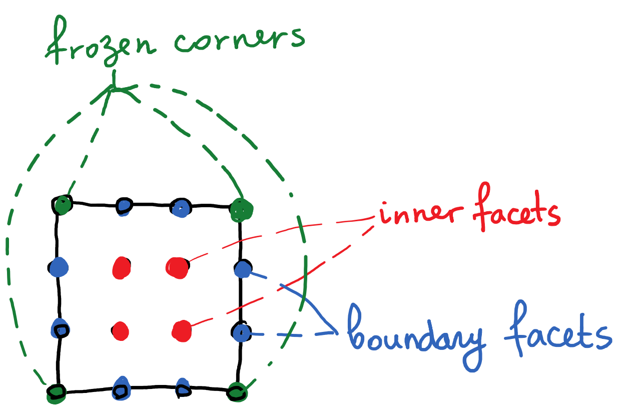
2-to-1

$\{(z,w)\in\mathbb{C}^2 : P(z,w)=0\} \longrightarrow \mathbb{R}^2$

$(z,w) \longmapsto (\log|z|, \log|w|)$



... with amoeba as an intermediary



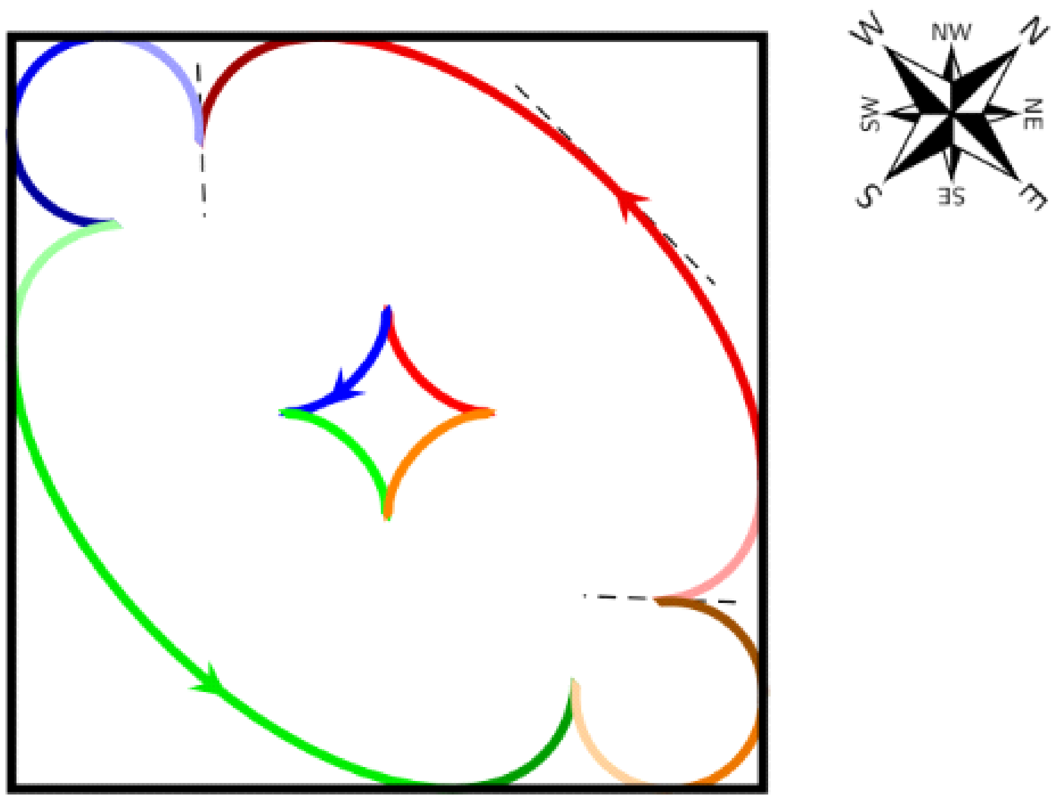
Newton polygon

coord's of Z-points are **slopes** of the facets

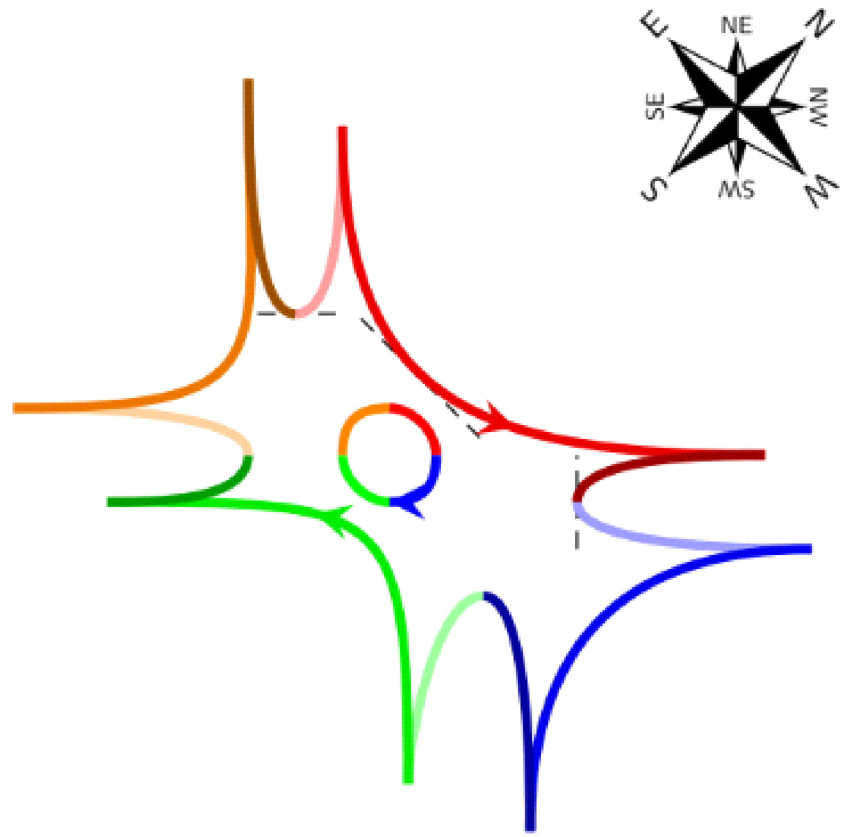
Two copies of the amoeba glued along the boundary is the Riemann surface.

The restriction of the map to the boundaries, of the facets and of the amoeba, **preserves the tangent lines.**

Hence, $4 = (2\pi + 2\pi)/\pi$ cusps on inner components in the Aztec.

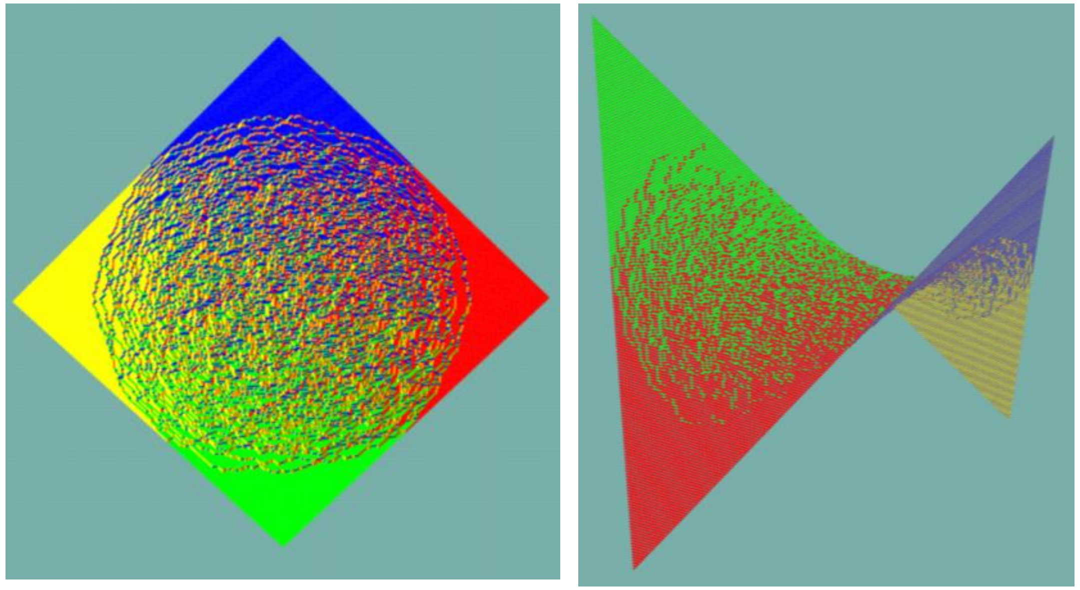


Aztec

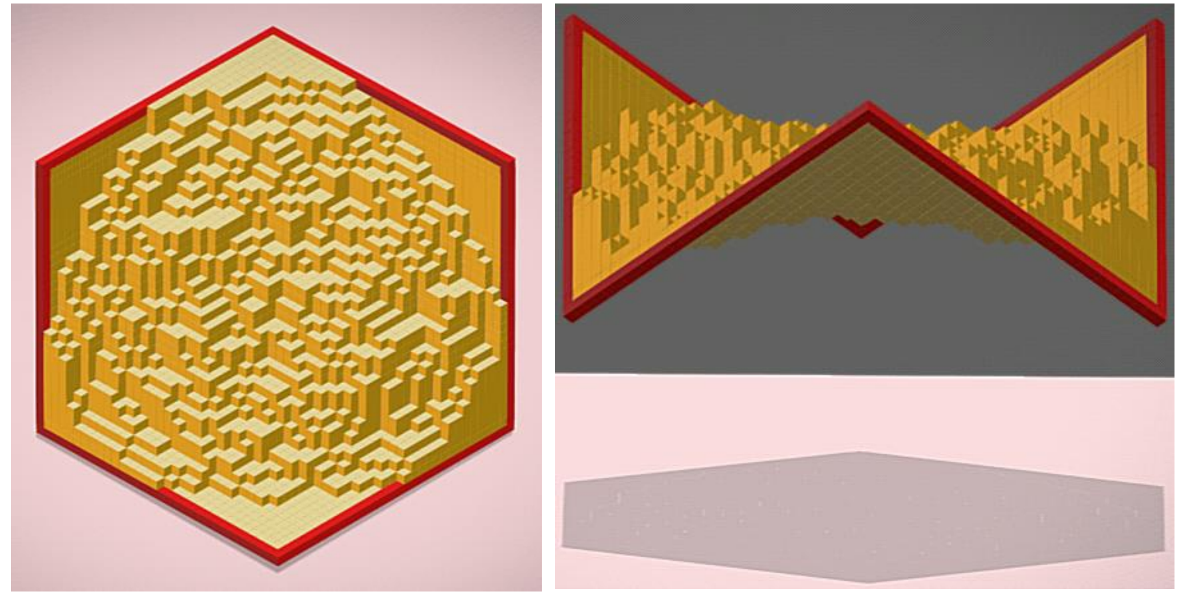


Amoeba

The first path from dimers to the Riemann surfaces goes via a variational problem.



Dominoes on the square lattice



Dominoes on the triangular lattice
(Lozenges)

Basic fact:

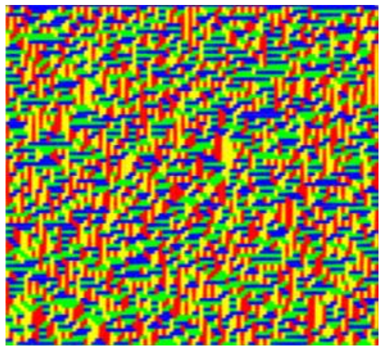
The random surface concentrates, with overwhelming probability, around a deterministic surface, limit shape, which is a unique solution of the corresponding variational problem.

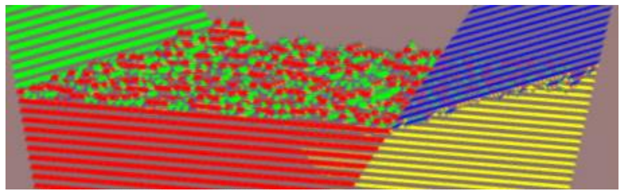
maximization of an integral functional

Why variational problem?

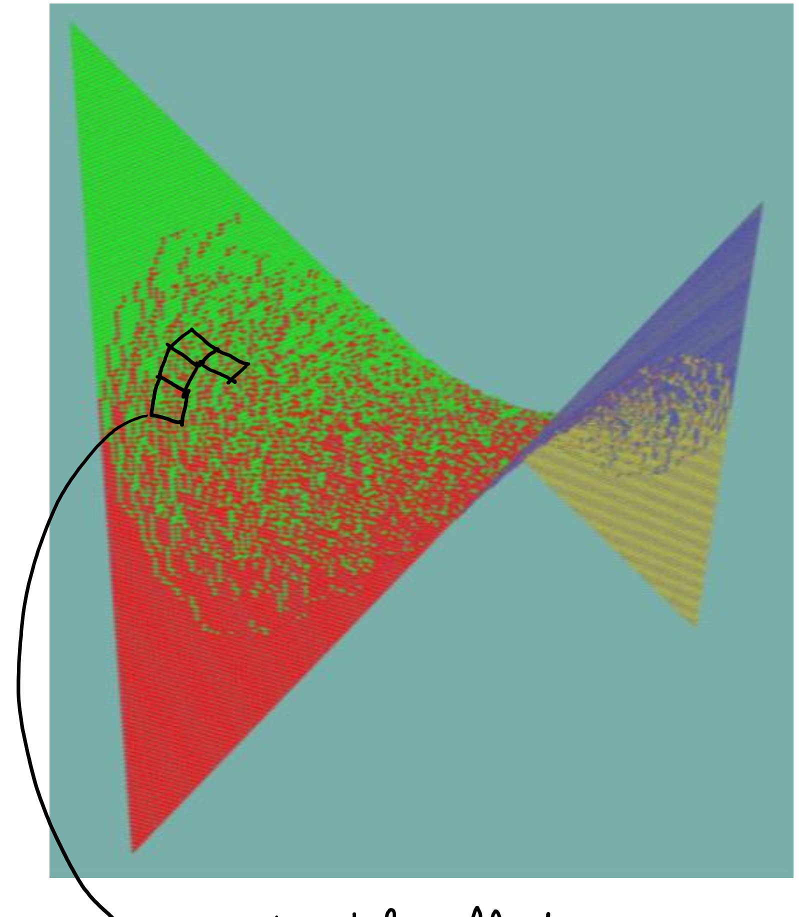
- Of all surfaces, the limit shape has the largest number of domino surfaces in its neighborhood. But how many?

- Cut the surface into small, approximately flat pieces. For each of them, fix the boundary and compute the number of ways to fill them with dominoes.





$$\prod_{k=1}^{m}\prod_{l=1}^{n}\left|2\cos\frac{\pi k}{m+1}+2i\cos\frac{\pi l}{n+1}\right|^{1/2}$$

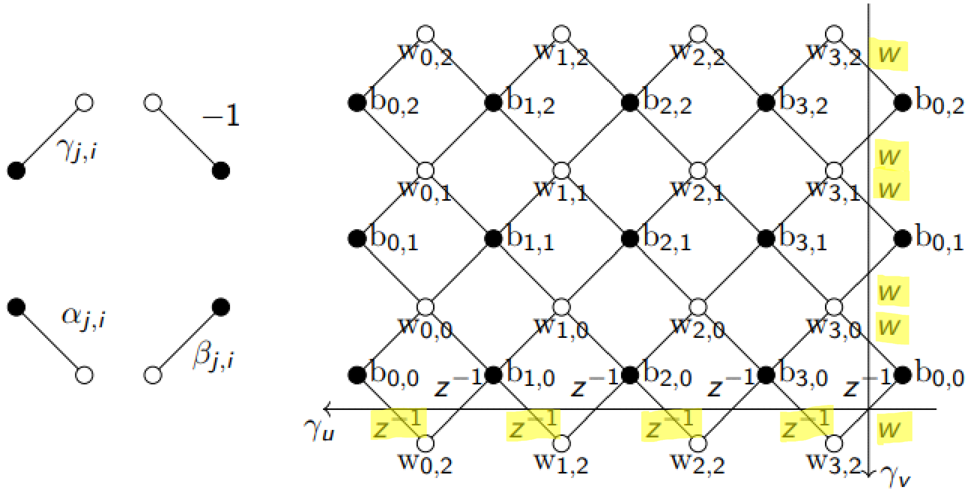


approximately flat pieces

- The product over flat pieces leads to the exponential of the integral of "surface tension" or "free energy" — the logarithm of the count of almost flat tilings with a given slope.

G. Wulff, 1901
H. Cohn, R. Kenyon, J. Propp, 2000
R. Cerf, R. Kenyon, 2001
R. Kenyon, A. Okounkov, S. Sheffield, 2003
N. Kuchumov, 2017
V. Gorin, 2021

The free energy computation is due to Kenyon-Okounkov-Sheffield, 2003.



$$P(z,w) = \det\left[\text{Weight}\left(\underset{w_m}{\circ}\cdots\cdots\underset{b_n}{\bullet}\right)\right]_{m,n=1}^{kl}$$

Spectral curve

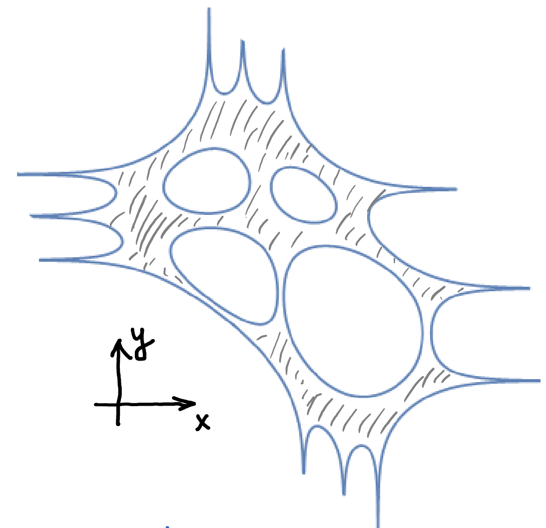
$$\{(z,w)\in\mathbb{C}^2 : P(z,w)=0\}.$$

Amoeba is its image under

$$\text{Log}(z,w) = (\log|z|, \log|w|) = (x,y)$$

<u>Theorem</u> The free energy (a.k.a. surface tension) is the Legendre transform of the Ronkin function

$$F(x,y) = \frac{1}{(2\pi i)^2}\int_{|z|=e^x}\int_{|w|=e^y}\log|P(z,w)|\,\frac{dz}{z}\frac{dw}{w}.$$



It is convex on the amoeba and linear on each connected component of the complement.

Thus, one "just" needs to maximize this functional. NOT EASY!

The second path to the (same!) Riemann surface is based on the specific geometric form of the Aztec diamond. Duits-Kuijlaars '17 and Berggren-Duits '19 show that the large size asymptotics can be reduced to a **matrix factorization** problem. In the simplest nontrivial case (2×2 periodicity) it looks like this:

$$\text{Elementary step:}\quad \begin{bmatrix}\alpha & \gamma z\\ \beta & \delta\end{bmatrix}\cdot\begin{bmatrix} a & c\\ b/z & d\end{bmatrix} = \begin{bmatrix} d & xc \\ b/zx & a\end{bmatrix}\begin{bmatrix}\delta x & \gamma z\\ \beta & \alpha/x\end{bmatrix}, \quad x=\frac{\alpha a+\gamma b}{\delta d+\beta c}.$$

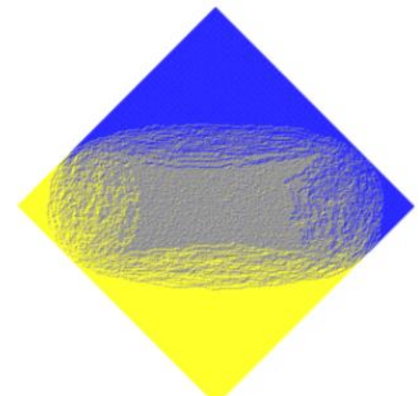
same eigenvalue

same eigenvalue

$$(P_{0,-}P_{0,+})^N = P_{0,-}P_{0,+}P_{0,-}P_{0,+}\cdots P_{0,-}P_{0,+} = P_{0,-}(P_{0,+}P_{0,-})^{N-1}P_{0,+} = P_{0,-}(P_{1,-}P_{1,+})^{N-1}P_{0,+} =$$

$$P_{1,-}P_{1,+} \xrightarrow{\text{swap}} P_{1,+}P_{1,-} \xlongequal{\text{re-factorize}} P_{2,-}P_{2,+}$$

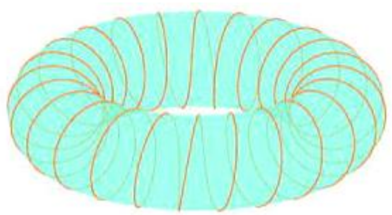
$$= P_{0,-}P_{1,-}(P_{2,-}P_{2,+})^{N-2}P_{1,+}P_{1,+} = \dots = \underbrace{(P_{0,-}P_{1,-}\cdots P_{N-1,-})(P_{N-1,+}\cdots P_{1,+}P_{0,+})}_{\text{Wiener-Hopf factorization}}$$



Linearization

$$P_-(z)\,P_+(z) = P(z) = \begin{bmatrix} a_{11} & a_{12}+b_{12}z \\ a_{21}+b_{21}z^{-1} & a_{22} \end{bmatrix} \longmapsto \tilde{P}(z) = P_+(z)P_-(z) = P_+(z)\,P(z)\,P_+^{-1}(z).$$

Central idea of Integrable Systems: Represent a nonlinear flow as a compatibility condition of linear problems (Lax pair).



Jacobian of $\det(P(z)-w)=0$

$$\begin{cases} (P(z)-w)\,\Psi(z,w) = 0 \\ \tilde{\Psi}(z,w) = R(z)\,\Psi(z,w) \end{cases}$$

Solution of $\det R(z)=0$ is the $|z|<1$ part of $\det P(z)=0$

If $\tilde{\Psi}$ satisfies $(\hat{P}(z)-w)\tilde{\Psi}=0$ with similar $\hat{P}(z)$, then $R(z)=P_+(z)$ and $\hat{P}=\tilde{P}$ (up to conjugations by scalar diagonal matrices).

Compare KdV equation $\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$ is a compatibility condition for its Lax pair $\left(\frac{\partial^2}{\partial x^2}+u\right)\varphi = \lambda\varphi$, $\frac{\partial \varphi}{\partial t} = \left(\frac{\partial^3}{\partial x^3} + \frac{3}{2}u + \frac{3}{4}u_x\right)\varphi$.

Bird's eye view on linearization

$$P(z) = \begin{bmatrix} a_{11} & a_{12}+b_{12}z \\ a_{21}+b_{21}z^{-1} & a_{22} \end{bmatrix}$$

The space of our flow

$$\begin{cases} (P(z)-w)\,\Psi(z,w) = 0 \\ \tilde{\Psi}(z,w) = P_+(z)\,\Psi(z,w) \end{cases}$$

The Lax pair

$$P(z) \longmapsto P_+(z)\,P(z)\,P_+^{-1}(z).$$

The flow

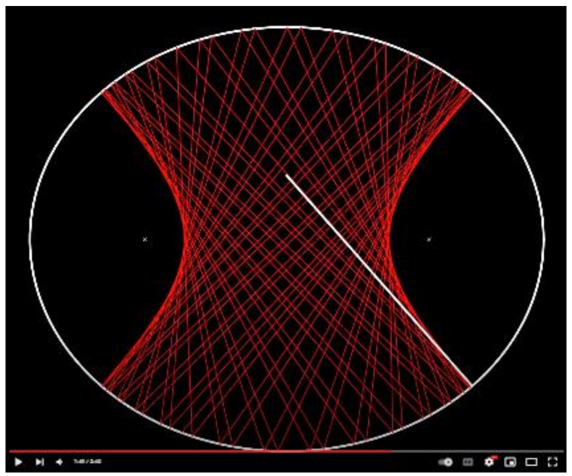
Note that $\det(P(z)-w)$ does not change under the evolution (<u>isospectrality</u>).
Hence, $\{(z,w)\in\mathbb{C}^2 : \det(P(z)-w)=0\}$ is an invariant. Its natural compactification $\mathcal{E}$ is the spectral curve; it has genus 1 (elliptic curve).

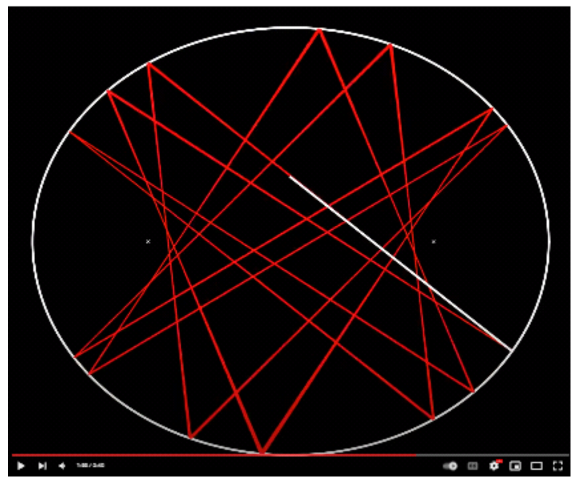
Moser-Veselov, 1991
finite-gap method, 1976+
Dubrovin, Its, Krichever

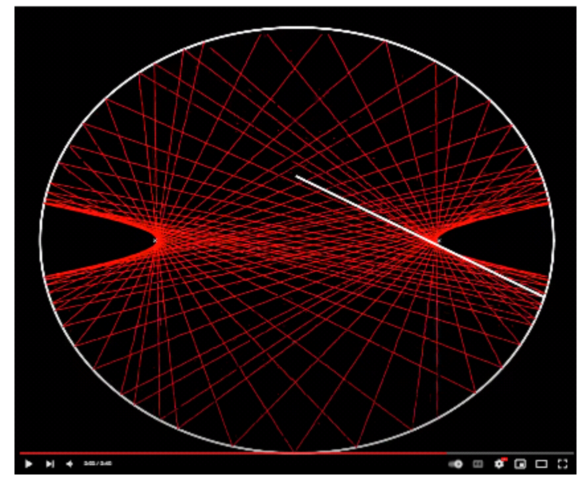
- Normalize $\Psi(z,w) = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$ by $\Psi_1(z,w)+\Psi_2(z,w) \equiv 1$.
- One shows that $\Psi_1(z,w)$, $\Psi_2(z,w)$ span the space of meromorphic functions on $\mathcal{E}$ with two fixed simple poles.
- One zero of Ψ_1 is at 0, one zero of Ψ_2 is at ∞.
- The second zero of $\Psi_{1/2}$ evolves by linear shifts on the Jacobian of $\mathcal{E}$.
- The linearity follows from singularity structure of P_+ and Abel's theorem.

Classical integrable systems solved by similar techniques

- QR-algorithm of numeric linear algebra
- Billiards inside an ellipsoid and geodesics on the ellipsoid.
- Discretization of the Euler equations of free rigid body motion







@AlexanderGustafssonAnimations

SUMMARY