Representations of the orthosymplectic Yangians

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The vector e_i has the parity $\bar{\imath} \mod 2$ and

$$\bar{\imath} = \begin{cases} 1 & \text{for} \quad i = 1, \dots, m, m', \dots, 1', \\ 0 & \text{for} \quad i = m+1, \dots, (m+1)', \end{cases}$$

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The endomorphism algebra $\operatorname{End}\mathbb{C}^{N|2m}$ is equipped with $\mathbb{Z}_2$-gradation, the parity of the matrix unit e_{ij} is $\bar{\imath} + \bar{\jmath} \mod 2$.

A standard basis of the Lie superalgebra $\mathfrak{gl}_{N|2m}$ is formed by elements E_{ij} of parity $\bar{\imath} + \bar{\jmath} \mod 2$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

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where

$$\theta_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, N+m, \\ -1 & \text{for} \quad i = N+m+1, \dots, N+2m. \end{cases}$$

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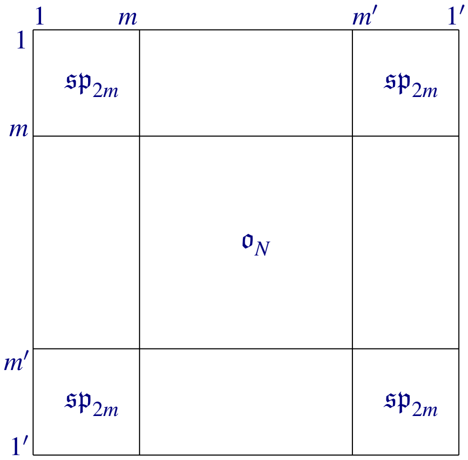

The permutation operator P takes the form

$$P = \sum_{i,j=1}^{N+2m} e_{ij} \otimes e_{ji} (-1)^{\bar{\jmath}} \in \mathrm{End}\, \mathbb{C}^{N|2m} \otimes \mathrm{End}\, \mathbb{C}^{N|2m}.$$

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The R-matrix associated with $\mathfrak{osp}_{N|2m}$ is the rational function in u given by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa}, \qquad \kappa = \frac{N}{2} - m - 1.$$

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[A. B. Zamolodchikov and Al. B. Zamolodchikov, 1979]

The extended Yangian $\mathrm{X}(\mathfrak{osp}_{N|2m})$ as a $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar{\imath}+\bar{\jmath} \mod 2$, where $1 \leqslant i,j \leqslant N+2m$ and $r = 1, 2, \dots$, satisfying the following defining relations.

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Introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \in \mathrm{X}(\mathfrak{osp}_{N|2m})[[u^{-1}]]$$

and combine them into the matrix $T(u) = [t_{ij}(u)]$.

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[D. Arnaudon, J. Avan, N. Crampé, L. Frappat, E. Ragoucy, '03]

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The Verma module $M(\lambda(u))$ is the quotient of $\mathrm{X}(\mathfrak{osp}_{N|2m})$ by the left ideal generated by all the coefficients of the series $t_{ij}(u)$ for $1 \leqslant i < j \leqslant 1'$ and $t_{ii}(u) - \lambda_i(u)$ for $1 \leqslant i \leqslant 1'$.

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Theorem. The Verma module $M(\lambda(u))$ is nonzero if and only if

$$\lambda_i(u)\, \lambda_{i'}\Big(u - \frac{N}{2} - (-1)^{\bar{\imath}}(m-i) + 1\Big) = \lambda_{i+1}(u)\, \lambda_{(i+1)'}\Big(u - \frac{N}{2} - (-1)^{\bar{\imath}}(m-i) + 1\Big)$$

for $1 \leqslant i < m + N/2$.

Hence we can re-define the highest weight by

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The irreducible highest weight representation $L(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{N|2m})$ is the quotient of the nonzero Verma module $M(\lambda(u))$ by the unique maximal proper submodule.

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Theorem. Every finite-dimensional irreducible representation of the Yangian $\mathrm{X}(\mathfrak{osp}_{N|2m})$ is isomorphic to a unique irreducible highest weight representation $L(\lambda(u))$.

Open problem. Find necessary and sufficient conditions on $\lambda(u)$ for the representation $L(\lambda(u))$ to be finite-dimensional.

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The solution relies on an explicit construction of the modules $L(\alpha)$.

Small Verma modules

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Let K be the submodule of $M(\lambda(u))$ generated by all vectors

$$t_{21}^{(r)}\,\xi \quad \text{for} \quad r \geqslant 2 \quad \text{and} \quad \big(t_{31}^{(r)} + (\alpha - 1/2)\,t_{31}^{(r-1)}\big)\,\xi \quad \text{for} \quad r \geqslant 3,$$

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Proposition. The elementary module $L(\alpha)$ is a quotient of the small Verma module $M(\alpha)$.

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For any $r, s \in \mathbb{Z}_+$ introduce vectors in $M(\alpha)$ by

$$\xi_{rs} = T_{21}(-\alpha-r+3/2)\dots T_{21}(-\alpha-1/2)\,T_{21}(-\alpha+1/2)$$
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Proposition. For any $\alpha \in \mathbb{C}$ the vectors ξ_{rs} with $0 \leqslant r \leqslant s$ form a basis of $M(\alpha)$.

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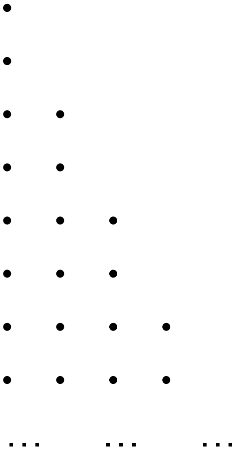

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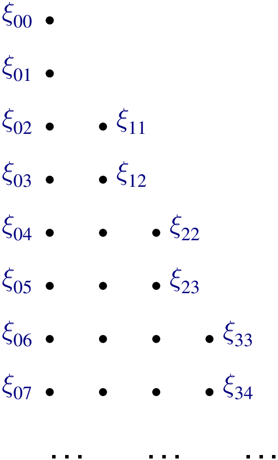

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- The $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $M(\alpha)$ is irreducible if and only if $-\alpha \notin \mathbb{Z}_+$ and $-\alpha + 1/2 \notin \mathbb{Z}_+$.

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$$\dim L(-k) = \binom{k+2}{2}.$$

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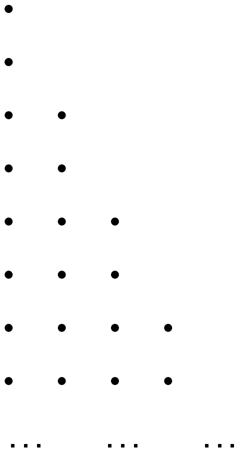

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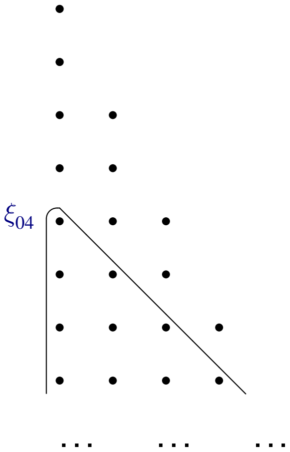

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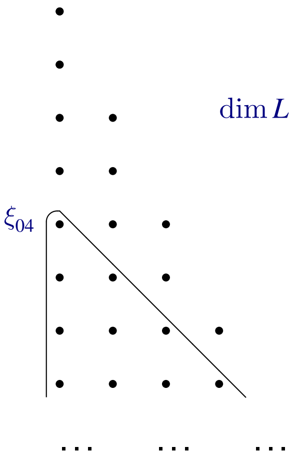

$$\dim L(-3) = \binom{5}{2} = 7 + 3 = 10.$$

For any given $\mu \in \mathbb{C}$ denote by $V(\mu)$ the irreducible highest weight module over $\mathfrak{osp}_{1|2}$ generated by a nonzero vector ξ such that $F_{11}\xi = \mu\,\xi$ and $F_{12}\xi = 0$.

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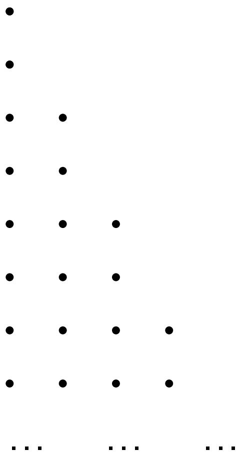

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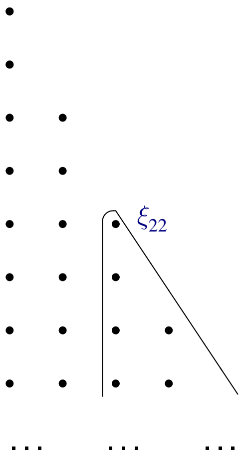

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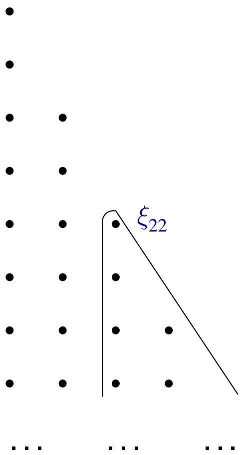

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Notation: for formal series $\mu(u)$ and $\nu(u)$ we write

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if there exists a monic polynomial $P(u)$ in u such that

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Theorem [2023]. The representation $L(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{1|2m})$ is finite-dimensional if and only if

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Odd reflections

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Definition [2022]. Let $\alpha(u)$ and $\beta(u)$ be formal series in u^{-1},

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The odd reflection is the transformation

$$\big(\alpha(u), \beta(u)\big) \mapsto \big(\beta^{[1]}(u), \alpha^{[1]}(u)\big),$$

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where $\alpha_i \neq \beta_j$ for all i, j, and $\gamma(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$.

The odd reflection is the transformation

$$\big(\alpha(u), \beta(u)\big) \mapsto \big(\beta^{[1]}(u), \alpha^{[1]}(u)\big),$$

where

$$\alpha^{[1]}(u) = (1+(\alpha_1+1)\,u^{-1})\dots(1+(\alpha_p+1)\,u^{-1})\,\gamma(u),$$

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to define the series $\lambda^{[n]}_m(u)$.

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- $n = 1$ and any $m \geqslant 1$,
- generic highest weights $\lambda(u)$ with $n, m \geqslant 1$.