SCGP 2

Friday, Augu1s:t5 11 9P,M 2022

$\dim X = 3$ target space description of GW in terms of

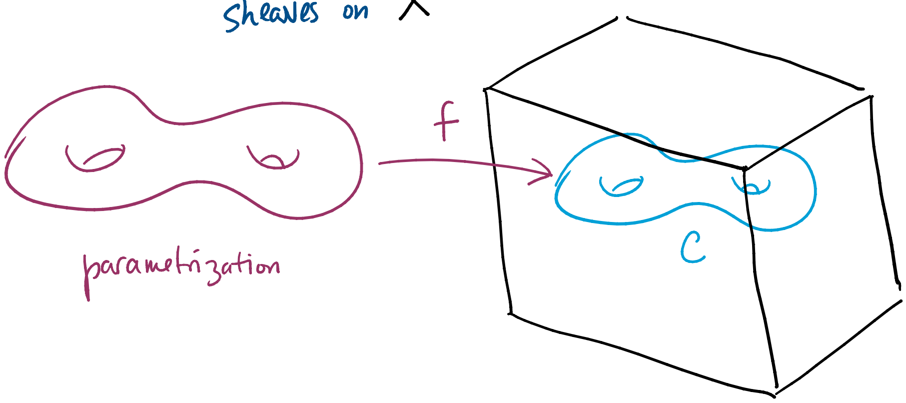
Donaldson-Thomas theories of X

MNOP [2003, 2004]

precursors: The topological vertex of Aganagic, Klemm, Mariño, Vafa

The melting crystal of A.O., Reshetikhin, Vafa

theories that count sheaves on X



or equations

$I_C \in \mathrm{Hilb}(X, \text{curves})$

ideal sheaf

components indexed by degree $H_2(X,\mathbb{Z})$ and Euler char.

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

$\chi = \chi(\mathcal{O}_C) = 1-g$ for a smooth curve

the Hilbert scheme has a natural obstruction theory computed by $\mathrm{Ext}^i(I_C, \mathcal{O}_C)$ $i \geq 0$.

do not want this! Not a "Hilbert" theory, but DT theory

Instead, interpret $I_C \to \mathcal{O}_X = I^{\vee\vee}$

Instead, interpret $\mathcal{I}_C \to \mathcal{O}_X = \mathcal{I}_C^{\vee\vee}$

torsion-free sheaf with $\det = \mathcal{O}_X$

has obstruction theory

$$\mathrm{Def} = \mathrm{Ext}^1_0(\mathcal{I}_C, \mathcal{I}_C)$$
$$\mathrm{Obs} = \mathrm{Ext}^2_0(\mathcal{I}_C, \mathcal{I}_C)$$

almost dual for general X

$$\mathrm{Ext}^3_0(\mathcal{I}_C, \mathcal{I}_C) = 0$$ for general X

shown by Zijun Zhou very generally, including orbifolds

virtual class constructed by R. Thomas

no CY, or other assumption on $\mathcal{K}_X$ are needed to count curves in X!

Descendents in DT: Künneth components of $\mathrm{ch}(\text{universal sheaf } \mathcal{I}_C) \in H^\bullet(X \times \mathrm{Hilb}(X))$

flat over Hilb

One may think that counts in fixed degree and fixed genus $g = 1 - \chi$ may be related

In fact:

this works, and in fact one may compare pushforwards of $[\ \ ]^{vir}$ to $\mathrm{Chow}(X)$

this doesn't work at all since the moduli spaces are empty for $g \ll 0$ or $\chi \ll 0$.

Instead

Conjecture (MNOP, 2003)

$$\sum_g u^{2g-2} \ldots \xleftrightarrow{\text{corresponds}} \sum_\chi (-z)^\chi \ldots$$

because $\mathrm{Obs} \approx \mathrm{Def}^*$

with the change of variables $z = e^{iu}$ ← Galois symmetric with the change of variables $z = e^{iu}$ ← Galois symmetric

DT series have coefficients in $\mathbb{Z}$ ($\Longleftrightarrow$ BPS integrality)

by F. Carlsson must be rational if we want equality of analytic functions

general conjecture

($\sim$ BPS finiteness)

for this to work out, virtual dimensions have to agree

$$\text{GW-vir-dim} = (\deg, c_1(X)) + (g-1)(3 - \cancel{\dim X}) + \#\text{ of marked pts}$$

$$\text{DT-vir-dim} = (\deg, c_1(X))$$

"Corresponds" includes
1) identificaton of insertions
2) divison by degree 0 ← some universal series when no descendents
3) prefactors like

$$z^{-\frac{\text{vir dim}}{2}}\, DT \longleftrightarrow (-iu)^{\text{vir dim}}\, GW$$

see arXiv 1802.00779 "Takagi lectures on DT theory"

Division by degree 0 is achieved <u>geometrically</u> in Pandharipande-Thomas theory

$$\mathcal{O}_X \to \mathcal{O}_C \to 0 \qquad\qquad \mathcal{O}_X \to \mathcal{F} \to \text{Coker}$$

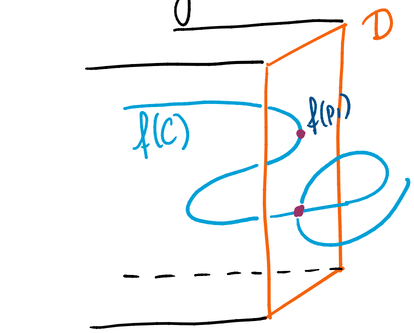
Hilbert scheme

pure 1-dim ($\mathcal{F}$) 0-dim. (Coker)

in 1) Some parts are easy, some intricate in 1) Some parts are easy, some intricate → Alexei ?

- primary = primary ← these are pulled back from Chow
- relative = relative

in GW theory



$$f^{-1}(D) = 2p_1 + p_2 + 2p_3$$

evaluation map to $S^n D \ni 2f(p_1) + 3f(p_2)$ $\quad f(p_3) = f(p_2)$

or better to the inertia stack of

(2,1)

really integrate orbifold cohomology classes

$\Updownarrow$

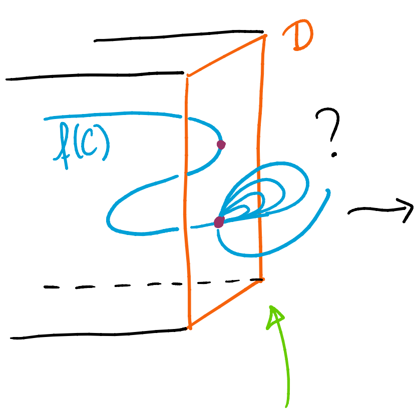
classes in $H^\bullet(D)$ additionally colored by an integer 1, 2, 3, ...

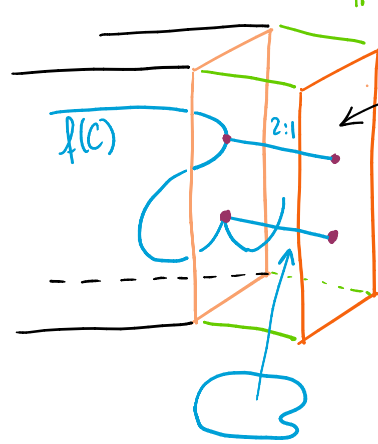
$t, t^2, t^3, \ldots$

$$S^\bullet\left(H^\bullet(D) \otimes t\,\mathbb{C}[t]\right)$$

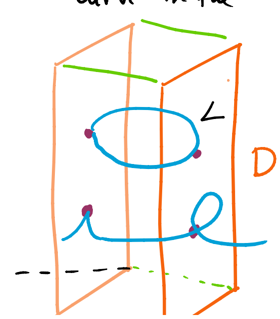
symmetric algebra

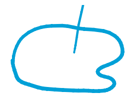
$\mathbb{P}$ bubble or a whole accordeon.

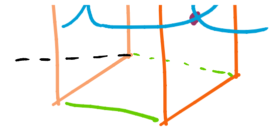




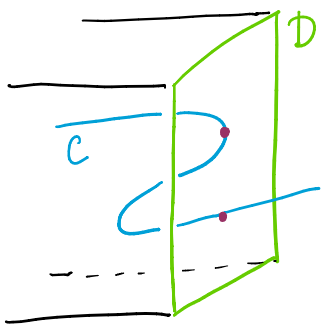
or a more general curve in the bubble.







in DT theory we have



$$\mathcal{O}_C(-D) \xrightarrow[\text{equation}]{\text{mult. by the}} \mathcal{O}_C \longrightarrow \mathcal{O}_{C\cap D} \in \mathrm{Hilb}(D,n)$$

regular as long as , is injective

Take relative conditions from $H^*(\mathrm{Hilb}(D,n))$

described by Nakajima

$\bigoplus_n H^*(\mathrm{Hilb}(D,n))$

$\cong$

$S^\bullet\left(H^\bullet(D) \otimes t\mathbb{C}[t]\right)$

Has Nakajima basis $\prod \alpha_{-m_i}(\gamma_i)\,|\mathrm{vac}\rangle$

$|\mathrm{vac}\rangle = 1 \in H^0(\mathrm{Hilb}(D,0))$

add a subscheme of length m_i along $\gamma_i \in H^*(D)$

$\Big\|$ GW/DT ?

tangency of order m_i along γ_i

GW = DT is by now proven for many, many, but not all 3-folds

Key step in the proof, and for the logic of these lectures

[MOOP]

$$\begin{array}{ccc} A_r & \hookrightarrow & X \\ & & \downarrow \\ & & \mathcal{B} \end{array}$$

A_0, A_1, A_2 enough to span cobordism

crepant resolution of $\mathbb{C}^2/\begin{pmatrix}\zeta & \\ & \zeta^{-1}\end{pmatrix}$

$\zeta \in \mu_{r+1}$

crepant resolution of

$$\begin{array}{c} \mathcal{L}_1 \oplus \mathcal{L}_2 \,/\, \mu_{r+1} \\ \downarrow \\ \mathcal{B} \end{array}$$

ADE surfaces special for many reasons ...

$\zeta \in \mu_{r+1}$

in particular, $r=0$ is ok, $X = \begin{array}{c} \mathcal{L}_1 \oplus \mathcal{L}_2 \\ \downarrow \\ B \end{array}$ local curve

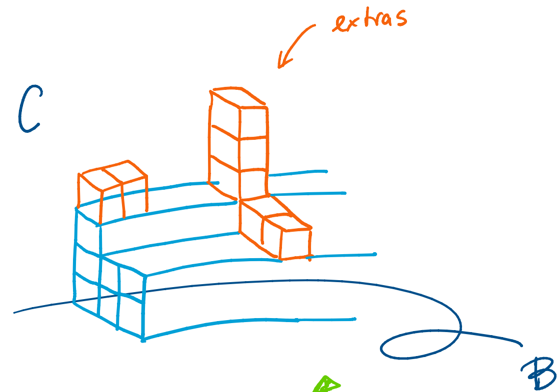
ADE surfaces special for many reasons, e.g. enhanced symmetries $H^\bullet(\mathrm{Hilb})$ or $D^b\,\mathrm{Coh}$

for instance, $B \times \underset{\varepsilon_1}{\mathbb{C}} \times \underset{\varepsilon_2}{\mathbb{C}}$ is ok and we should find the Hurwitz theory for $\varepsilon_1 + \varepsilon_2 = 0$

Torus fixed points on Hilb( )

closure of generic point $\hookrightarrow C$

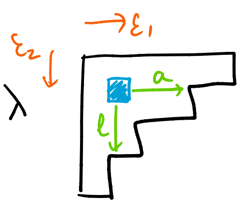
$\hookrightarrow$



minimal χ, component of $\mathrm{Hilb}(X) \simeq \mathrm{Hilb}(\mathbb{C}^2)$

DT analog of Mumford relations: if $\varepsilon_1 + \varepsilon_2 = 0$, the virtual class vanishes for all non minimal

what is the virtual class here?



$$T_\lambda \mathrm{Hilb}(\mathbb{C}^2, n) = \sum_{\square} \big(a(\square)+1\big)\varepsilon_1 - \ell(\square)\,\varepsilon_2 + \text{the other way around}$$

$|\lambda| = n$

$$\mathrm{Ext}^\bullet_{X,\circ}(\mathcal{I}_\lambda, \mathcal{I}_\lambda) = H^\bullet(B, \quad)$$

in this instance, a trivial bundle on B, but nontrivial in the case $X = \begin{array}{c} \mathcal{L}_1 \oplus \mathcal{L}_2 \\ \downarrow \\ B \end{array}$

$$\text{Virtual class} = \prod_{\square} \Big((a+1-\ell)(\ell+1-a) \Big)^{g(B)-1}$$

$$\text{at } \varepsilon_1 + \varepsilon_2 = 0 \qquad (-1)^{n(1-g(B))}\, \varepsilon_1^{2n(1-g(B))} \prod_{\square} \mathrm{hook}(\square)^{2g(B)-2}$$

$$\hookrightarrow \left(\frac{\dim \lambda}{n!}\right)^{2-2g(B)}$$

$$\frac{\dim \lambda}{n!} = \frac{1}{\prod \text{hooks}}$$

$\qquad)^{2-2g(B)}$

$$\frac{\quad}{n!} = \frac{\quad}{\prod \text{hooks}} \qquad \Big(\quad n! \quad \Big)$$

$$\text{localization in DT} = \sum_{|\lambda|=n} \left(\frac{\dim \lambda}{n!}\right)^{2-2g(B)}$$

$\varepsilon_1 + \varepsilon_2 = 0$

characteristic classes of the universal sheaf on $\mathrm{Hilb}(\mathbb{C}^2, n)$

$\|$ Vershik-Kerov-Olshanski

central characters of symmetric group

Back to
$$\begin{tikzcd} A_r \arrow[r, hook] & X \arrow[d] \\ & B \end{tikzcd}$$

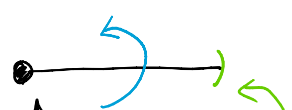
$\leftarrow$ primary and relative

GW/DT understood, originally, by [OP] + [Maulik-Oblomkov] leading up to [MOOP]

a different perspective developed subsequently, in particular for computations in K-theory

some kind of CohFT on the stack of $(\mathbb{C}^\times)^2$- bundles on curves B

has various building blocks all related to



$B = \mathbb{P}^1$, descendent relative

$S^\bullet(H^\bullet(A_r) \otimes \mathrm{poly}(t))$ $H^\bullet(\mathrm{Hilb}(A_r, n))$ fixed number

$$\begin{tikzcd} \mathcal{O}(1) \oplus \mathcal{O} \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

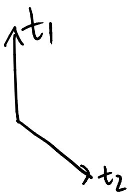
Nekrasov & Shatashvili $\sim$ 2007/8 (+ Nakajima + Bezrukavnikov ...)

this structure should be described by a quantum group

And this is now known, also in K-theory (depending on the exact after 10+ years of work

And this is now known, also in K-theory (depending on the exact flavor of DT theory $\longrightarrow$ Henry?)
and the quantum group is

$$\mathcal{U}_\hbar\left(\widehat{\widehat{\mathfrak{gl}}}(r+1)\right) \text{ in K-theory} \quad \text{and Yangian for } H^\bullet \qquad \hbar = -\varepsilon_1 - \varepsilon_2$$

$A_r \to$ 

$/\mu_{r+1}$

$\hbar = \dfrac{1}{t_1 t_2}$ the weight of symplectic form

in hindsight shoul have been the weight of the Poisson tensor

the difference between $\widehat{\widehat{\mathfrak{sl}}}(r+1)$ and $\widehat{\widehat{\mathfrak{gl}}}(r+1)$ is noticeable and becomes really dramatic for other quivers

In particular $\displaystyle Y(\widehat{\mathfrak{gl}}(1))\Big|_{\varepsilon_1+\varepsilon_2=0} \simeq \mathcal{U}(\mathfrak{gl}(\infty))$ as Hopf algebra

$\hookrightarrow$ recover the description of Hurwitz theory from Lecture 1.

<u>for example</u>: recall from Lecture 1 that

$\overset{\bullet}{\longleftrightarrow}$ $\mathbb{P}^1$ $\in \mathrm{End}(H^\bullet(\mathrm{Hilb}(A_r, \mathrm{pts})))$ <u>commute</u>

Nekrasov & Shatashvili $\Rightarrow$ these should be Baxter Q-operators

$\uparrow$ true with small corrections, see [Pushkar-Smirnov-Zeitlin, Koroteev-PSZ]

<u>Quantum loop algebras</u> $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ = Hopf algebra deformation of $\mathcal{U}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$

$\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ coproduct

$S: \mathcal{A} \to \mathcal{A}$ antipode, unique when exists

has a loop rotation automorphism

$$V \longrightarrow V(a)$$

$\uparrow$ precompose with $t \mapsto at$

Modules over a Hopf algebra have

$\uparrow$ $\otimes$, duals

Modules over a Hopf algebra have

$\otimes$, duals

category, from which $\mathcal{A}$ can be reconstructed

Morphisms, i.e. maps that commute with the action

one particularly important :

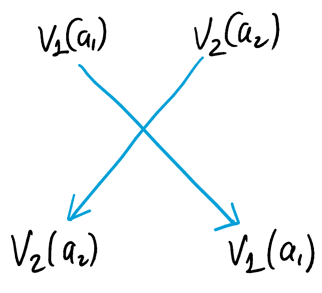
irreducible if V_i irred. and $a_1 \neq a_2$

for $\hat{\mathfrak{g}}$, $\quad V_1(a_1) \otimes V_2(a_2) \xrightarrow{(12)} V_2(a_2) \otimes V_1(a_1)$

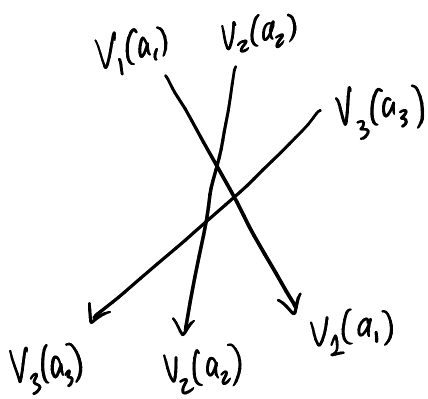
for $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ $\quad V_1(a_1) \otimes V_2(a_2) \xrightarrow[\check{R}_{12}(a_1/a_2)]{} V_2(a_2) \otimes V_1(a_1)$

rational operator

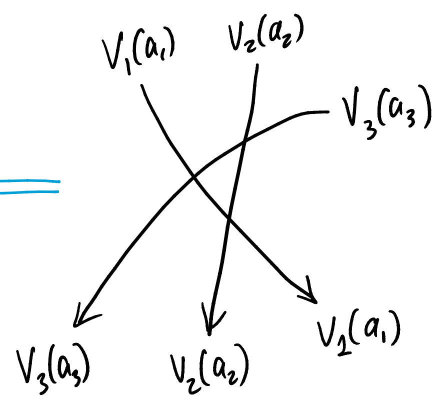
$R_{V_1,V_2}(a_1/a_2)$



$V_1(a_1) \otimes V_2(a_2) \longrightarrow V_1(a_1) \otimes_{opp} V_2(a_2)$.

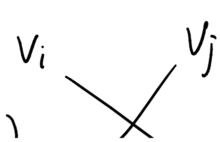


$\stackrel{?}{=}$

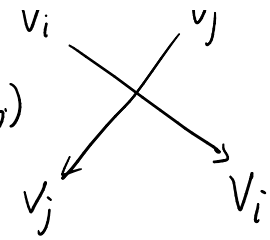


we assume equality on the nose, a.k.a. YBE

Reconstruction : given a family of $R_{\ldots}(a/a)$

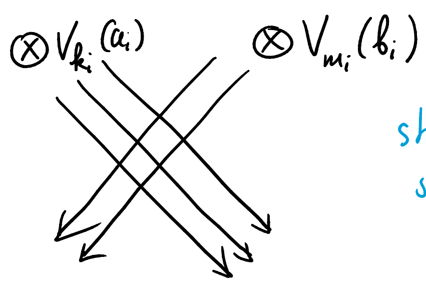


Reconstruction: given a family of solutions $R_{ij}(a_i/a_j)$



Hopf algebra $\hookrightarrow \bigotimes V_{k_i}(a_i)$.

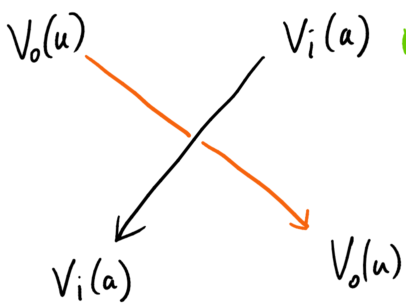
1)



still satisfies YB, so may assume V_i is closed under $\otimes$

2)

auxiliary



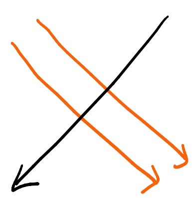
physical

matrix elements in V_0 give operators, depending on u in $V_i(a)$

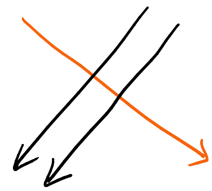
loop rotation aut.

generate $U_\hbar$

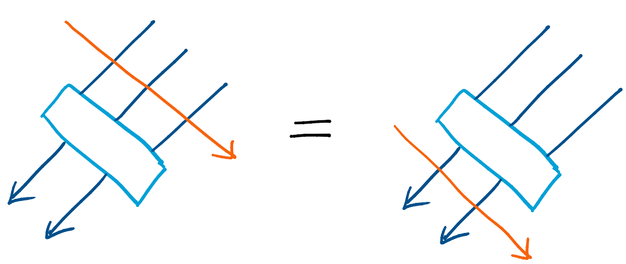
Product



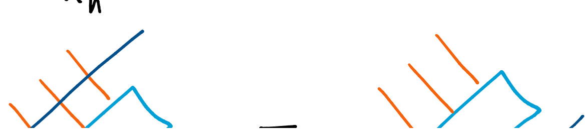
Coproduct

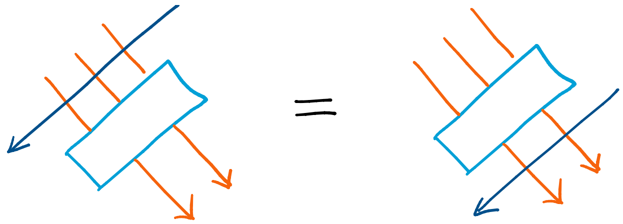


Morphisms: maps that commute with R-matrices

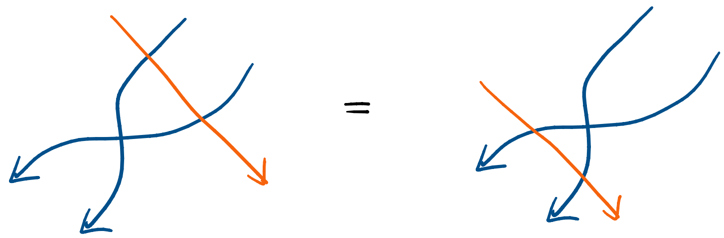


give relations in $U_\hbar$





in particular, by YB equation



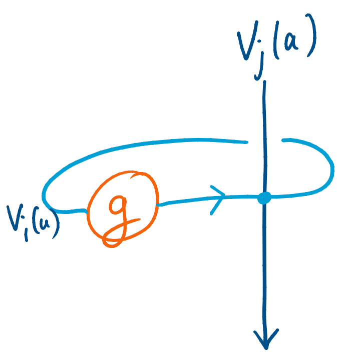
thus R-matrix indeed gives
1) braiding
2) commutation relations

Baxter subalgebra:

Let $g \in End(V_i(a_i))$ be such that $[g \otimes g, R] = 0$

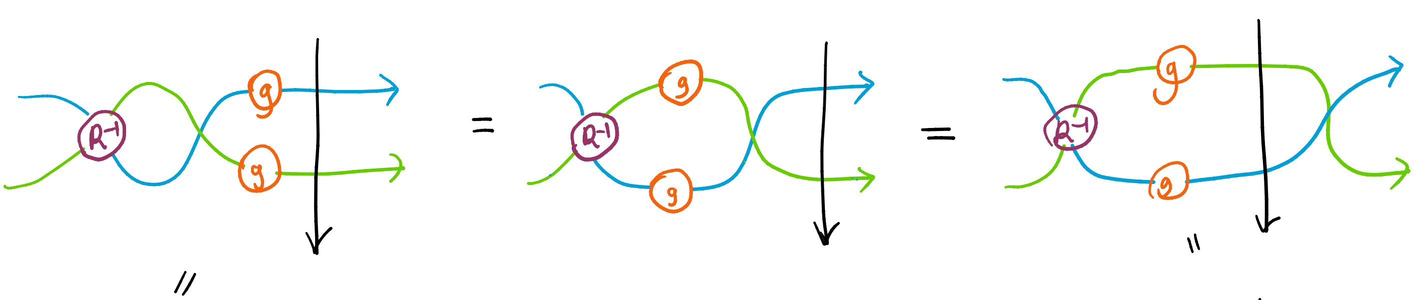
come from the Cartan torus

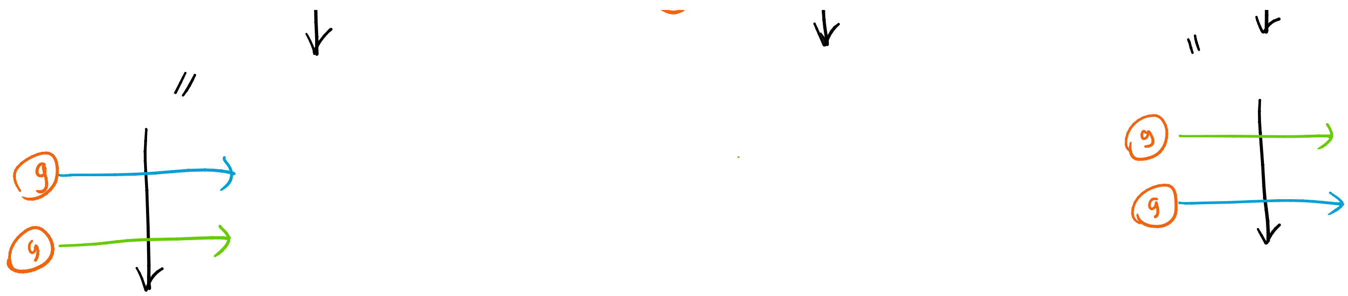
from discrete invariants for geometrically constructed quantum groups



$= tr_1 (g \otimes 1) R_{V_i V_j}(u/a)$ "Transfer matrix"

commute for fixed g and all V_i and u





Hence, get a family of commutative subalgebras in $U_\hbar(\hat{\mathfrak{g}})$

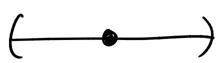
$\cup$

Baxter operators

parametrized by a max. torus in $\mathfrak{g}$

$$0 \to H_2(X, D) \to H_2(X,\mathbb{Z}) \to \underset{\approx \mathbb{Z}}{H_2(B,\mathbb{Z})} \to 0.$$

number of points n



commute for fixed z and degree variables $\leftarrow H^2(D)\otimes \mathbb{C}^*$

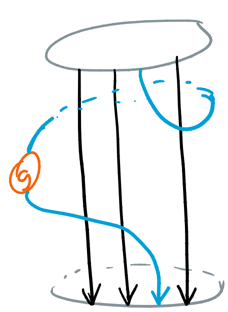
$z = e^{iu}$

$H^2(D)\otimes \mathbb{C}^* = (\mathbb{C}^*)^r$ for A_r

$$H_2(\mathrm{Hilb}(D)) = H_2(D) + \mathbb{Z}$$

cocharacters of max torus in $U_\hbar(\widehat{\mathfrak{gl}(r+1)})$.

Similarly:



commute for any $[g \otimes g, R(a_1/a_2)]$

including $g(a_i) = q\, a_i$

new variable

$\Rightarrow$ canonical flat q-difference connection on $V_1(a_1)\otimes \cdots \otimes V_n(a_n)$.

$\Longrightarrow$ [illegible]

qKZ of I. Frenkel - Reshetikhin. $\qquad\qquad$ $\downarrow$ qa_1