SCGP 4

Thursday, August 25, 2022 2:55 PM

Enumerative geometry $\longleftarrow\!\!\longrightarrow$ QIS

Nakajima quiver var $\coprod_v \mathcal{M}(v,w)$

commutative algebra $\subset \mathrm{End}\left(H^\bullet\left(\mathrm{Hilb}(A_r,*)\right)\right)$

find eigenvectors and eigenvalues

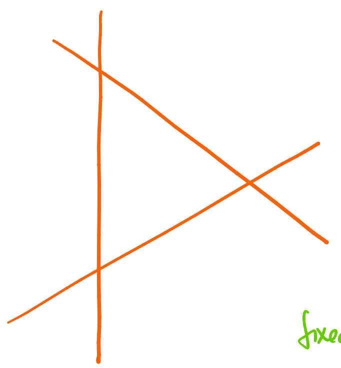
$q\to 1$

flat q-difference connection

find an integral solution.

differential GM

difference eq.



$\langle \beta_i, x\rangle + y_i = 0$

fixed

vary

$$\int_{\gamma_j} \omega_i(x) \prod_m \left(\langle \beta_m, x\rangle + y_m\right)^{c_m}$$

rational

$H_\bullet$(vector \ hyper, local system)

Euler $\int x^{a-1}(x-1)^{b-1}(y-x)^{c-1}\,dx$

q-analog

replace y and c

fundamental solution of q-diff. eq Ψ_{ij} $=$ $$\int_{\gamma_j} \omega_i(x) \underbrace{\prod_m \frac{\Gamma_q(x^{\beta_m} a_m)}{\Gamma_q(x^{\beta_m} b_m)}}$$

Mellin-Barnes

H_0(torus \ translates of subtori)

$$e^{\frac{S(\ldots\ldots)}{\ln q}}$$

$$\Psi(qa) = M(a)\,\Psi(a)$$

as $q\to 1$ solutions $e^{\frac{\int \ln \lambda_j \frac{da}{a}}{\ln q}}\, \psi_j(a)$

of $M(a)$

eigenvector with eigenvalue $\lambda_j(a)$

eigenvectors & eigenvalues $\Longleftrightarrow$ critical points

eigenvalues $e^{\frac{\partial S}{\partial a}}$

$e^{\sigma u}$

eigenvectors: evaluate $\omega_i(x)$ at critical pts of S in x

off-shell Bethe eigenfunctions

depend on x ← auxiliary var.

substitute solutions of $\partial_x S = 0$.

Bethe equations

small but important detail:

$$\int_{\gamma_j} \omega_i(x) \prod_m \frac{\Gamma_q(x^m a_m)}{\Gamma_q(x^{p_m} b_m)} \quad \xrightarrow{\text{better}} \quad \int_{\|x\|=1} \omega_i(x)\, e_j(x) \prod_m \frac{\Gamma_q(x^m a_m)}{\Gamma_q(x^{p_m} b_m)}$$

γ_j ← H_0(torus \ translates of subtori)

$e_j(x)$ ← elliptic ← numbers solution

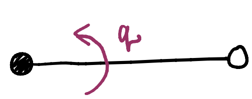
Geometrically (← work with quasimaps to Hilb / Nakajima q.v.)

Fundamental solution: j, numbers solutions

Yesterday:

one row of

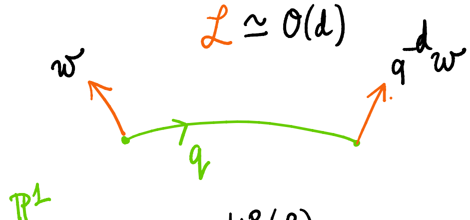
By contrast:



= Mellin-Barnes integral

all features are at 0 for q-fixed locus

$\mathcal{L} \simeq \mathcal{O}(d)$



Euler$(H^\bullet(\mathbb{P}^1, \mathcal{L}))$

$$H^0(\mathcal{L}) = w + q^{-1}w + \dots + q^{-d}w \qquad H^1(\mathcal{L}) = 0$$

$$\Gamma \ldots = (1 - w^{-1})(1 - q w^{-1}) \ldots (1 - q^d w^{-1})$$

locus

is a GIT quotient by some group G

$$\int_{\text{cycle} \subset \text{max torus of } G} \text{rational} \; \frac{\Gamma_q}{\Gamma_q}$$

$H^{\cdot}(\alpha) = w + q\, w + \dots + q^{d} w \qquad H^{\cdot}(\alpha) = 0$

$$\text{Euler} = (1-w^{-1})(1-qw^{-1})\dots(1-q^{d}w^{-1}).$$

$$\frac{\Gamma_q(\;)}{\Gamma_q(\;)}$$

need to find an elliptic term

elliptic cohomology..

 $=$ quasimaps $(\mathbb{P}^1 \longrightarrow \text{Hilb} / 中島)$

$$\begin{tikzcd}
 & \text{quasimaps} \arrow[dl, "ev_0"'] \arrow[dr, "ev_\infty"] & \\
\text{ambient stack} & & \text{Hilb}
\end{tikzcd}$$

$$\begin{tikzcd}
K(\text{ambient stack}) & & K(\text{Hilb}) \arrow[ll, "ev_0(\hat{\mathcal{O}}_{vir} \otimes ev_\infty \cdot z^{\deg})"'] \\
Ell(\quad) \arrow[u, "\frac{\Gamma_q}{\Gamma_q}"] & & Ell(\text{Hilb}) \arrow[ll, "!"'] \arrow[u, "\frac{\Gamma_q}{\Gamma_q}"']
\end{tikzcd}$$

入谷

$$\int_{\|x\|=1} \boxed{\omega_i(x)} \; e_j(x) \; \boxed{\prod_m \frac{\Gamma_q(x^{m} a_m)}{\Gamma_q(x^{b_m} b_m)}}$$

also in $q \to 0$ terms

index in Hilb

controls the spectrum

can see it here

index in Hilb

these details do not affect

... spectrum
can see it here

observe already in
NS ~ 2007
basis of their conjectures

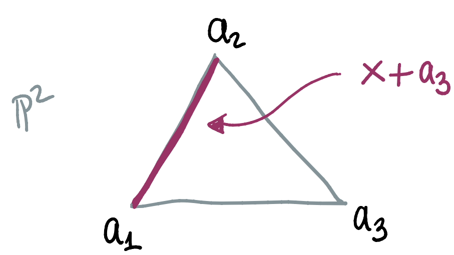
Elliptic cohomology

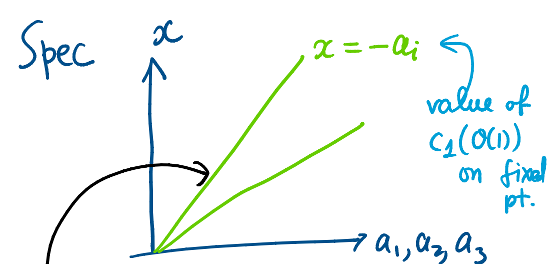
$$H^*(\mathbb{P}^{n-1}, \mathbb{C}) = \mathbb{C}[x]/x^n$$

$x = c_1(\mathcal{O}(1))$

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \hookrightarrow \mathbb{P}(\mathbb{C}^n) \qquad H^*_A(\mathbb{P}^{n-1}, \mathbb{C}) = \mathbb{C}[x, a_1, \dots, a_n] \Big/ \prod_{i=1}^n (x + a_i)$$

$a \in \mathrm{Lie}$

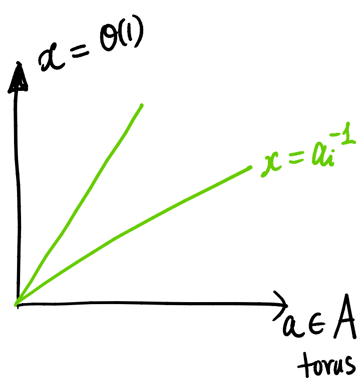




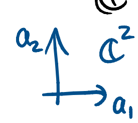
each hyperplane is the cohomology of $pt \in (\mathbb{P}^{n-1})^A$

Similarly:

$\mathrm{Spec}\, K_A(\mathbb{P}(\mathbb{C}^n))$



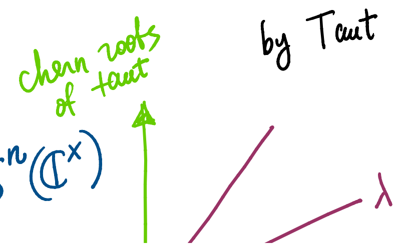
$\mathrm{Spec}\, K_{(\mathbb{C}^\times)^2}(\mathrm{Hilb}(\mathbb{C}^2, n))$



generated by Taut (and its Λ^i Taut).

Clearly, it makes sense to replace $\mathbb{C}^*$ by

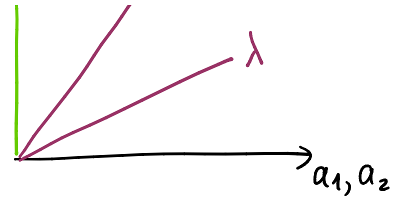
$S^n(\mathbb{C}^\times)$

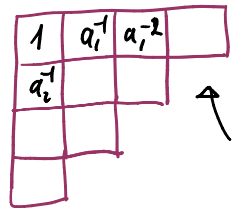


to replace $\mathbb{C}^*$ by

$$E = \mathbb{C}^*/q^{\mathbb{Z}}$$

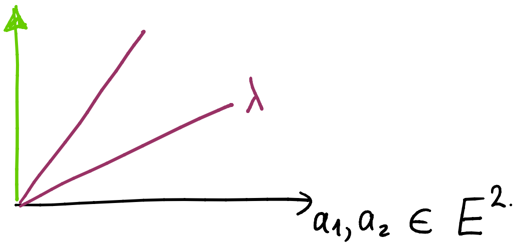
$S^n(\mathbb{C}^\times)$





contets

$S^n E$



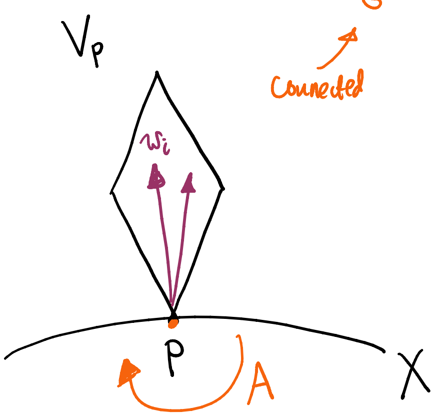
Important: $\mathrm{Ell}_G(X)$ is a scheme not an algebra

covariant in both G and X

$\mathrm{Ell}_G(pt) = \mathrm{Ell}(pt/G) =$ semistable degree 0 G-bundles on $E^\vee$

connected

$\mathrm{Ell}_{(\mathbb{C}^*)^n}(pt) = E^n$



functions on Spec H/Spec K.

$$\mathrm{Euler}(V)\Big|_p \;=\; \text{in cohomology}\quad \prod w_i$$

$$= \text{in K-theory}\quad \prod(1 - w_i^{-1})$$

$$= \prod \vartheta(w_i)$$

odd theta.

section of a line bundle

of rank n

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{C}^n/GL(n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & pt/GL(n) \end{array}$$

$\mathrm{Ell}(X) \longrightarrow \mathrm{Ell}(pt/GL(n)) = S^n E$ $\qquad w_1, \dots, w_n \qquad \prod \vartheta(w_i) = 0$

$\Theta(V) \xleftarrow{\text{pull back}}$ $\qquad \Theta$ divisor $= \{$ one of the numbers $\}$

$\Theta(V) \xleftarrow{\text{pull back}}$ Θ divisor $= \{$ one of the numbers is $0 \}$.

Euler(V) is a section

s.s. degree 0 bundles with a section

Pushforwards involve Euler classes and hence twists by line bundles

	in H	in K	in Ell
pushforward by localization:	$\frac{1}{\prod w_i}$	$\frac{1}{\prod(1-\frac{1}{w_i})}$	$\frac{1}{\vartheta(w_i)}$

tangent weight

push forward $X \longrightarrow pt$ $\quad \Theta(TX) \longrightarrow Ell(pt)$.

inclusion $pt \longrightarrow X$ $\quad Ell(pt) \longrightarrow \Theta(TX)$.

$\Theta(TX) \cong$ canonical on Ell

$Y \subset X$

smooth

$$Pic(X) \otimes E \longrightarrow Pic_0(Ell(X))$$

Kähler variables/q

quantum computation in K $\longleftrightarrow$ classical computation in Ell