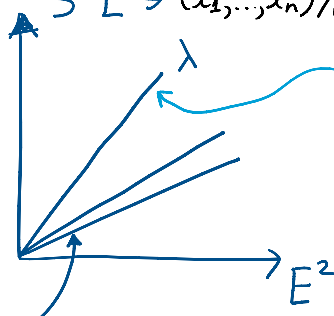
SCGP 5

Friday, August 26, 2022 12:29 PM

$\mathrm{Ell}_{(\mathbb{C}^*)^2} \mathrm{Hilb}(\mathbb{C}^2, n) \subset$

$S^n E \ni (x_1, \dots, x_n)/\mathrm{perm}$



$(x_k) = \left(t_1^{1-j}\, t_2^{1-i}\right)$ $\quad \square = (i,j) \in \lambda$

Chern roots of the tautological bundle Taut

intersect in a complicated way

t_1, t_2 mod q $\quad \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \hookrightarrow \mathbb{C}^2, \ \mathrm{Hilb}(\mathbb{C}^2)$

$$\Theta(\mathrm{Taut}) = \left(\prod \vartheta(x_i)\right)$$

one of these numbers is $= 1$

$$\deg \Theta(\mathrm{Taut})\Big|_\lambda = \sum_{\square} S^2 \begin{pmatrix} 1-j \\ 1-i \end{pmatrix} = \sum_{\square} \begin{pmatrix} (j-1)^2 & (i-1)(j-1) \\ (i-1)(j-1) & (i-1)^2 \end{pmatrix} \in S^2 \mathbb{Z}^2 = NS(E^2)$$

$$\deg \Theta(\mathrm{Tan})\Big|_\lambda = \sum_{\square} S^2 \begin{pmatrix} a+1 \\ -\ell \end{pmatrix} + S^2 \begin{pmatrix} -a \\ \ell+1 \end{pmatrix}$$

if we restrict $t_1 t_2 = 1$ even, has $\sqrt{\ }$

$$\mathrm{Tan} = T^{1/2} + t_1 t_2 \left(T^{1/2}\right)^\vee$$

polarization

$\mathrm{Pic}_0(\mathrm{Ell})$
$\Downarrow$

We will need sections of a line bundle $\mathcal{J} = \sqrt{\Theta(\mathrm{Tan})} \otimes$ something of degree 0

$'' \ \Theta(T^{1/2})$

Kähler variables

$$\mathrm{Taut} \longrightarrow \mathcal{O}(1) = \det \mathrm{Taut}$$

$$\underset{(x_1, \dots, x_n)}{S^n E} \longrightarrow \underset{\prod x_i}{E}$$

$\mathrm{Pic}_0(E) = E^\vee \simeq E$
$\Downarrow$
z

u

$$\frac{\vartheta(\prod x_i\, z)}{\vartheta(\prod x_i)\, \vartheta(z)} \quad \text{meromorphic section}$$

$$\frac{\vartheta(\prod x_i z)}{\vartheta(\prod x_i)\,\vartheta(z)}$$ meromorphic section

more generally $z \in \mathrm{Pic}(X) \otimes E$

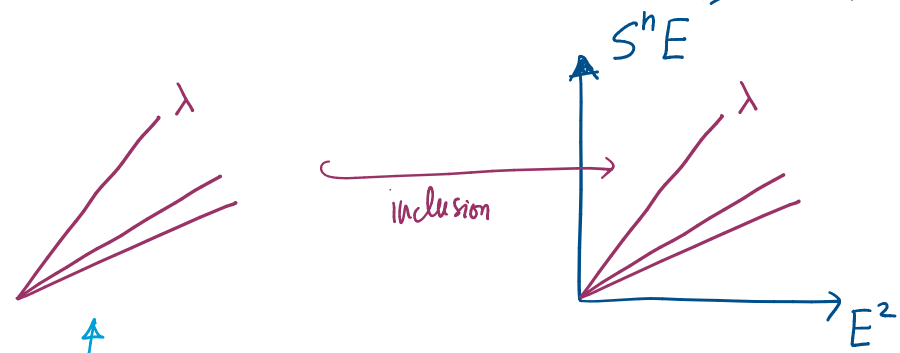
Hilb, 中島

Lagrangian class $\longrightarrow$ — " —

$$\mathrm{Ell}(\mathrm{Hilb}) \longrightarrow \mathrm{Ell}(\text{Ambient stack})$$

stable locus

$= \mathrm{Ell}(pt/GL(n))$



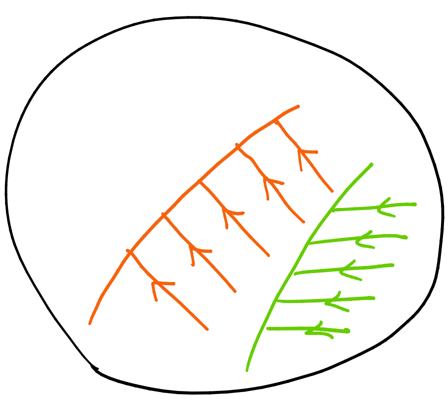
a section of $\mathcal{J}$ $\xrightarrow{\text{interpolate}}$ a section of $\mathcal{J}$

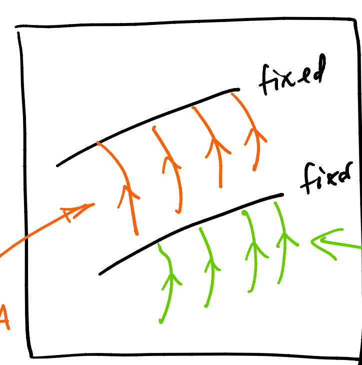
Stack has a stratification in which the stable locus is the first piece

" Harder-Nara.. "

BB

$X \supseteq A$ abelian





$\mathrm{Attr} \subset X \times X^A$

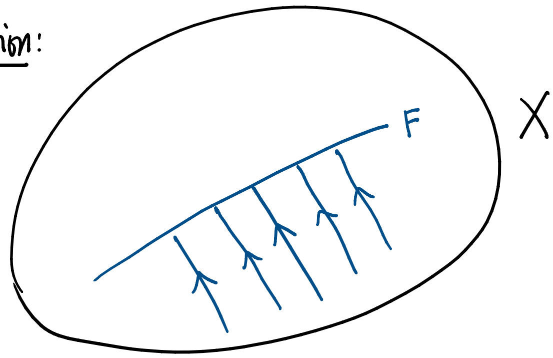
Stable envelopes extend Attr further down.

in elliptic coh will explicitly depend on Kähler variables

<u>One step of the induction:</u>

on Kähler variables

One step of the induction:



$$\underbrace{X \setminus \mathrm{Attr}(F)}_{U} \hookrightarrow X$$

in ordinary cohomology

$$\cancel{\to} H^i(X,U) \to H^i(X) \to H^i(U) \cancel{\to}$$

AB $\quad N_{X/\mathrm{Attr}(F)}$ has nontrivial action

replace by embedding of F into a Vector bundle. $\mathcal{V}$ over F

$$0 \to H^i(\mathrm{Thom}(V)) \to H^i(\text{total } \mathcal{V}) \to H^i(\mathcal{V} \setminus 0) \to 0.$$

functions on Spec $H^\bullet$

dual to the embedding of $GL(r-1) \hookrightarrow GL(r)$ of the embedding of $\Theta(\mathcal{V})$

in elliptic cohomology will be an exact sequence of line bundles

$$\begin{array}{ccccccc} V & \longrightarrow & \mathbb{C}^n/GL(r) & \supset (\mathbb{C}^n \setminus 0)/GL(r) & = \mathrm{pt}\Big/\begin{pmatrix} 1 & * \\ & \boxed{*} \end{pmatrix} & = \mathrm{pt}/GL(r-1). \\ \downarrow & & \downarrow & & & \\ F & \longrightarrow & \mathrm{pt}/GL(r) & & & \end{array}$$

$$\mathrm{Ell}(\mathcal{V} \setminus 0) \hookrightarrow \mathrm{Ell}(\mathcal{V})$$

$$\text{divisor } \Theta(\mathcal{V}) \longleftrightarrow \mathrm{Ell}(\mathcal{V})$$

i.e. interpolation from a section of $\mathcal{S}\big|_{\Theta(\mathcal{V})}$

$$0 \to \mathcal{S} \otimes \Theta(-V) \to \mathcal{S} \to \mathcal{S}\big|_{\Theta(V)} \to 0$$

sheaves on $\mathrm{Ell}(F)$ we want global sections

$$0 \to H^0(\mathcal{S}\otimes\Theta(-V)) \to H^0(\mathcal{S}) \to H^0\big(\mathcal{S}\big|_{\Theta(V)}\big) \to$$

$$H^1(\underbrace{\mathcal{S}\otimes\Theta(-V)}_{\mathcal{L}}) \to$$

want this $\Longleftarrow$ if $\mathcal{L}$ is a line bundle on an abelian variety $\mathcal{E}$, $\deg \mathcal{L} = 0$, $\mathcal{L} \neq \mathcal{O}_{\mathcal{E}}$ then $H^0(\mathcal{L}) = 0$

because $\mathcal{S} = \Theta(T^{1/2}) \otimes$ degree 0

$N_{X/\mathrm{Attr}(F)}$ picks exactly half directions

Kähler variables

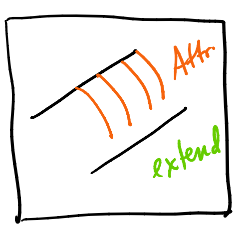
poles here related to poles in curve counting

Summary: this gives

$$\mathrm{Ell}(\text{stable}) \xrightarrow{\text{Stable envelope}} \mathrm{Ell}(\text{stack})$$

interpolation for sections of line bundles

and also



to Stab: $\mathrm{Ell}(X^A) \to \mathrm{Ell}(X)$

as we change attracting / repelling dir

$\hookrightarrow$ elliptic R-matrices

$\rightsquigarrow$ K-theory, H

- Important point

T^* / Vector Space

• Important point

for Stack $= T^*\left(\text{vector space} / \prod GL(v_i) \right)$ $\rightsquigarrow$ K-theory, H

the first reduces to the second $\rightsquigarrow$ 1704.08746

Solve quantum q-difference equations by integrals

$\underbrace{\text{Solve quantum q-difference}}$ $\longleftarrow$ o

quotient by G

0 •——o ∞

$$K(\text{Stable}) \xrightarrow{ev_0\left(ev_\infty^{-1} \otimes \hat{\mathcal{O}}_{vir}\, z^{\deg}\right)} K(\text{stack})[[z]]$$

入谷の Γ_q

expand.

$$K(\text{Stable}) \xrightarrow{ch(\text{elliptic stable env})} K(\text{Stack})(z)$$

fund solution

Kähler var.

rational

$$\Psi_{ij} = \int_{|x|=1} \omega_i(x)\; e_i(x)\; \prod \frac{\Gamma_q}{\Gamma_q}$$

elliptic

as $q \to 1$ this gives Bethe eq.

max compact torus in G

image under

in K(Stable)

in 1704.08746 this is the $q=0$ limit of $e_i(x)$

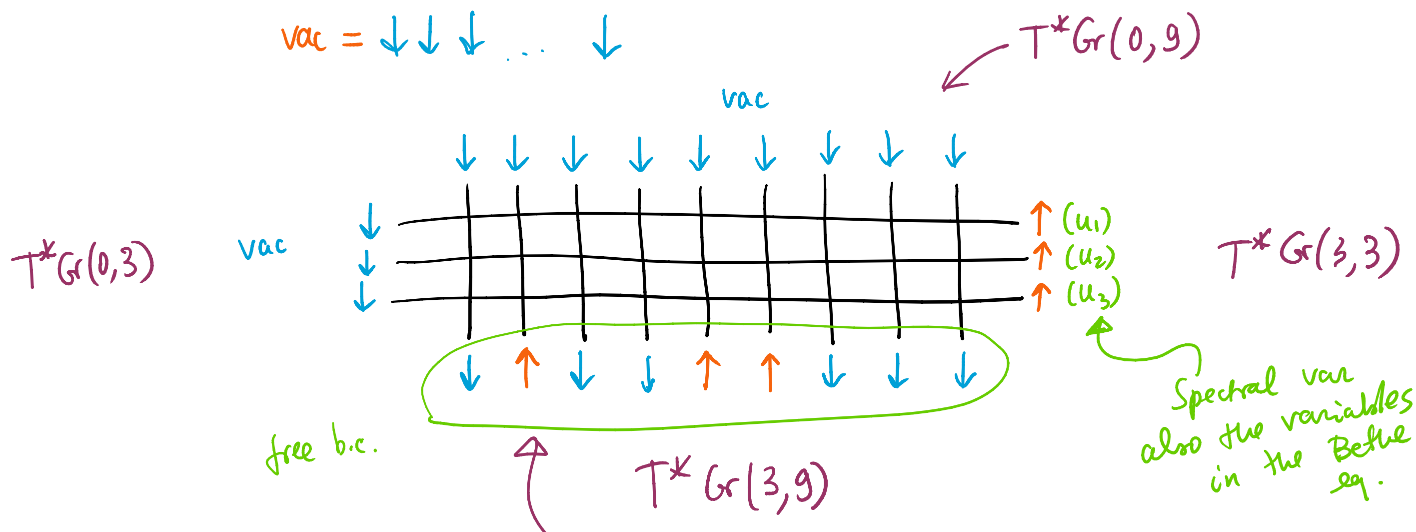
Stable envelope in K-theory.

this is the off-shell Bethe eigenfunction

For $/\prod GL(v_i)$, the nonabelian stable env reduces to the abelian one

hence $\exists$ formula for in terms of R-matrices

for XXZ $\uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \downarrow$



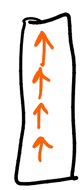
if u_i solve the Bethe eq. then is an eigenvector

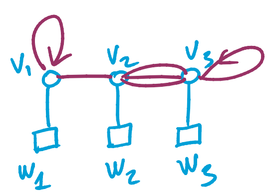
origin of algebraic Bethe Ansatz

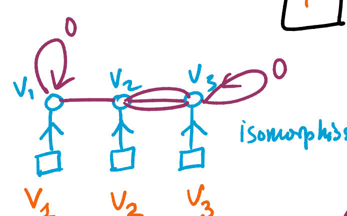
$$\begin{tikzcd} & \mathcal{M}(0,w) \arrow[dd] & \\ \mathcal{M}(0,v) \arrow[rr] & & \mathcal{M}(v,v) \\ & \mathcal{M}(v,w) & \end{tikzcd}$$

vacuum

v in place of w

what is the analog of  here

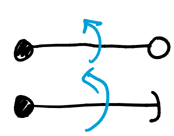


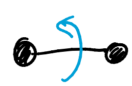


Bethe roots sit in a max torus

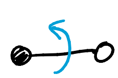
$\prod GL(\text{framing}) = \prod GL(V_i)$

Open (as far as I know) problem

in dim 1, we computed  in terms of $GL(\infty)$

and concluded that  is a τ-function for 2-Toda.

$GL(\infty) \curvearrowright \text{Fock} \otimes \text{Fock} \ni \sum_k \psi_k \otimes \psi_k^*$

in dim 3: know both of  in terms of $U_\hbar(\widehat{\mathfrak{gl}}(1)) \xrightarrow{\hbar=1} U(\mathfrak{gl}(\infty))$

in dim 3: know both of in terms of $U_\hbar(\widehat{\mathfrak{gl}}(1)) \xrightarrow[\hbar=1]{} U(\mathfrak{gl}(\infty))$

$3 \to 1$
by Mumford

$U_\hbar \hookrightarrow \mathrm{Fock} \otimes \mathrm{Fock}(\text{shift}) \leftrightarrows$ screening operator

$\mathrm{Fock} = \bigoplus_{n \geq 0} K(\mathrm{Hilb}(\mathbb{C}^2, n))$

$M(z)$

slice

what is the deformation of the 2-Toda ?

can investigate using Henry's code

In H^*, $U_\hbar \rightsquigarrow Y$

hyperspom in z

2 lectures

2 minutes $\Longleftarrow$ A. Smirnov "Rationality ...."

rational in z

3 key facts: 1) $=$ classical. for $M(v,w)$ for $w \gg 0$

a fixed descendent

"large framing vanishing" [PCMi]

2) $w = w_0 \oplus w_1$

$a \in \mathbb{C}^*$ acts on W_1 by scalar.

fixed large

$M(\cdot, w)^a = M(\cdot, w_0) \times M(\cdot, w_1)$

descendent is a Laurent poly in chern roots of taut V

$V = V_0 \oplus a V_1$

$\otimes \det V$

polynomial

$a \to 0$

same desc. for $w = w_0$

want to know

product

localiz on $\mathbb{P}^1$ and a, together with $a \to 0$

no descendent

3) $\xrightarrow{a \to 0}$

[Smirnov]

fund. solut. for

some explicit operator

fund solut. for
q- diff. eq.

[Smirnov]

some explicit operator
"fusion" ≈ glue
matrix

in terms of $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}(1)})$

only one hat