Disclaimer

These notes were taken during the lectures using neovim. Any errors are mine and not the speakers’. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. Also, notation may differ between lecturers. If you find any errors, please contact me at plei@math.columbia.edu.

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Seminar Website: https://math.columbia.edu/~plei/s24-birat.html
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1 Preliminaries

1.1 Givental formalism (Patrick, Feb 01)

1.1.1 Introduction  Let $X$ be a smooth projective variety. Then for any $g, n \in \mathbb{Z}_{\geq 0}, \beta \in H_2(X, \mathbb{Z})$, there exists a moduli space $\overline{M}_{g,n}(X, \beta)$ (Givental’s notation is $X_{g,n,\beta}$) of stable maps $f: C \to X$ from genus-$g$, $n$-marked prestable curves to $X$ with $f_*[C] = \beta$. It is well-known that $\overline{M}_{g,n}(X, \beta)$ has a virtual fundamental class $[\overline{M}_{g,n}(X, \beta)]_{\text{vir}} \in A^*(\overline{M}_{g,n}(X, \beta))$, $\delta = \int_\beta c_1(X) + (\dim X - 3)(1-g) + 3$.

In addition, there is a universal curve and sections $\pi: C \to \overline{M}_{g,n}(X, \beta)$.

In this setup, there are tautological classes $\psi_i := c_1(\sigma_i^*\omega_\pi) \in H^2(\overline{M}_{g,n}(X, \beta))$.

This allows us to define individual Gromov-Witten invariants by

$$\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle_{g,n,\beta}^X = \int_{[\overline{M}_{g,n}(X, \beta)]_{\text{vir}}} \prod_{i=1}^n \text{ev}_i^* \phi_i \cdot \psi_{a_i}.$$ 

These invariants satisfy various relations. The first is the string equation:

$$\langle \tau_0(1)\tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle_{g,n+1,\beta}^X = \sum_{i=1}^n \left\langle \tau_{a_i-1}(\phi_i) \prod_{j \neq i} \tau_{a_j}(\phi_j) \right\rangle_{g,n,\beta}^X.$$ 

The next is the dilaton equation:

$$\langle \tau_1(1)\tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle_{g,n+1,\beta}^X = (2g - 2 + n) \langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle_{g,n,\beta}^X.$$
Finally, we have the divisor equation when one insertion is a divisor $D \in H^2(X)$:

$$\langle \tau_0(D) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle_{g,n+1,\beta}^X = \left( \int_D \right) \cdot \langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle_{g,n,\beta}^X + \sum_{i=1}^n \langle \tau_{a_{i-1}}(\phi_i \cdot D) \prod_{j \neq i} \tau_{a_j}(\phi_j) \rangle_{g,n,\beta}^X.$$

It is often useful to package Gromov-Witten invariants into various generating series.

**Definition 1.1.1.** The quantum cohomology $\text{QH}^*(X)$ of $X$ is defined by the formula

$$(a \ast_t b, c) := \sum_{\beta,n} \frac{Q^\beta}{n!} \langle a, b, c, t, \ldots, t \rangle_{0,3+n,\beta}^X$$

for any $t \in H^*(X)$. This is a commutative and associative product.

The small quantum cohomology is obtained by setting $t = 0$ and the ordinary cohomology is obtained by further setting $Q = 0$.

**Remark 1.1.2.** Convergence of the formula does not hold in general, so quantum cohomology needs to be treated as a formal object.

**Definition 1.1.3.** Let $\phi_i$ be a basis of $H^*(X)$ and $\phi^i$ be the dual basis. Then the $J$-function of $X$ is the cohomology-valued function

$$J_X(t, z) := z + t + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!} \langle \frac{\phi_i}{z-\psi}, t, \ldots, t \rangle_{0,n+1,\beta}^X \phi^i.$$

**Definition 1.1.4.** The genus-0 GW potential of $X$ is the (formal) function

$$\mathcal{F}_X(t(z)) = \sum_{\beta,n} \frac{Q^\beta}{n!} \langle t(\psi), \ldots, t(\psi) \rangle_{0,n,\beta}^X.$$

The associativity of the quantum product is equivalent to the PDE

$$\sum_{e,f} \mathcal{F}_X^{abef} \eta^{ef} \mathcal{F}_c^{def} = \sum_{e,f} \mathcal{F}_X^{abef} \eta^{ef} \mathcal{F}_c^{bcf}$$

for any $a, b, c, d$, which are known as the WDVV equations. Here, we choose coordinates on $H^*(X)$ and set $z = 0$ (only consider primary insertions). In addition, set $\eta_{ef}$ to be the components of the Poincaré pairing and let $\eta^{ef}$ be the inverse matrix.

**1.1.2 Frobenius manifolds** A Frobenius manifold can be thought of as a formalization of the WDVV equations.

**Definition 1.1.5.** A Frobenius manifold is a complex manifold $M$ with a flat symmetric bilinear form $\langle -, - \rangle$ (meaning that the Levi-Civita connection has zero curvature) on $TM$ and a holomorphic system of (commutative, associative) products $\ast_t$ on $T_tM$ satisfying:

1. The unit vector field $1$ is flat: $\nabla 1 = 0$;
2. For any \( t \) and \( a, b, c \in T_t M \), \( \langle a \ast_1 b, c \rangle = \langle a, b \ast_1 c \rangle \);

3. If \( c(u, v, w) := \langle u \ast_1 v, w \rangle \), then the tensor \((\nabla_z c)(u, v, w)\) is symmetric in \( u, v, w, z \in T_t M \).

If there exists a vector field \( E \) such that \( \nabla \nabla E = 0 \) and complex number \( d \) such that:

1. \( \nabla \nabla E = 0 \);
2. \( \mathcal{L}_E (u \ast v) - \mathcal{L}_E u \ast v - u \ast \mathcal{L}_E v = u \ast v \) for all vector fields \( u, v \);
3. \( \mathcal{L}_E \langle u, v \rangle - \langle \mathcal{L}_E u, v \rangle - \langle u, \mathcal{L}_E v \rangle = (2 - d) \langle u, v \rangle \) for all vector fields \( u, v \),

then \( E \) is called an \textit{Euler vector field} and the Frobenius manifold \( M \) is called \textit{conformal}.

\textbf{Example 1.1.6.} Let \( X \) be a smooth projective variety. Then we can give \( \mathbb{H}^*(X) \) the structure of a Frobenius algebra with the Poincaré pairing and the quantum product. Note that the quantum product does not converge in general, so we must treat this as a formal object. The Euler vector field is given by

\[
E_X = c_1(X) + \sum_i \left( 1 - \frac{\deg \phi_i}{2} \right) t^i \phi_i,
\]

where a general element of \( \mathbb{H}^*(X) \) is given by \( t = \sum_i t^i \phi_i \). We will also impose that \( \phi_1 = 1 \). There is another very important structure, the \textit{quantum connection}, which is given by the formula

\[
\nabla t^i := \frac{\partial}{\partial t^i} + \frac{1}{z} \phi_i \ast t
\]

\[
\nabla z \frac{d}{dz} := \frac{d}{dz} - \frac{1}{z} E_X \ast t + \mu_X.
\]

Here, \( \mu_X \) is the \textit{grading operator}, defined for pure degree classes \( \phi \in \mathbb{H}^*(X) \) by

\[
\mu_X(\phi) = \frac{\deg \phi - \dim X}{2} \phi.
\]

Finally, in the direction of the Novikov variables, we have

\[
\nabla \xi Q \partial Q = \xi Q \partial Q + \frac{1}{z} \xi \ast t.
\]

\textbf{Remark 1.1.7.} For a general conformal Frobenius manifold \((\mathbb{H}, (-, -), \ast, E)\), there is still a \textit{deformed flat connection} or Dubrovin connection given by

\[
\nabla t^i := \frac{\partial}{\partial t^i} + \frac{1}{z} \phi_i \ast
\]

\[
\nabla z \frac{d}{dz} := \frac{d}{dz} - \frac{1}{z} E \ast.
\]

\textbf{Definition 1.1.8.} The \textit{quantum D-module} of \( X \) is the module \( \mathbb{H}^*(X)[z][Q, t] \) with the quantum connection defined above.

\textbf{Remark 1.1.9.} It is important to note that the quantum connection has a fundamental solution matrix \( S^X(t, z) \) given by

\[
S^X(t, z) \phi = \phi + \sum_i \sum_{n, \beta} \frac{Q^\beta}{n!} \phi^i \left( \frac{\phi_i}{z - \psi}, \phi, t, \ldots, t \right)^X_{0, n + 2, \beta}.
\]

It satisfies the important equation

\[
S^X(t, -z) S(t, z) = 1.
\]

Using this formalism, the \( J \)-function is given by \( S^X(t, z) J = z^{-1} J_X(t, z) \).
1.1.3 Givental formalism  The Givental formalism is a geometric way to package enumerative (CohFT) invariants cleanly. We begin by defining the symplectic space

$$\mathcal{H} := \mathcal{H}(X, \Lambda)(z^{-1})$$

with the symplectic form

$$\Omega(f, g) := \text{Res}_{z=0}(f(-z)g(z))$$.

This has a polarization by Lagrangian subspaces

$$\mathcal{H}^+ := \mathcal{H}(X, \Lambda)[z], \quad \mathcal{H}^- := z^{-1}\mathcal{H}(X, \Lambda)[z^{-1}]$$

giving $\mathcal{H} \cong T^*\mathcal{H}^+$ as symplectic vector spaces. Choose Darboux coordinates $p, q$ on $\mathcal{H}$. For example, there is a choice in Coates’s thesis which gives a general element of $\mathcal{H}$ as

$$X_k \geq 0 X_i q^i k \phi_i z^k + X_\ell \geq 0 X_j p^j \ell \phi_j (-z)^{-\ell - 1}.$$  

Taking the dilaton shift

$$q(z) = t(z) - z = -z + t_0 + t_1 z + t_2 z^2 + \cdots,$$

we can now think of $\mathcal{F}^X$ has a formal function on $\mathcal{H}^+$. This convention is called the dilaton shift.

Before we continue, we need to recast the string and dilaton equations in terms of $\mathcal{F}^X$. Write $t_\chi = \sum t_k^i \phi_i$. Then the string equation becomes

$$\partial_{0}^1 \mathcal{F}(t) = \frac{1}{2} (t_0, t_0) + \sum_{n=0}^\infty \sum_{j} t_{n+1}^j \partial_n^j \mathcal{F}(t)$$

and the dilaton equation becomes

$$\partial_1^1 \mathcal{F}(t) = \sum_{n=0}^\infty t_n^j \partial_n^j \mathcal{F}(t) - 2 \mathcal{F}(t).$$

There are also an infinite series of topological recursion relations

$$\partial_{k+1}^l \partial_{\ell m}^j \mathcal{F}(t) = \sum_{a,b} \partial_k^a \mathcal{F}(t) \eta^{ab} \partial_0^b \partial_{\ell m}^j \mathcal{F}(t).$$

We can make sense of these three relations for any (formal) function $\mathcal{F}$ on $\mathcal{H}^+$.

Now let

$$\mathcal{L} = \{(p, q) \in \mathcal{H} \mid p = d_q \mathcal{F}\}$$

be the graph of $d\mathcal{F}$. This is a formal germ at $q = -z$ of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}^+$ and is therefore a formal germ of a Lagrangian submanifold in $\mathcal{H}$.

**Theorem 1.1.10.** The function $\mathcal{F}$ satisfies the string equation, dilaton equation, and topological recursion relations if and only if $\mathcal{L}$ is a Lagrangian cone with vertex at the origin $q = 0$ such that its tangent spaces $L$ are tangent to $\mathcal{L}$ exactly along $zL$.  


Because of this theorem, $\mathcal{L}$ is known as the Lagrangian cone. It can be recovered from the $J$-function by the following procedure. First consider $\mathcal{L} \cap (-z + z\mathcal{H}(-))$. Via the projection to $-z + H$ along $\mathcal{H}$, this can be considered as the graph of the $J$-function. Next, we consider the derivatives $\frac{\partial J}{\partial t^i}$, which form a basis of $\mathcal{L} \cap z\mathcal{H}(-)$, which is a complement to $z\mathcal{L}$ in $\mathcal{L}$. Then we know that
\[
z \frac{\partial J}{\partial t^i} \in z\mathcal{L} \subset \mathcal{L},
\]
so
\[
z \frac{\partial^2 J}{\partial t^i \partial t^j} \in \mathcal{L} \cap z\mathcal{H}(-).
\]
Writing these in terms of the first derivatives $\frac{\partial J}{\partial t^i}$ and using the fact that $J$ is a solution of the quantum connection, so we recover the Frobenius structure of quantum cohomology.

We will now express some classical results in this formalism. Let $X$ be a toric variety with toric divisors $D_1, \ldots, D_N$ such that $D_1, \ldots, D_k$ form a basis of $H^2(X)$ and Picard rank $k$. Then define the $I$-function
\[
I_X = ze^{\sum_{i=1}^k t_i D_i} \sum_\beta \prod_{j=1}^N \prod_{m=0}^{\infty} (D_j + mz) \prod_{j'=1}^N \prod_{m=0}^{\infty} (D_j + mz).
\]

**Theorem 1.1.11 (Mirror theorem).** The formal functions $I_X$ and $J_X$ coincide up to some change of variables, which if $c_1(X)$ is semi-positive is given by components of the $I$-function.

**Theorem 1.1.12 (Mirror theorem in this formalism).** For any $t$, we have
\[
I_X(t,z) \in \mathcal{L}.
\]

Another direction in Gromov-Witten theory is the Virasoro constraints. In the original formulation, these involved very complicated explicit differential operators, but in the Givental formalism, there is a very compact formulation.

Define $\ell^{-1} = z^{-1}$ and
\[
\ell_0 = z \frac{dz}{d^2z} + \frac12 + \mu + \frac{c_1(X) \cup -}{z}.
\]
Then define
\[
\ell_n = \ell_0(\ell_0)^n.
\]

**Theorem 1.1.13 (Genus-0 Virasoro constraints).** Suppose the vector field on $\mathcal{H}$ defined by $\ell_0$ is tangent to $\mathcal{L}$. Then the same is true for the vector fields defined by $\ell_n$ for any $n \geq 1$.

*Proof.* Let $L$ be a tangent space to $\mathcal{L}$. Then if $f \in z\mathcal{L} \subset \mathcal{L}$, the assumption gives us $\ell_0 f \in L$. But then $z\ell_0 f \in z\mathcal{L}$, so $\ell_0 z\ell_0 f = \ell_1 f \in L$. Continuing, we obtain $\ell_n f \in L$ for all $n$. \qed

Later, we will learn that the Quantum Riemann-Roch theorem can be stated in this formalism. Let $\mathcal{L}^{tw}$ be the twisted Lagrangian cone (where the twisted theory will be defined next week).

**Theorem 1.1.14 (Quantum Riemann-Roch).** For some explicit linear symplectic transformaiton $\Delta$, we have $\mathcal{L}^{tw} = \Delta \mathcal{L}$.
1.1.4 Quantization In the last part of the talk, we will briefly discuss the quantization formalism, which encodes the higher-genus theory. In Darboux coordinates $p_a, q_b$, we will quantize symplectic transformations by the standard rules
\[
q_a q_b = \frac{q_a q_b}{\hbar}, \quad q_a p_b = q_a \frac{\partial}{\partial q_b}, \quad p_a p_b = \hbar \frac{\partial^2}{\partial q_a \partial q_b}.
\]
This determines a differential operator acting on functions on $\mathcal{H}_+$. We also need the genus-$g$ potential
\[
\mathcal{F}_g^X := \sum_{\beta, n} \frac{Q^\beta}{n!} \langle t(\psi), \ldots, t(\psi) \rangle^X_{g,n,\beta}
\]
and the total descendent potential
\[
\mathcal{D} := \exp \left( \sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_g^X \right).
\]

In this formalism, the Virasoro conjecture can be expressed as follows. Let $L_n = \hat{t}_n + c_n$, where $c_n$ is a carefully chosen constant.

**Conjecture 1.1.15** (Virasoro conjecture). If $L_{-1} \mathcal{D} = L_0 \mathcal{D} = 0$, then $L_n \mathcal{D} = 0$ for all $n \geq 1$.

In this formalism, the higher-genus version of the Quantum Riemann-Roch theorem takes the very simple form

**Theorem 1.1.16** (Quantum Riemann-Roch). Let $\mathcal{D}^{tw}$ be the twisted descendent potential. Then
\[
\mathcal{D}^{tw} = \hat{\Delta} \mathcal{D}.
\]

1.2 Quantum Riemann-Roch (Shaoyun, Feb 08)

We will state and prove the Quantum Riemann-Roch theorem in genus 0, following Coates-Givental.

1.2.1 Twisted Gromov-Witten invariants Again, let $X$ be a smooth projective variety. Let $E$ be a vector bundle on $X$. We should note that
\[
\overline{M}_{0,n+1}(X, \beta) \xrightarrow{\pi} \overline{M}_{0,n}(X, \beta)
\]
is the universal curve, and the universal morphism is simply $\text{ev}_{n+1}$. We will consider the sheaf
\[
E_{0,n,\beta} := R\pi_* \text{ev}_{n+1}^* E \in K^0(\overline{M}_{0,n}(X, \beta)).
\]

We need to check that this is a well-defined K-theory class. Choose an ample line bundle $L \to X$. By definition, for $N \gg 1$, the cohomology
\[
H^i(X, E \otimes L^N) = 0
\]
whenever $i \geq 1$. This gives us an exact sequence
\[
0 \to \ker(= A) \to H^0(X, E \otimes L^N) \otimes L^{-N} (= B) \to E \to 0.
\]
For any stable map \( f: \Sigma \to X \) of positive degree, we obtain a long exact sequence

\[
0 \to H^0(\Sigma, f^*E) \to H^1(\Sigma, f^*A) \to H^1(\Sigma, f^*B) \to H^1(\Sigma, f^*E) \to 0,
\]

so we obtain

\[
R^0 f_* ev^* E - R^1 f_* ev^* A = R^1 f_* ev^* B - R^1 f_* ev^* A.
\]

This expresses \( E_{0,n,\beta} \) as a difference of vector bundles.

We will now introduce a universal characteristic class

\[
c(-) = \exp \left( \sum_{k=0}^{\infty} s_k ch_k(-) \right),
\]

where \( s_0, s_1, s_2, \ldots \) are formal variables and \( ch_k \) is the \( k \)-th Chern character

\[
\frac{x_1^k}{k!} + \cdots + \frac{x_i^k}{k!},
\]

where \( x_i \) are the Chern roots.

**Example 1.2.1.** Let \( E \to X \) be a vector bundle and equip it with the fiberwise \( \mathbb{C}^* \)-action by scaling. Let \( \lambda \) be the equivariant parameter and \( \rho_i \) be the Chern roots. Then

\[
e(E) = \sum_i (\lambda + \rho_i).
\]

We then rewrite

\[
\prod (\lambda + \rho_i) = \exp \left( \sum_i \left( \log \lambda - \sum_k \frac{(-\rho_i)^k}{k!} \right) \right)
\]

\[
= \exp \left( ch_0(E) \log \lambda + \sum_{k>0} \frac{(-1)^{k-1}(k-1)!}{\lambda^k} ch_k(E) \right),
\]

so for the (equivariant Euler class), we obtain

\[
s_0 = \log \lambda
\]

\[
s_k = \frac{(-1)^{k-1}(k-1)!}{\lambda^k}, \quad k > 0.
\]

We are now ready to define the \((E, c)\)-twisted Gromov-Witten invariants.

**Definition 1.2.2.** Define the twisted Gromov-Witten invariants by

\[
\langle \alpha_1 \psi_1^{k_1}, \ldots, \alpha_n \psi_n^{k_n} \rangle^{X/(E,c)}_{0,n,\beta} := \int_{[\overline{M}_{0,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i) \psi_i^{k_i} \cup c(E_{0,n,\beta})
\]

for \( \alpha_i \in H^*(X) \) and \( k_i \in \mathbb{Z}_{\geq 0} \).

We will now construct the Lagrangian cone for the twisted theory. Let \( R \) be the coefficient ring containing \( s_0, s_1, \ldots \) and define

\[
\mathcal{H}_X^{tw} := H^*(X) \otimes R[z^{-1}][Q].
\]
We also introduce the twisted Poincaré pairing
\[(a, b)_{(E,c)} = \int_X a \cup b \cup c(E).\]
The symplectic structure is defined by
\[\Omega_{tw}(f, g) = \text{Res}_{z=0}(f(-z)g(z))_{(E,c)}.\]
There is a polarization
\[\mathcal{H}_{tw}^+ = H^*(X) \otimes \mathbb{R}[z][Q]\]
with
\[\mathcal{H}_{tw}^- = H^*(X) \otimes \mathbb{R}[z][Q].\]
Finally, we have the twisted genus-0 descendent potential
\[\mathcal{F}_{0,X,tw}(t) := \sum_{\beta,n} \frac{Q^\beta}{n!} (t, \ldots, t)_{0,n,\beta}^{X(E,c)}.\]

1.2.2 Proof of Theorem 1.2.3 The idea is to use the Grothendieck-Riemann-Roch theorem.

Proposition 1.2.4 We can write
\[[\overline{M}_{0,n}(X, \beta)]^{vir} \cap \text{ch}_k(E_{0,n,\beta}) = \pi_* \left( \sum_{r+\ell=k+1} \frac{B_r}{r!} \text{ch}_\ell(e_{n+1}^* E) \Psi(r) \right),\]
where
\[\Psi(r) = \psi_{n+1}^r \cap \overline{M}_{0,n+1}(X, \beta)^{vir}\]
\[- \sum_{i=1}^n (\sigma_1)_* (\psi_i^{n-1} \cap \overline{M}_{0,n}(X, \beta)^{vir})\]
\[+ \frac{1}{2} \sum_{a+b=r-2, a, b \geq 0} (-1)^a \psi_i^a \psi_j^b \cap \overline{Z}_{0,n+1,\beta}^{vir}.\]
Here, $Z_{0,n+1,\beta}$ is formed by the nodes of $\pi$, $\tilde{Z}_{0,n+1,\beta}$ is a double cover of $Z_{0,n+1,\beta}$ formed by a choice of branch of the nodes, $\psi_+$ and $\psi_-$ are the $\psi$-classes at the two branches of the nodes, and

$$j: \tilde{Z}_{0,n+1,\beta} \to Z_{0,n+1,\beta} \to \overline{M}_{0,n+1}(X, \beta)$$

is the "inclusion."

**Proof.** We will first assume that $\overline{M}_{0,n+1}(X, \beta)$, $\overline{M}_{0,n}(X, \beta)$, and $Z_{0,n+1,\beta}$ are all smooth and that $\pi(Z_{0,n+1,\beta})$ is a normal crossings divisor. In general, we need a Cartesian diagram

![Cartesian diagram](image)

Continuing in the ideal situation, we apply Grothendieck-Riemann-Roch to obtain

$$ch(E_{0,n,\beta}) = ch(R\pi_* ev_{n+1}^* E) = \pi_* (ch(ev_{n+1}^* E) \cdot td^\vee \Omega_{\pi}),$$

where $td^\vee$ is the dual Todd class, defined by $\frac{x}{1-e^{tx}}$, and $\Omega_{\pi}$ is the sheaf of relative differentials.

We then have two short exact sequences

$$0 \to \Omega_{\pi} \to \omega_{\pi} \to \mathcal{O}_{Z_{0,n+1,\beta}} \to 0$$

and

$$0 \to \omega_{\pi} \to L_{n+1} \to \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \to 0,$$

where $D_i$ is the divisor where the marked points $i, n + 1$ collide and their component has exactly three special points. Now we obtain

$$\Omega_{\pi} = L_{n+1} - \sum_{i=1}^{n} \mathcal{O}_{D_i} - \mathcal{O}_{Z_{0,n+1,\beta}}$$

in K-theory. Using the facts that $c_1(L_{n+1}) = \psi_{n+1}$, $D_i \cap D_j = \emptyset$ for $i \neq j$, and $D_i \cap Z_{0,n+1,\beta} = \emptyset$, we see that $L_{n+1}$ is trivial when restricted to $D_i$ and $Z_{0,n+1,\beta}$. Now we apply the dual Todd class.

**Lemma 1.2.5.** If $x_1 \cup x_2 = 0$, then

$$(td^\vee(x_1) - 1)(td^\vee(x_2) - 1) = 0.$$
Using the lemma, we obtain
\[
\text{td}^\vee(\Omega_\tau) = \text{td}^\vee(L_{n+1}) \prod_{i=1}^n \text{td}^\vee(-\mathcal{O}_{D_i}) \text{td}^\vee(\mathcal{O}_{\mathcal{Z}_{0,n+1,\beta}}) - 1
\]
\[
= 1 + (\text{td}^\vee(L_{n+1}) - 1) + \sum_{i=1}^n \left( \frac{1}{\text{td}^\vee(\mathcal{O}_{D_i})} - 1 \right) + \left( \frac{1}{\text{td}^\vee(\mathcal{O}_{\mathcal{Z}_{0,n+1,\beta}})} - 1 \right).
\]
The first term in the statement comes from the dual Todd class of $L_{n+1}$, the second comes from
\[
0 \to \mathcal{O}(-D_i) \to 0 \to \mathcal{O}_{D_i} \to 0
\]
and the relation between $\mathcal{O}(-D_i)$ and $L_i$, and the last term can be found in Appendix A of Coates-Givental.

To obtain the Quantum Riemann-Roch theorem, we use the previous proposition and manipulate the generating function. If $E$ is convex and $Y \subset X$ is a complete intersection defined by $E$, then $\mathcal{L}^w_X$ is closely related to $\mathcal{L}_Y$, so we are able to study the Gromov-Witten theory of $Y$ using this.

1.3 Shift operators (Melissa, Feb 15)

Let $X$ be a semiprojective smooth variety. This means that $X$ is projective over its affinization. Also assume that $X$ has an action by $T = (\mathbb{C}^\times)^m$ such that all $T$-weights in $H^0(X, \mathcal{O})$ are contained in a strictly convex cone in $\text{Hom}(T, \mathbb{C}^\times)_R$ and $H^0(X, \mathcal{O})^T = \mathbb{C}$. All such $X$ imply that

(a) The fixed locus $X^T$ is projective;

(b) The $T$-variety $X$ is equivariantly formal. This means that $H^*_T(X)$ is a free module over $H^*_T(\text{pt}) = \mathbb{Q}[\lambda] := \mathbb{Q}[\lambda_1, \ldots, \lambda_m]$ and there is a non-canonical isomorphism
\[
H^*_T(X) \cong H^*(X) \otimes H^*_T(\text{pt})
\]
as $H^*_T(\text{pt})$-modules.

(c) The evaluation maps $\text{ev}_i : X_{0,n,d} \to X$ are proper.

Using (b), we may choose a basis $\langle \phi_i \rangle_{i=0}^N$ of $H^*_T(X)$ over $H^*_T(\text{pt})$. Let $\tau^i$ be the dual coordinates.

1.3.1 Equivariant big quantum cohomology Let $(-,-)$ be the $T$-equivariant Poincaré pairing, which in general takes values in $\mathbb{Q}(\lambda)$. Then the $T$-equivariant big quantum product is defined by
\[
\langle \phi_i \ast_T \phi_j, \phi_k \rangle = \langle \phi_i, \phi_j, \phi_k \rangle_{0,3}^{X,T}
\]
\[
= \sum_{d,n} \frac{Q}{n!} \langle \phi_i, \phi_j, \tau, \ldots, \tau \rangle_{0,n+3,d}^{X,T}.
\]
This can also be defined using the evaluation maps
\[
(\text{ev}_i)_* : H^*_T(X_{0,n+3,d}) \to H^*_{T_n}(-c_1(X) \cdot d_n, X)
\]
as
\[
\phi_i \ast_T \phi_j = \sum_{d,n} \frac{Q}{n!} (\text{ev}_3)_* \left( \text{ev}_1^*(\phi_i) \text{ev}_2^*(\phi_j) \prod_{i=4}^{n+3} \text{ev}_i^*(\tau) \cap [X_{0,n+3,d}]^{\text{vir}} \right) \in H^*_T(X)[[\mathbb{Q}][[\tau_0, \ldots, \tau_n]].
\]
1.3.2 Quantum connection  We will define
\[ \nabla_i : \mathcal{H}^* T(X)[\tau] \to z^{-1} \mathcal{H}^* T(X)[\tau^0, \ldots, \tau^N] \]
by setting
\[ \nabla_i = \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i^*). \]
We can view \( z \) as the loop variable by setting \( \tilde{T} = T \times C^\times \). If the extra copy of \( C^\times \) acts trivially on \( X \), then
\[ \mathcal{H}^*_T(X) = \mathcal{H}^*_T(X)[z]. \]
This has a fundamental solution \( M(\tau) : \mathcal{H}^*_T(X)[Q, \tau] \to \mathcal{H}^*_T(X)_{\text{loc}}[Q, \tau] \)
where
\[ \mathcal{H}^*_T(X)_{\text{loc}} := \mathcal{H}^*_T(X) \otimes Q(\lambda, z) Q(\lambda(z)). \]
This satisfies the differential equation
\[ z \frac{\partial}{\partial \tau^i} M(\tau) = M(\tau)(\phi_i^*), \]
which is equivalent to
\[ \frac{\partial}{\partial \tau^i} \circ M(\tau) = M(\tau) \circ \nabla_i. \]
The solution has the form
\[ (M(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j) + \left\langle \phi_i, \frac{\phi_j}{z - \psi} \right\rangle_{X,T}^{X, T}. \]

1.3.3 Shift operators  Let \( k : C^\times \to T \) be a cocharacter of \( T \). Then define a \( \tilde{T} \)-action \( \rho_k \) on \( X \) by
\[ \rho_k(t, x) = tu^k \cdot x \]
for \( t \in T, u \in C^\times, x \in X \). Under the group automorphism
\[ \phi_k : \tilde{T} \to \tilde{T} \quad \phi_k(t, u) = (tu^{-k}, u), \]
the identity map \((X, \rho_0) \to (X, \rho_k)\) is \( \tilde{T} \)-equivariant, so we obtain isomorphisms
\[ \Phi_k : \mathcal{H}^*_T(X)_{\rho_0} \to \mathcal{H}^*_T(X)_{\rho_k}. \]

Now define the bundle \( E_k = (X \times (C^2 \setminus 0))/C^\times \)
where \( C^\times \) acts by
\[ s \cdot (x, v_1, v_2) = (s^k x, s^{-1} v_1, s^{-1} v_2). \]
This is an \( X \)-bundle over \( \mathbb{P}^1 \) with an action on \( \tilde{T} \) by
\[ (t, u) \cdot [x, (v_1, v_2)] = [t \cdot x, (v_1, uv_2)]. \]
Setting \( 0 = [1, 0] \) and \( \infty = [0, 1] \), we see that \( \tilde{T} \) acts on \( X_0 \) by \( \rho_0 \) and \( X_\infty \) by \( \rho_k \).
**Definition 1.3.1.** A cocharacter \( k : \mathbb{C}^\times \to T \) is seminegative if all weights of \( H^0(X, O) \) are nonpositive with respect to \( k \) and is negative if all nonzero weights of \( H^0(X, O) \) are negative.

**Lemma 1.3.2.** If \( k \) is seminegative, then \( E_k \) is semiprojective.

Now let \( \tau : E_k \to \mathbb{P}^1 \) be the projection. We now consider section classes, which are those effective classes in \( H_0(E_k, \mathbb{Z}) \) satisfying \( \tau_0 \cdot d = [\mathbb{P}^1] \). For the \( \mathbb{C}^\times \)-action on \( X \) given by \( k \), there is a unique fixed component \( F_{\min} \) whose normal weights are all positive (one way to see this is to consider the moment map of the corresponding circle action). Therefore, there is a minimal section class \( \sigma_{\min} \) corresponding to \( F_{\min} \).

**Lemma 1.3.3.** Given \( \tau \in H^*_T(X) \), there exists \( \tilde{\tau} \in H^*_T(E_k) \) such that \( \tilde{\tau}|_{X_0} = \tau \) and \( \tilde{\tau}|_{X_\infty} = \Phi_k(\tau) \).

**Lemma 1.3.4.** If \( k \) is seminegative, then

\[
\text{Eff}(E_k)^{\sec} = \sigma_{\min} + \text{Eff}(X).
\]

**Definition 1.3.5.** Let \( k : \mathbb{C}^\times \to T \) be seminegative. Given \( \tau \in H^*_T(X) \), we define the shift operator

\[
\hat{S}_k : H^*_T,\rho_0 \cap (Q) \to H^*_T,\rho_k \cap (Q)
\]

by the formula

\[
(\hat{S}_k(\alpha, \beta)) = \sum_{\Phi \in \text{Eff}(E_k)^{\sec}} \frac{Q^d \sigma_{\min}}{n!} \langle (t_0, \alpha, \tau, \beta, \tau, \ldots, \tau)_{\rho_k} \tau \rangle
\]

where \( \alpha \in H^*_T,\rho_0 \cap (Q) \) and \( \beta \in H^*_T,\rho_k \cap (Q) \). We also define

\[
S_k(\tau) = \Phi_k^{-1} \circ \hat{S}_k(\tau).
\]

**Theorem 1.3.6.** We have the formula

\[
M(\tau) \circ S_k(\tau) = S_k \circ M(\tau),
\]

where \( S_k \) is defined via the commutative diagram

\[
\begin{array}{ccc}
H^*_T(X)_\text{loc} & \xrightarrow{S_k} & H^*_T(X)_\text{loc} \\
\downarrow & & \downarrow \Theta
\end{array}
\]

\[
\begin{array}{ccc}
H^*_T(X^T)_\text{loc} & \xrightarrow{\Theta \circ \Delta_1(k) \circ e^{-2k\delta_k}} & H^*_T(X^T)_\text{loc}
\end{array}
\]

Here, we define

\[
\Delta_1(k) = Q^{\sigma_{\min}} \prod_{\alpha} \prod_{j=1}^{r_k} \prod_{c=-\infty}^{0} \prod_{\rho_{i,\alpha,j} + \alpha + cz} \in H^*_T(F_i)_{\text{loc}}[Q],
\]

where

\[
N_i = N_{F_i/X} = \bigoplus_{\alpha} N_{i,\alpha}
\]

is the normal bundle of \( F_i \) in \( X \) and \( \rho_{i,\alpha,j} \) are its Chern roots.
The idea of the proof is to decompose

\[
E^T_{k,0,n+2,\delta} = \bigsqcup_i I_{1+i} \cup I_{2}= [n+2] d_0 + d_\infty + \tilde{\delta} = \tilde{\delta}
\]

Using the exact sequence

\[
0 \to \text{Aut}(C, x) \to \text{Def}(f) \to T^1 \to \text{Def}(C, x) \to \text{Obs}(f) \to T^2 \to 0,
\]

we obtain the explicit formulae

\[
\text{Aut}(C, x)^m = \text{Aut}(C_0, x_0)^m + \text{Aut}(C_\infty, x_\infty)^m
\]

\[
\text{Def}(C, x)^m = \text{Def}(C_0, x_0)^m \oplus \text{Def}(C_\infty, x_\infty)^m \oplus T_p C_0 \otimes T_p \mathbb{P}^1 \oplus T_q C_\infty \otimes T_q \mathbb{P}^1.
\]

This gives the virtual normal bundle, and using virtual localization, we obtain

\[
[\tilde{S}_k(\tau) \alpha, \beta] = (\tilde{S}_k M(\tau, z) \alpha, M'(\tau', -z) \beta),
\]

where

\[
M'(\tau', z) = \Phi_k \circ M(\tau, z) \circ \Phi_k^{-1}.
\]

Using the unitarity property of $M$, we obtain the desired result.
Quantum cohomology of projective bundles

2.1 Mirror theorem (Che, Feb 22)

Setup  Let \( X \) be a smooth projective variety, \( \{ \phi_i \}_{i=0}^s \) be a basis of \( H^*(X) \), \( \{ \phi^1_i \}_{i=0}^s \) be the dual basis, and

\[
\tau = \sum_{i=0}^s \tau^i \phi_i \in H^*(X).
\]

We will let

\[
J_X(\tau) = 1 + \frac{\tau}{z} + z^{-1} \sum_{d,n} \sum_{j=0}^s \left( \tau_{j} \ldots, \tau \frac{\phi_{j}}{z - \psi} \right)_{0,n+1,d} \frac{Q^d}{n!},
\]

which is the \( J \)-function in Definition 1.1.3 multiplied by \( z^{-1} \). Also, recall the inverse of the fundamental solution of the quantum D-module

\[
M_X(\tau) \in \text{End}(H^*(X))[z^{-1}][Q, \tau],
\]

which is defined by

\[
(M_X(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j)_X + \sum_{d,n} \sum_{j=0}^s \left( \phi_{i}, \tau_{j} \ldots, \tau \frac{\phi_{j}}{z - \psi} \right)_{0,n+2,d} \frac{Q^d}{n!}.
\]

Remark 2.1.1. By the string equation, we have

\[
J_X(\tau) = M_X(\tau) \cdot 1.
\]

The vector bundle case  Now let \( V \to B \) be a vector bundle with \( \text{rk} \, V \geq 2 \). This has an action of \( \mathbb{C}^\times \) scaling the fibers. Then we have

\[
H^*_\mathbb{C}^\times(V) = H^*(B) \otimes \mathbb{C}[\lambda].
\]

Now we may take \( \tau^0, \ldots, \tau^s \) to be \( \mathbb{C}[\lambda] \)-valued coordinates.

Remark 2.1.2. Equivariant localization is required to define the Gromov-Witten invariants of \( V \), which lie in \( \mathbb{C}[\lambda, \lambda^{-1}] \).

\[\text{This is in fact the older definition of the } J \text{-function, but the one in Definition 1.1.3 lies on the Lagrangian cone}\]
In order to avoid this issue, we will assume that \( V^\vee \) is globally generated. This implies that \( V \) is semiprojective, meaning that the evaluation maps \( \text{ev} : V_{0,n,d} \to V \) are proper. As before, we may define the fundamental solution

\[
M_V(\tau) \in \text{End}(H^\bullet(B)[\lambda, z^{-1}][Q, \tau])
\]

and the J-function

\[
J_V(\tau) = M_V(\tau) \cdot 1.
\]

Because the evaluation maps are proper, they can be defined without localization.

**Statement and discussion of the mirror theorem**

**Theorem 2.1.3.** Define the \( H^\bullet(\mathbb{P}(V)) \)-valued function

\[
I_{\mathbb{P}(V)}(\tau, t) = \sum_{k=0}^{\infty} \frac{e^{p_t/z} q^k t^k}{\prod_{l=1}^k (p + \delta + cz)} J_{\mathbb{P}(V)}^{p+kz}(\tau),
\]

where \( \delta \) are the Chern roots of \( V \), \( q \) is the Novikov variable, and \( p = c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \). Then \( zI_{\mathbb{P}(V)}(\tau, t) \) lies on the Lagrangian cone of \( \mathbb{P}(V) \).

Let \( L_X^{\text{orig}} \) be the Lagrangian cone for \( X \), which has the explicit form

\[
-z + t(z) + \sum_{d,n,k \geq 0} \sum_{i=0}^{s} \frac{\phi_i^k}{(-z)^{k+1}} \left\langle t(\psi), \ldots, t(\psi), \phi_i \psi^k \right\rangle_{0,n+1,d} \frac{Q^d}{n!}.
\]

**Definition 2.1.4.** For a set of variables \( x = (x_1, x_2, \ldots) \), we say that \( f \in \mathcal{H}_x[\bar{x}] \) is a \( C[\bar{Q}, \tau] \)-valued point on \( L_X^{\text{orig}} \) if \( f \) is of the form 2.1 for some \( t(z) \in \mathcal{H}_x[\bar{x}] \) with \( t(z)|_{Q=x=0} = 0 \).

**Example 2.1.5.** The point \( zI_X(\tau)|_{z \to z^{-1}} \) is a \( C[\bar{Q}, \tau] \)-valued point on \( L_X^{\text{orig}} \).

Given this, define \( L_X := L_X^{\text{orig}}|_{z \to z^{-1}} \). By Theorem 1.1.10, we obtain

\[
L_X = \bigcup_{\tau} zM_X(\tau)\mathcal{H}_x[\bar{x}],
\]

which means that any \( C[\bar{Q}, x] \)-valued point on \( L_X \) can be written as \( zM_X(\tau)f \) for some \( \tau \in H^\bullet(X)[\bar{Q}, x] \) and \( f \in \mathcal{H}_x[\bar{x}] \) such that \( \tau|_{Q=x=0} = 0 \) and \( f|_{Q=x=0} = 1 \). This property will be used to construct the Fourier transform later.

**2.1.1 Proof of Theorem 2.1.3** We will now sketch a proof of Theorem 2.1.3. First, we will need Quantum-Riemann-Roch for a vector bundle \( W \to X \) in two cases:

(a) When the vector bundle \( W \) is convex, which means that \( H^1(C, f^*W) = 0 \) for all stable maps \( f : C \to X \) of genus 0, and \( c = e(\lambda) \) is the equivariant Euler class, which corresponds to setting

\[
s_k = \begin{cases} 
\log \lambda & k = 0 \\
(-1)^{k-1}(k-1)!\lambda^{-k} & k > 0.
\end{cases}
\]

(b) When \( W \) is globally generated and \( c = e(\lambda)^1 \).
In the first case, we obtain the Gromov-Witten invariants of the zeroes of a regular section $Z \subset X$ of $W$ via
\[
\lim_{\lambda \to 0} \left( \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \right)_{0, n, d}^{X(W, e_{\lambda})} = \sum_{i + d = d} \left( i^* \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \right)_{0, n, d}^{Z}.
\]

In the second case, we obtain the Gromov-Witten invariants of $W$ via
\[
\left( \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \right)_{0, n, d}^{X(W, e_{\lambda}^{-1})} = \left( i^* \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \right)_{0, n, d}^{W}.
\]

We are now ready to begin the proof. Because $V^\vee$ is globally generated, there is a surjection
\[
\mathcal{O}^\oplus N \to V^\vee.
\]
This gives an exact sequence
\[
0 \to V \to \mathcal{O}^\oplus N \to Q \to 0
\]
embedding $\mathbb{P}(V) \hookrightarrow B \times \mathbb{P}^{N-1}$. By a result of Brown-Elezi, we have
\[
J_{B \times \mathbb{P}^{N-1}}(\tau, t) = \sum_{k=0}^{\infty} \frac{e^{pt/z} q^k e^{kt}}{\prod_{c=1}^{k} (p + cz)^N} J_B(\tau).
\]
Now define
\[
Q(1) := \pi^*_1 Q \otimes \pi^*_2 \mathcal{O}(1)
\]
on $B \times \mathbb{P}^1$. This has a section $s$ given by
\[
\pi^*_2 \mathcal{O}(-1) \to \mathcal{O}^\oplus N_{B \times \mathbb{P}^{N-1}} \to \pi^*_1 Q
\]
which satisfies $s^{-1}(0) = \mathbb{P}(V)$. Because $Q(1)$ is convex, we use Quantum-Riemann-Roch in case (a) to relate the Gromov-Witten theory of $\mathbb{P}(V)$ to the $(Q(1), e_{\lambda})$-twisted Gromov-Witten theory. We now require two more technical ingredients.

**Moving points on the Lagrangian cone via differential operators**

**Lemma 2.1.6.** Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be formal variables. Let
\[
F \in \mathbb{C}[z][x]\langle z \partial x_1, z \partial x_2, \ldots \rangle[Q, y]
\]
be a differential operator. Then $\exp(F/z)$ preserves $\mathbb{C}[Q, x, y]$-valued points on $\mathcal{L}_X$.

**Definition 2.1.7.** A $\mathbb{C}[Q, \tau, y]$-valued point $f$ on $\mathcal{L}_X$ is called a miniversal slice if
\[
f|_{Q=y=0} = z + \tau + O(z^{-1}).
\]
For example, the $J$-function is a miniversal slice.

**Lemma 2.1.8.** Any miniversal slice on $\mathcal{L}_X$ can be obtained from $z|\mathcal{L}_X(\tau)$ by applying $\exp(F/z)$ for some differential operator $F$ as in the previous lemma satisfying $F|_{Q=y=0} = 0$. 
The rest of the proof (ignoring convergence issues) First, we introduce

\[ \Delta^\lambda_W := e^{\lambda(W)(\log \lambda - \lambda)/z} \Delta_{(W,e^{-1}_{\lambda})}. \]

Because \( \log \Delta^\lambda_W \) and \( \log \Gamma(x) \) have similar asymptotic expansions, we have

\[ \Delta^{\lambda + cz}_W / \Delta^\lambda_W = \prod_{c=1}^{k} \delta^{(\lambda + \delta + cz)}. \]

Using the exact sequence

\[ 0 \rightarrow V \rightarrow \mathcal{O}^\oplus N \rightarrow Q \rightarrow 0, \]

we see that

\[ \Delta^\lambda_V \Delta^\lambda_Q = \Delta^\lambda_{\mathcal{O}^\oplus N}, \]

which preserves the Lagrangian cone \( \mathcal{L}_B \). We see that

\[ \Delta^\lambda_Q : \mathcal{L}_{B,(V,e^{-1}_{\lambda})} \rightarrow \mathcal{L}_B. \]

Applying Quantum-Riemann-Roch in case (b), we see that

\[ zJ^\lambda_V(z) \in \mathcal{L}_{B,(V,e^{-1}_{\lambda})}, \]

and thus

\[ \Delta^\lambda_Q zJ^\lambda_V(z) \in \mathcal{L}_B. \]

By Lemma 2.1.8, there exists \( F \) such that

\[ \Delta^\lambda_Q zJ^\lambda_V(z) = e^{F(\lambda)/z} zJ_B(\tau). \]

By Lemma 2.1.6, we obtain

\[ e^{F(\lambda + z\delta_1)/z} zJ_B(\tau) \in \mathcal{L}_{B \times \mathbb{P}^{N-1}}. \]

Now we compute

\[ I^\lambda(\tau, t) := (\Delta^\lambda_Q(\tau))^{-1} e^{F(\lambda + z\delta_1)/z} J_{B \times \mathbb{P}^{N-1}}(\tau) \]

\[ = \sum_{k=0}^{\infty} \frac{e^{pt/z} q^k e^{kt}}{\prod_{c=1}^{k} (p + cz)^N} (\Delta^\lambda_Q(\tau))^{-1} e^{F(\lambda + p + kcz)} J_B(\tau) \]

\[ = \sum_{k=0}^{\infty} \frac{e^{pt/z} q^k e^{kt}}{\prod_{c=1}^{k} (p + cz)^N} (\Delta^\lambda_Q(\tau))^{-1} (\Delta^\lambda_Q + p)^{-1} \Delta^\lambda + p + kcz \]

\[ = \sum_{k \geq 0} e^{pt/z} q^k e^{kt} \prod_{c=1}^{k} \prod_{\epsilon=1}^{\infty} (\lambda + p + \epsilon + cz)^{\lambda + p + kcz} \prod_{c=1}^{k} (p + cz)^N, \]

where \( \epsilon \) runs over the Chern roots of \( Q \). Taking the non-equivariant limit \( \lambda \rightarrow 0 \), we obtain the \( I(\tau, t) \) in the statement of Theorem 2.1.3.