Complex Analysis
Math 621

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Disclaimer

These notes are a transcription of handwritten notes that were taken during lecture. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style (omit lengthy computations, use category theory) and that of the instructor. If you find any errors, please contact me at `plei@umass.edu`.

Contents

1

Basic Notions

A *complex number* is a sum $z = x + iy$, where $x, y \in \mathbb{R}$ and i is a symbol satisfying the identity $i^2 = -1$. Addition and multiplication work as one would expect. The set $\mathbb{C}$ of complex numbers is a *field*. This means that $(\mathbb{C}, +)$ is an abelian group, $(\mathbb{C} \setminus \{0\}, \cdot)$ is an abelian group, and that multiplication distributes over addition.

Remark 1.0.1. If F is a field and $f \in F[x]$ is irreducible, then we can construct a field extension K/F such that K has a root of f by setting $K = F[x]/(f)$. In this way, we have $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$. The Galois group $\operatorname{Gal}\mathbb{C}/\mathbb{R}$ is generated by complex conjugation.

1.1 Holomorphic Functions

Let $\Omega \subset \mathbb{C}$ be an open set. Here, the topology is the Euclidean topology on $\mathbb{C} = \mathbb{R}^2$. Then $f : \Omega \to \mathbb{C}$ is *holomorphic* at a point $z_0 \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we write $f'(z_0)$ for the derivative at z_0.

Example 1.1.1. The function $f(z) = \bar{z}$ is **not** holomorphic. To see this, the difference quotient has different limits on the real and imaginary axes.

Example 1.1.2. The function $f(z) = z^n$ is holomorphic for $n \in \mathbb{N}$, and $f'(z) = nz^{n-1}$.

Remark 1.1.3. The usual formulas for differentiation (chain rule, product rule, linearity) hold in this case.

We will now compare holomorphic and real differentiability. Rewrite $F = u + iv$. Recall that F is *real differentiable* at $\mathbf{a} = (x_0, y_0)$ if

$$\lim_{\mathbf{h} \to 0} \frac{\|F(\mathbf{a} + \mathbf{h}) - F(\mathbf{a}) - A\mathbf{h}\|}{\|\mathbf{h}\|} = 0$$

for some linear map A. We say that A is the derivative of F at $\mathbf{a}$. Moreover, A is given by the *Jacobian matrix*

$$J_F = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Note that F is differentiable if and only if the partials exist and are continuous. Then recalling that holomorphic means that

$$\lim_{h\to 0}\frac{f(z_0+h)=f(z_0)-f'(z_0)h}{h}=0,$$

we see that f is holomorphic if and only if f is differentiable and $J_{Z_0}F$ corresponds to multiplication by a complex number.

Remark 1.1.4. The standard definition of holomorphic is that the differentiability condition is satisfied in an open neighborhood around z_0.

Now recall that if $\lambda = a+ib \in \mathbb{C}$, multiplication by λ is given by the real matrix $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Therefore, we must have $u_x = v_y$ and $u_y = -v_x$. These are the *Cauchy-Riemann equations*.

Theorem 1.1.5 (Cauchy-Riemann)**.** *The function* $f\colon U \to \mathbb{C}$ *is holomorphic if and only if it is real differentiable and its Jacobian satisfies the Cauchy-Riemann equations.*

We can also write $\lambda = r(\cos\theta + i\sin\theta) = re^{i\theta} \in \mathbb{C}$ in polar form. We know that multiplication by λ is rotation by θ followed by dilation by r, so in particular it preserves angles. Suppose $f\colon U\to\mathbb{C}$ is holomorphic at z_0 and $\gamma_1,\gamma_2\colon(-1,1)\to U$ are parameterized curve with $\gamma_i(0)=z_0$, then we define the angle between γ_1,γ_2 to be the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$. We also obtain the curves $f\gamma_1, f\gamma_2$. If $f'(z_0\neq 0)$, then the angle between $f\gamma_1, f\gamma_2$ equals the angle between γ_1,γ_2. To see this, the chain rule gives us

$$(f\gamma_i)'(0) = f'(\gamma_i(0))\gamma_i'(0) = f'(z_0)\gamma_i'(0).$$

Remark 1.1.6. Note that the condition that $f'(z_0)\neq 0$ is necessary. In fact, f is conformal at z_0 if and only if $f'(z_0)\neq 0$. For example, consider $f(z)=z^2$, which satisfies $f'(0)=0$.

1.2 Power Series

Theorem 1.2.1. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *be a power series with* $a_n\in\mathbb{C}$*. Then there exists* $R\in R_{\geqslant 0}\cup\infty$ *such that*

1. *The series converges absolutely for* $|z|<R$*;*
2. *The series diverges for* $|z|>r$.

Moreover, we have $\frac{1}{R}=\limsup_{n\to\infty}|a_n|^{1/n}$.

Recall that if b_n is a sequence of real numbers, we define

$$\limsup b_n = \lim_{m\to\infty}\sup_{n\geqslant m} b_n.$$

This exists if b_n is bounded below because the sequence of supremums is decreasing.

Proof. Define R by the formula. Given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $|a_n|^{1/n}\leqslant\frac{1}{R}+\varepsilon$ for $n>N$. Then

$$|a_n z^n| = (|a_n|^{1/n}|z|)^n < \left(\left(\frac{1}{R}+\varepsilon\right)|z|\right)^n,$$

so if $|z|<\frac{1}{\frac{1}{R}+\varepsilon}$, then the series converges absolutely by the comparison test. Then ε is arbitrary, so the series converges absolutely for $|z|<R$.

If $|z| > R$, then fix ρ such that $|z| > \rho > R$. For all $N \in |N|$, there exists $n > N$ such that $|a_n|^{1/n} > \frac{1}{\rho}$. Now

$$|a_n z^n| > \left(\frac{|z|}{\rho}\right)^n \to \infty$$

as $n \to \infty$, because $\frac{|z|}{\rho} > 1$. $\square$

Theorem 1.2.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]$ *have radius of convergence* R *and* $\Omega = \{z \mid |z| < R\}$. *Then* $f\colon \Omega \to C$ *is holomorphic and* $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ *has the same radius of convergence.*

Here is the most important example of a power series.

Example 1.2.3. Define $e^z - \sum_{n=0}^{\infty} \frac{z^n}{n!}$. This has radius of convergence $R = \infty$ because

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} \geqslant \limsup_{n\to\infty} \left(\frac{n}{2}\right)^{1/2} = \infty.$$

Alternatively, we may use the ratio test instead of the root test.

It is easy to see that $\frac{d}{dz} e^z = e^z$ from the power series. Then we have

$$\begin{aligned} e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} \\ &= \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{\ell=0}^{\infty} \frac{w^\ell}{\ell!}\right) \\ &= e^z e^w \end{aligned}$$

because if $\sum a_n, \sum b_n$ are absolutely convergent and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $\sum c_n$ is absolutely convergent and $\sum c_n = \sum a_n \cdot \sum b_n$ (see baby Rudin for a reference).

Now we may define

$$\begin{aligned} \cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{e^{iz} - e^{-iz}}{2i}. \end{aligned}$$

It is easy to see from the power series that $e^{\pm iz} = \cos z \pm i \sin z$. The formulas for the derivatives from basic calculus hold.

Example 1.2.4. Consider the series $\sum_{n=0}^{\infty} z^n$. This has radius of convergence $R = 1$ and diverges for $|z| \geqslant 1$. In fact, $f(z) = \frac{1}{1-z}$ for $|z| < 1$. In this case, the power series $f(z)$ extends to a holomorphic $g(z) = \frac{1}{1-z}$ defined on $\mathbb{C} \setminus \{1\}$. More generally, if $g\colon \Omega \to C$ is holomorphic and $z_0 \in \Omega$ is a point, then it has a power series expansion

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)(z-z_0)^n}{n!}$$

valid in the largest open disc contained in Ω. This means that g is an *analytic continuation* of f.

Note that convergence for $|z| = R$ is very delicate.

1. The series $\sum z^n$ has $R = 1$ and diverges when $|z| = 1$;
2. The series $\sum \frac{1}{n^2} z^n$ has $R = 1$ and converges absolutely for $|z| = 1$;
3. The series $\sum \frac{1}{n} z^n$ has $R = 1$ and diverges for $z = 1$ but converges for $|z| = 1, z \neq 1$.

1.3 Integration along curves

A *paramaterized curve* is a continuous function $z\colon [a, b] \to \mathbb{C}$. We will call it *smooth* if z is continuously differentiable. We also assume that $z'(t) \neq 0$ for all $t \in [a, b]$.

Example 1.3.1. The circle centered at z_0 with radius r is given by

$$z\colon [0, 2\pi] \to \mathbb{C} \qquad z(t) = z_0 + re^{it} = z_0 + r(\cos t + i \sin t).$$

Two parameterized curves $z_1\colon [a, b] \to \mathbb{C}, z_2\colon [c, d] \to \mathbb{C}$ are *equivalent* if there exists a C^1 homeomorphism $t\colon [c, d] \to [a, b]$ with $t'(s) = 0$ for all $s \in [c, d]$ and $z_2 = z_1 \circ t$. Therefore, we can define a *smooth curve* to be an equivalence class of parameterized curves. It is *closed* if $z(a) = z(b)$ and simple if $z(s) \neq z(t)$ for all $s \neq t$ unless $s, t = a, b$. Given a curve γ with parameterization $z\colon [a, b] \to \mathbb{C}$, we write $-\gamma$ for the curve

$$\widetilde{z}\colon [a, b] \to \mathbb{C} \qquad t \mapsto z(a + b - t).$$

We may finally define integration along a curve. If f is continuous and γ is a smooth curve, we define

$$\int_\gamma f(z)\, dz \coloneqq \int_a^b f(z(t)) z'(t)\, dt$$

for some parameterization $z\colon [a, b] \to \mathbb{C}$ of γ. We need to check that this is well-defined, and we have

$$\begin{aligned} \int_a^b f(z_1(t)) z_1'(t) &= \int_c^d f(z_1(t(s))) z_1'(t(s)) t'(s)\, ds \\ &= \int_c^d f(z_2(s)) z_2'(s). \end{aligned}$$

Example 1.3.2. Let γ be a circle of radius r centered at the origin with parameterization $z\colon [0, 2\pi] \to \mathbb{C}$ given by $z(t) = re^{it}$. Then we have

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{re^{it}} (re^{it})' = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\, dt = 2\pi i.$$

It is easy to see that $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$. We may also define the length

$$\text{length}(\gamma) \coloneqq \int_a^b \left|z'(t)\right| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt\,.$$

This has an easy bound given by

$$\left|\int_a^b f(z(t))z'(t)\,dt\right| \leqslant \int_a^b |f(z(t))|\left|z'(t)\right| dt \leqslant \sup_{z\in\gamma}|f(z)| \int_a^b \left|z'(t)\right| dt = \sup_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma).$$

Theorem 1.3.3 (Fundamental Theorem of Calculus)**.** *Let* $f\colon \Omega \to \mathbb{C}$ *be continuous ans assume there exists a holomorphic* $F\colon \Omega \to \mathbb{C}$ *such that* $F' = f$ *(in other words, a primitive for* f*). Then*

$$\int_\gamma f(z)\,dz = F(z(b)) - F(z(a)).$$

In particular, if γ *is closed, then* $\int_\gamma f(z)\,dz = 0$*.*

Proof. By definition, we have

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\
&= \int_a^b F'(z(t))z'(t)\,dt \\
&= \int_a^b (F\circ z)'(t)\,dt \\
&= F(z(b)) - F(z(a))
\end{aligned}$$

by the fundamental theorem of calculus over $\mathbb{R}$. □

2

Local Theory

2.1 Integrals of Holomorphic Functions

Theorem 2.1.1 (Cauchy)**.** *Let* $f\colon \Omega \to \mathbb{C}$ *be holomorphic and* γ *be a simple closed curve in* Ω *such that the interior of* γ *is contained in* Ω*. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Warning 2.1.2. It is very tricky to precisely describe what is meant by "interior" and we need the Jordan curve theorem from algebraic topology to do so.

Sketch of Proof. We write

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma (u + iv)(dx + i\,dy) \\
&= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy \\
&= \int_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\int_\gamma \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \\
&= 0
\end{aligned}$$

by Green's theorem and the Cauchy-Riemann equations. □

Of course, this assumes that the partial derivatives of u, v are C^∞, which we do not know yet. Instead, we will give a more careful proof of a weaker theorem.

Theorem 2.1.3 (Goursat)**.** *Let* $f\colon \Omega \to \mathbb{C}$ *be holomorphic and* $T \subset \Omega$ *be a triangle. Then*

$$\int_{\partial T} f(z)\,dz = 0.$$

Proof. Bisect each side of T and create four smaller triangles T_i^1. Then we have

$$\int_{\partial T} f(z)\,dz = \sum_{i=1}^{4} \int_{\partial T_i^1} f(z)\,dz$$

and thus

$$\left|\int_{\partial T} f(z)\,dz\right| \leqslant 4\left|\int_{\partial T_i^1} f(z)\,dz\right|$$

for some i. Now write $T^1 = T_i^1$. We may repeat this process to obtain

$$T \supset T^1 \supset T^2 \supset \cdots \supset T^n \supset \cdots,$$

and thus we have

$$\left|\int_{\partial T} f(z)\,dz\right| \leqslant 4^n\left|\int_{\partial T^n} f(z)\,dz\right|$$

and $\operatorname{length}(\partial T^n) = 2^{-n}\operatorname{length}(\partial T)$. Now set $z_0 = \bigcap_{n\geqslant 1} T^n$. We now use the holomorphicity of f to estimate the integral, and we have $f(z_0+h) = f(z_0) + hf'(z_0) + h\psi(h)$, where $\lim_{h\to 0}\psi(h) = 0$. Therefore

$$\begin{aligned}
\left|\int_{\partial T^n} f(z)\,dz\right| &= \left|\int_{\partial T^n} f(z_0) + (z-z_0)f'(z_0) + (z-z_0)\psi(z-z_0)\,dz\right| \\
&= \left|\int_{\partial T^n} (z-z_0)\psi(z-z_0)\,dz\right| \\
&\leqslant \operatorname{length}(\partial T^n)\cdot \sup_{z\in\partial T^n} |z-z_0|\psi(z-z_0) \\
&\leqslant \operatorname{length}(\partial T^n)^2 \sup_{z\in\partial T^n} |\psi(z-z_0)|.
\end{aligned}$$

Therefore, we have

$$\left|\int_{\partial T} f(z)\,dz\right| \leqslant 4^n(2^{-n}\operatorname{length}(\partial T^n)) \sup_{z\in\partial T^n} |\psi(z-z_0)| \to 0$$

as $n\to\infty$. □

Corollary 2.1.4. *The same result holds for a rectangle.*

Now we want to discuss the existence of local primitives.

Proposition 2.1.5. *Let* $f\colon D\to\mathbb{C}$ *be holomorphic on an open disc* $D = D_r(z_0)$*. Then there exists* $F\colon D\to\mathbb{C}$ *holomorphic such that* $F' = f$*. In particular,* $\int_\gamma f(z)\,dz = 0$ *for any closed curve* $\gamma\subset D$*.*

Proof. Define F as an integral

$$F(z) = \int_{\gamma_z} f(w)\,dw.$$

Here, we define γ_z as

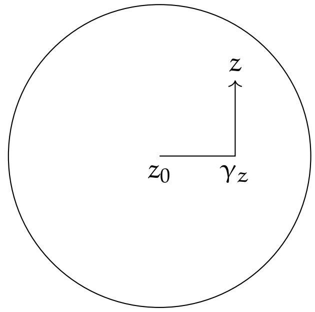

Figure 2.1: Path of integration

Now we need to prove that $F'(z) = f(z)$. Note that

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw\,.$$

Then note that the path of integration is given by

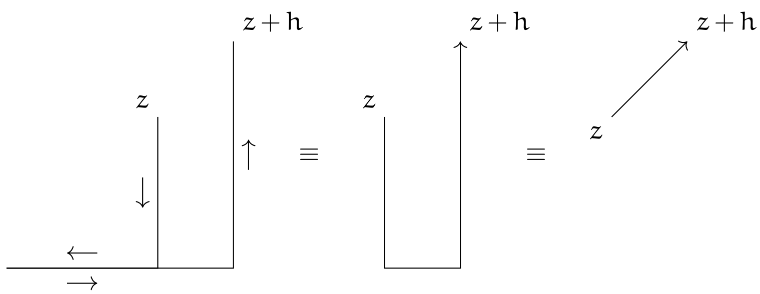

Figure 2.2: Equivalent paths

Call the final path $[z, z+h]$. Therefore,

$$\begin{aligned} F(z+h) - F(z) &= \int_{[z,z+h]} f(w)\,dw \\ &= \int_{[z,z+h]} (f(z) + \psi(w-z))\,dw \\ &= f(z)\cdot h \int_{[z,z+h]} \psi(w-z)\,dw\,, \end{aligned}$$

so

$$F'(z) = f(z) + \lim_{h\to 0} \frac{1}{h} \int_{[z,z+h]} \psi(w-z)\,dw\,,$$

but then

$$\left| \frac{1}{h} \int_{[z,z+h]} \psi(w-z)\,dw \right| \leqslant \left|\frac{1}{h}\right| |h| \sup_{w\in[z,z+h]} |\psi(w-z)| \to 0$$

as $h \to 0$. $\square$

Now we will relate the function f itself to some integral of points neearby f.

Theorem 2.1.6 (Cauchy integral formula). *Let* $f\colon \Omega \to \mathbb{C}$ *be holomorphic and* D *be an open disc such that* $\overline{D} \subset \Omega$. *Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw$$

for all $z \in D$.

Proof. First, if Ω' is a disk with a segment of a radius removed, then any holomorphic f has a primitive on Ω' by the same argument as before. Now define the path $\gamma_{\varepsilon,\delta}$ by

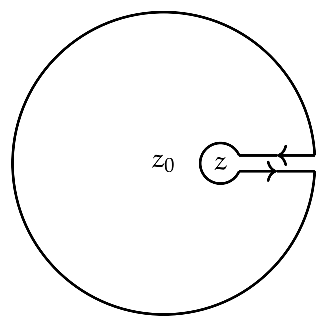

Figure 2.3: Contour $\gamma_{\varepsilon,\delta}$

where the radius of the circle around z is δ and the width of the corridor is 2ε. Now $\frac{f(z)}{z-2}$ is holomorphic on the interior of $\gamma_{\varepsilon,\delta}$, so there eixists a primitive. Therefore, $\int_{\gamma_{\varepsilon,\delta}} \frac{f(w)}{w-z}\, dz = 0$. Now we let $\varepsilon \to 0$. Because $\frac{f(w)}{w-z}$ is continuous on the rectangular corridor, we have

$$\int_{\gamma_{\varepsilon,\delta}} \frac{f(w)}{w-z}\, dw \to \int_{\gamma_\delta} \frac{f(w)}{w-z}\, dw$$

as $\varepsilon \to 0$. This is because if f is continuous on some rectangle, then

$$\int_a^b f(s,t)\, dt \to \int_a^b f(s_0,t)\, dt$$

as $s \to s_0$ by uniform continuity on compact sets. Now we see that

$$0 = \int_{\gamma_\delta} \frac{f(w)}{z-w}\, dw = \int_{\partial D} \frac{f(w)}{z-w}\, dw - \int_{\alpha_\delta} \frac{f(w)}{z-w}\, dw\,,$$

where α_δ is the circle of radius δ centered at z. Then we can rewrite

$$\frac{f(w)}{w-z} = \frac{f(z)}{w-z} + \frac{f(w)-f(z)}{w-z},$$

and the last term approaches $f'(z)$ as $w \to z$. Therefore it is bounded, so

$$\int_{\alpha_\delta} \frac{f(w)-f(z)}{w-z}\, dw \to 0$$

as $\delta \to 0$ because $\text{length}(\alpha_\delta) \to 0$ as $\delta \to 0$. Therefore, we have

$$\begin{aligned}
\int_{\delta D} \frac{f(w)}{w-z}\, dw &= \lim_{\delta \to 0} \int_{\alpha_\delta} \frac{f(w)}{w-z}\, dw \\
&= \lim_{\delta \to 0} \int_{\alpha_\delta} \frac{f(z)}{z-w}\, dw \\
&= f(z) \lim_{\delta \to 0} \int_{\alpha_\delta} \frac{1}{z-w}\, dw \\
&= 2\pi i f(z).
\end{aligned}$$

□

Corollary 2.1.7 (Higher derivatives)**.** *Let* $f\colon \Omega \to \mathbb{C}$ *be holomorphic and* $\overline{D} \subset \Omega$ *be a disk. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\, dw\,.$$

This formula is obtained by differentiating under the integral sign with respect to z, and this is Ok because in general

$$\frac{d}{dt} \int_a^b \varphi(x,t)\, dx = \int_a^b \frac{\partial}{\partial x} \varphi(x,t)\, dx$$

as long as φ, $\frac{\partial \varphi}{\partial x}$ are continuous. Again, a reference for this is baby Rudin. Alternatively, we can use the following theorem.

Theorem 2.1.8. *Let* $f\colon \Omega \to \mathbb{C}$ *be holomorphic and* $\overline{D} \subset \Omega$ *be a disk. Then*

$$f(z) - \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

for all $z \in D$.

Proof. Fix $z \in D$. By the Cauchy integral formula, we have $f(z) = \int_{\partial D} \frac{f(z)}{w-z}\, dw$. Expanding out

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\
&= \frac{1}{1-\frac{z-z_0}{w-z_0}} \frac{1}{w-z_0} \\
&= \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(w-z_0)^{n+1}} (z-z_0)^n,
\end{aligned}$$

we obtain

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\partial D} f(w) \sum_{n=0}^{\infty} \frac{1}{(w-z)^{n+1}} (z-z_0)^n\, dw \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\, dw \right) (z-z_0)^n.
\end{aligned}$$

Note that $f(w)$ is bounded on ∂D and the sum $\sum_{n=0}^{\infty} \frac{(w-z)^{n+1}}{(z-z_0)}^{n}$ converges uniformly on ∂D, so we can interchange the sum and the integral. □

Corollary 2.1.9 (Cauchy inequality). *Let z_0 be the center of our disk with radius R. Then we see that*

$$\left|f^{(n)}(z_0)\right| \leqslant \frac{n!}{2\pi} \cdot 2\pi R \sup_{w \in \partial D} |f(w)| \frac{1}{R^{n+1}} \leqslant \frac{n!}{R^n} \sup_{w \mid |w - z_0| < R} |f(w)|.$$

Theorem 2.1.10 (Liouville). *Let $f \colon \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant.*

Proof. Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and this is valid for all $z \in \mathbb{C}$. By the Cauchy inequality, we have

$$|a_n| = \frac{\left|f^{(n)}(0)\right|}{n!} \leqslant \frac{1}{R^n} \sup\{f(w) \mid |w| = R\} \leqslant \frac{M}{R^n} \to 0$$

as $R \to \infty$. Therefore, $a_n = 0$ for $n \neq 0$. □

Theorem 2.1.11 (Fundamental Theorem of Algebra). *Let $f \in \mathbb{C}[z]$ be a nonconstant polynomial. Then there exists $\alpha \in \mathbb{C}$ such that $f(\alpha) = 0$.*

Proof. Suppose no root exists, so we may consider the entire function $g(z) = \frac{1}{f(z)}$. Then there exists $R > 0$ such that $|f(z)| > \frac{1}{2}|a_n z^n|$ for $|z| \geqslant R$. Then

$$|g(z)| = \frac{1}{f(z)} < \frac{2}{|a_n|} \cdot \frac{1}{|z|^n} \leqslant \frac{2}{|a_n|} R^n.$$

On the other hand, g is bounded for $|z| \leqslant R$, so by Liouville's theorem g is constant. Therefore f is constant. □

In particular, if f has degree n, then it has n roots counting multiplicity.

2.2 Analytic Continuation

Theorem 2.2.1. *Let $f \colon \Omega \to \mathbb{C}$ be holomorphic with Ω open and connected. Suppose there exists a sequence z_n of distinct points of Ω such that $z_n \to \alpha \in \Omega$ as $n \to \infty$ and $f(z_n) = 0$. Then $f = 0$.*

Proof. Expand $f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n = \sum_{n=m}^{\infty} a_n (z - \alpha)^n$ for some m. In particular, we have $f(z) = (z - \alpha)^m g(z)$, where $g(z)$ is holomorphic on $D \subset \Omega$ centered at α. Thus $f(z) \neq 0$ for z close to α. But then $g(z) \neq 0$ for $|z - \alpha| < \varepsilon$ and $g(\alpha) = a_m \neq 0$. But this gives a contradiction, so we must have $a_n = 0$ for all n. Now we show that $f = 0$ on all of Ω. If Ω_1 is the interior of the set $\{z \in \Omega \mid f(z) = 0\}$, then we know Ω_1 is closed by the argument above, so by connectedness of Ω, we see that $\Omega = \Omega_1$. □

Given $f \colon \Omega \to \mathbb{C}$ holomorphic and $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}$, an *analytic continuation* of f to $\widetilde{\Omega}$ is a holomorphic function $\widetilde{f} \colon \widetilde{\Omega} \to \mathbb{C}$ such that $\widetilde{f}\big|_{\Omega} = f$. By the theorem, the analytic continuation is unique.

Example 2.2.2. The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$ from $\{s \mid \mathrm{Re}(s) > 1\}$.

Theorem 2.2.3. *Let $f \colon \Omega \to \mathbb{C}$ be holomorphic with Ω connected. Assume that $fneq0$. Given $z_0 \in \Omega$, then there exists an open neighborhood U of z_0, $g \colon U \to \mathbb{C}$ holomorphic, and $n \in \mathbb{Z}_{\geqslant 0}$ such that $f(z) = (z - z_0)^n g(z)$, where $g(z) \neq 0$ for all $z \in U$.*

Proof. Expand $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_n (z - z_0)^n + O(z^{n+1}) = (z - z_0)^n (a_n + O(z))$. □

We say that f has *a zero of order* n at $z = z_0$.

2.3 Poles

Let $f\colon \Omega \to \mathbb{C}$ be holomorphic and $D^\times = \{z \mid 0 < |z - z_0| < r\} \subset \Omega$. Then we say that f has an *isolated singularity* at $z = z_0$. We say that f has a *pole* at z_0 if

$$g(z) = \begin{cases} \frac{1}{f(z)} & z \neq n_0 \\ 0 & z = z_0 \end{cases}$$

is holomorphic in a neighborhood of z_0. In particular, if $z = z_0$ is a pole, then $\frac{1}{f(z)} \to 0$ as $z \to z_0$.

Better, write $g(z) = (z - z_0)^n h(z)$ for some holomorphic h with $h(z) \neq 0$ for z near z_0. Then we can write $f(z) = \frac{1}{g(z)} = (z - z_0)^{-n} k(z)$ for some holomorphic k. Then we say that f has a *pole of order* n.

Example 2.3.1. Consider $f(z) = \frac{g(z)}{h(z)} = \frac{(z-z_0)^n k(z)}{(z-z_0)^m \ell(z)} = (z - z_0)^{n+m} \frac{k(z)}{\ell(z)}$. Then if $m > n$ we have a pole of order $m - n$ and if $m \leqslant n$ we have a removable singularity and can extend to a holomorphic function with a zero of order $n - m$.

Now we say that f is *meromorphic* on Ω if there exists a discrete set $S \subset \Omega$ such that $f\colon \Omega \setminus S \to \mathbb{C}$ is holomorphic and f has a pole at each $s \in S$. If f has a pole of order n at z_0, then $f(z) = (z - z_0)^{-n} g(z)$ near z_0. Expanding g as a power series, we have

$$f(z) = \underbrace{a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1}}_{\text{principal part of } f \text{ at } z = z_0} + h(z),$$

where $h(z)$ is holomorphic. We call $a_{-1} =: \operatorname{Res}_{z_0} f$ the *residue of* f *at* z_0.

Theorem 2.3.2. *Let $f\colon \Omega \to \mathbb{C}$ be holomorphic with a pole at z_0. Let C be a small circle centered at z_0. Then*

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z_0} f.$$

Proof. We have

$$\int_C f(z)\,dz = \int_C a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1} + h(z)\,dz\,.$$

By Cauchy, the integral of $h(z)$ vanishes, and each $a_{-k}(z - z_0)^{-k}$ has a primitive for $k \neq 1$, so by the computation of the integral of $\frac{1}{z}$ from before, we obtain the desired result. □

Theorem 2.3.3 (Residue Theorem)**.** *Let $f\colon \Omega \to \mathbb{C}$ be holomorphic and assume γ is a simple closed curve in Ω and the interior of γ is contained in $\Omega \cup \{z_1, \ldots, z_N\}$, where the z_j are the poles of f. then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{z_j} f.$$

Proof. Recall the contour in Figure 2.3, except now with more than one keyhole. Then by Cauchy's theorem, we know

$$\int_{\gamma_{\varepsilon,r}} f(z)\,dz = 0,$$

so letting $\varepsilon \to 0$, we have

$$\int_{\gamma_r} f(z)\,dz = 0.$$

Now if c_j is a small circle around z_j, we have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{j=1}^N \int_{C_j} f(z)\,dz\\ &= 2\pi i \sum_{j=1}^N \mathrm{Res}_{Z_j} f(z).\end{aligned}$$ □

Remark 2.3.4. We gave a sketch of a proof of Cauchy relying on Green's theorem. This required the partial derivatives to be continuous, but this is now fine because we proved that f is infinitely differentiable. However, we still have the issue of the interior of a simple closed curve.

Example 2.3.5. Consider the integral

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}.$$

If we set $z = e^{i\theta}$, then $dz = ie^{i\theta}$, so we obtain

$$\int_\gamma \frac{1}{a + \frac{z+z^{-1}}{2}} \frac{dz}{iz} = \frac{1}{i}\int_\gamma \frac{2}{z^2+2az+1}\,dz = 2\pi \sum_i \mathrm{Res}_{z_i}\left(\frac{2}{z^2+2az+1}\right).$$

Now we would actually like to be able to compute residues. This is important if we want to actually be able to compute integrals.

Example 2.3.6. Consider $f(z) = \frac{\sin z}{z^6}$. Then there is a pole at $z = 0$, and the residue is $\frac{1}{5!} = \frac{1}{120}$ by inspeection of the power series.

Example 2.3.7. Consider $f(z) = \tan z = \frac{\sin z}{\cos z}$. We would like to compute the residue at $z_0 = \frac{\pi}{2}$. If $f(z)$ has a simple pole (order 1) at $z = z_0$, then

$$\mathrm{Res}_{z_0} f = \lim_{z\to z_0} (z - z_0) f(z).$$

In particular, if $f = \frac{g}{h}$ and h has a simple zero at z_0, then f has a simple pole at z_0. This gives us

$$\begin{aligned}\mathrm{Res}_{z_0} f &= \lim_{z\to z_0}(z-z_0)f(z)\\ &= \lim_{z\to z_0} \frac{(z-z_0)g(z)}{h(z)}\\ &= \lim_{z\to z_0} \frac{g(z)}{\left(\frac{h(z)-h(z_0)}{z-z_0}\right)}\\ &= \frac{g(z_0)}{h'(z_0)},\end{aligned}$$

so $\mathrm{Res}_{\pi/2} \tan z = \frac{\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1$.

Similarly, if f has a pole of order k at z_0, then

$$\operatorname{Res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}(z-z_0)^k f(z).$$

Example 2.3.8. Consider $f(z) = \frac{1}{e^z-(1+z)}$, which has a pole of order 2 at $z = 0$. In particular, we have

$$f(z) = \frac{1}{\frac{z^2}{2}+\frac{z^3}{6}+\cdots},$$

so

$$\begin{aligned}\operatorname{Res}_0 f &= \lim_{z\to 0}\frac{d}{dz}z^2 f(z)\\ &= \lim_{z\to 0}\frac{d}{dz}\frac{z^2}{\frac{z^2}{2}+\frac{z^3}{6}+\cdots}\\ &= \lim_{z\to 0}\frac{-\left(\frac{1}{6}+\frac{2z}{24}+\cdots\right)}{\left(\frac{1}{2}+\frac{z}{6}+\cdots\right)^2}\\ &= -\frac{2}{3}.\end{aligned}$$

Alternatively, we could formally invert the power series $\frac{1}{2}+\frac{z}{6}+\cdots$. Returning to our integral over a real variable θ, we see the poles of $z^2+2az+1=0$ are at $z = -a \pm \sqrt{a^2-1}$, so our integral is

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} &= 2\pi\sum_i \operatorname{Res}_{z_i}\left(\frac{2}{z^2+2az+1}\right)\\ &= 2\pi \operatorname{Res}_{-a+\sqrt{a^2-1}}\left(\frac{2}{z^2+2az+1}\right)\\ &= \frac{\pi}{\sqrt{a^2-1}}.\end{aligned}$$

Of course, we may consider a general form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$.

Consider the integral $\int_{-\infty}^{\infty} F(x)\, dx$, where $F \in \mathbb{R}(x)$ has the form $F = P/Q$ with $\deg P + 2 \leqslant \deg Q$ and no poles in $\mathbb{R}$. Now if we consider the contour

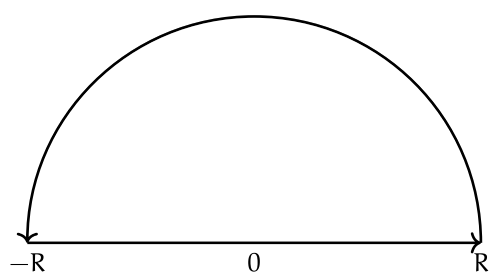

Figure 2.4: The contour γ_R

then we see that

$$\int_{\gamma_R} F(z)\,dz = \int_{-R}^{R} F(x)\,dx + \int_{\text{arc}} F(z)\,dz\,,$$

and the second term approaches 0 as $R \to \infty$, so

$$\int_{-\infty}^{\infty} F(x)\,dx = \lim_{R\to\infty} \int_{\gamma_R} F(z)\,dz = 2\pi i \sum_{\mathcal{H}} \operatorname{Res}_z f.$$

To see that the integral of the arc disappears, observe that

$$\left| \int_{\text{arc}} F(z)\,dz \right| \leqslant \pi R \sup_{z\in\text{arc}} |F(z)|.$$

But then if $\deg P = n, \deg Q = m$, $z^{m-n}F(z) \to \frac{a_n}{b_m}$ as $|z| \to \infty$, so we can bound this quantity by some constant C, and thus

$$\left| \int_{\text{arc}} F(z)\,dz \right| \leqslant \pi RC \cdot R^{n-m} = \pi C R^{n+1-m} \to 0$$

as $R \to \infty$ because $n+1-m < 0$.

Example 2.3.9. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 4}\,dx\,.$$

Then we have simple poles at $z = \pm i, \pm 3i$, so we have

$$\begin{aligned} I &= 2\pi i \operatorname{Res}_i F + \operatorname{Res}_{3i} F \\ &= 2\pi i \left(\frac{P(i)}{Q'(i)} + \frac{P(3i)}{Q'(3i)} \right) \\ &= \frac{5}{12}\pi. \end{aligned}$$

Now consider $F \in \mathbb{R}(x)$ with $\deg P + 1 \leqslant \deg Q$ and $Q(x) \neq 0$ for $x \in \mathbb{R}$. We want to compute the integral

$$\int_{-\infty}^{\infty} F(x)e^{ix}\,dx\,.$$

We will consider the contour

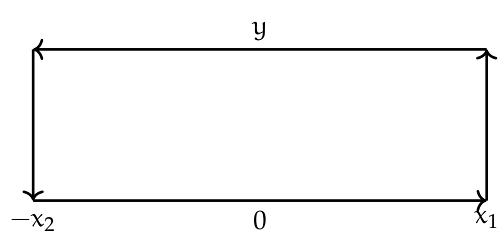

Figure 2.5: The contour of integration

Now we note that

$$\left|\int_{\text{up}}\right| F(z)e^{iz}\,dz \leqslant y \sup \left|F(z)e^{iz}\right| \leqslant y \sup |F(z)|.$$

Then we know $|F(z)| \leqslant C|z|^{\deg P - \deg Q}$, so

$$\left|\int_{\text{up}} F(z)e^{iz}\,dz\right| \leqslant \int_0^y \frac{C}{|z|}e^{-y}\,dy \leqslant \int_0^y \frac{C}{x_1}e^{-y}\,dy \leqslant \frac{C}{x_1} \to 0$$

as $x_1 \to \infty$. By a similar argument, $\int_{\text{down}}$ vanishes in the limit. Finally, we have

$$\left|\int_{\text{left}} F(z)e^{iz}\,dz\right| \leqslant (x_1 + x_2)\frac{C}{y}e^{-y},$$

and this clearly vanishes in the limit. This gives us

$$\int_{-\infty}^{\infty} F(x)e^{ix}\,dx = 2\pi i \sum_{z_j \in \mathcal{H}} \operatorname{Res}_{z_j} F(z)e^{iz}.$$

This is because

$$\left|\int_{-x_2}^{x_1} F(x)e^{ix}\,dx - 2\pi i \sum_{z_i \in \mathcal{H}} \operatorname{Res}_{z_j} F(z)e^{iz}\right| \leqslant (x_1 + x_2)\cdot\frac{C}{y}e^{iy} + \frac{C}{x_1} + \frac{C}{x_2}.$$

As $x_1, x_2, y \to \infty$, this vanishes.

Example 2.3.10. Consider the integral $I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\,dx$ for $a \in \mathbb{R}_{>0}$. We see that $\sin x = \operatorname{Im}(e^{ix})$, so we must have

$$\begin{aligned}
I &= \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2}\,dx\right) \\
&= \operatorname{Im}\left(2\pi i \sum_{\mathcal{H}} \operatorname{Res}\left(\frac{ze^{iz}}{z^2 + a^2}\right)\right) \\
&= \operatorname{Im}\left(2\pi i \frac{iae^{i^2 a}}{2ia}\right) \\
&= \pi e^{-a}.
\end{aligned}$$

Finally, let $\alpha \in [0, 1)$ and $F = P/Q$ with $\deg P + 2 \leqslant \deg Q$ and consider the integral

$$I = \int_0^{\infty} x^{\alpha} F(x)\,dx\,.$$

Suppose that $Q(x) \neq 0$ for $x \in \mathbb{R}_{>0}$ and at worst has a simple zero at $x = 0$. Then we will consider the contour $\gamma_{r,R,\varepsilon}$, where r is the inner radius, R the outer radius, and ε the width of the corridor.

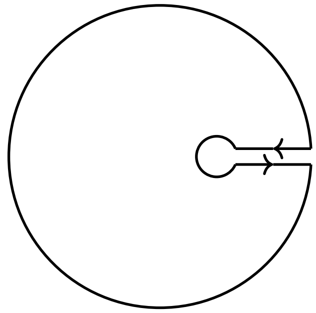

Figure 2.6: Contour $\gamma_{r,R,\varepsilon}$

Now as $\varepsilon \to 0$, we have

$$\int_{\gamma_{r,R,\varepsilon}} \to \int_{C_R} - \int_{C_r} + \int_0^\infty x^\alpha F(x)\,dx - e^{2\pi i\alpha}\int_0^\infty x^\alpha F(x)\,dx\,,$$

so we obtain the equation

$$(1-e^{2\pi i\alpha})I + \int_{C_R} - \int_{C_r} = 2\pi i \sum_p \operatorname{Res}_p z^\alpha F(x)\,dx\,.$$

But now we have

$$\left|\int_{C_R} z^\alpha F(z)\,dz\right| \leqslant 2\pi R \sup_{z\in C_R} |z^\alpha F(z)| \leqslant 2\pi R^{\alpha-1} \to 0$$

as $R \to \infty$ because $|z^\alpha F| \leqslant C \cdot R^\alpha R^{\deg P - \deg Q} \leqslant C \cdot R^{\alpha-2}$. We also have

$$\left|\int_{C_r} z^\alpha F(z)\,dz\right| \leqslant 2\pi r C r^{\alpha-1} = 2\pi C r^\alpha \to 0$$

as $r \to 0$ because $|z^\alpha F| \leqslant C r^{\alpha-1}$. Therefore, we have

$$(1-e^{2\pi i\alpha})I = 2\pi i \sum_{p\in\mathbb{C}} \operatorname{Res}_p\,.$$

Example 2.3.11. Consider the integral $I = \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$. Then we have

$$\begin{aligned}(1-e^{\pi i})I &= 2\pi i\left(\operatorname{Res}_i \frac{z^{1/2}}{1+z^2} + \operatorname{Res}_{-i} \frac{z^{1/2}}{1+z^2}\right)\\ &= 2\pi i\left(\frac{e^{\pi/4}}{2i} + \frac{e^{3\pi/4}}{-2i}\right)\\ &= \pi\sqrt{2}.\end{aligned}$$

Therefore, $I = \frac{\pi}{\sqrt{2}}$.

2.4 Singularities

We will now consider the next type of singularity. These are the removable singularities.

Theorem 2.4.1 (Riemann, removable singularity theorem)**.** *Let $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and assume f is bounded on $D \setminus \{z_0\}$ for some disk D centered at z_0. Then f extends to a holomorphic function on Ω and z_0 is a removable singularity.*

Proof. Consider the same keyhole as in Figure 2.3 except now with keyholes at z_0 and z.

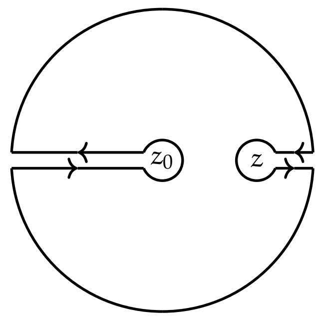

Figure 2.7: Contour γ

Then we have

$$\int_\gamma \frac{f(w)}{w-z}\,dz = 0$$

by Cauchy's theorem. Therefore, $\int_{\partial D} - \int_{C_1} - \int_{C_2} = 0$, so we see that

$$\int_{\partial D} \frac{f(w)}{w-z}\,dw = \int_{C_1} \frac{f(w)}{w-z}\,dw + \int_{C_2} \frac{f(w)}{w-z}\,dw\,,$$

where C_1 is a circle centered at z_0 and C_2 is a circle centered at z. Then as $r \to 0$, we see that $\int_{C_1} \to 0$. But now we can set $f(z) \coloneqq \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw$, so f is holomorphic at z. □

Corollary 2.4.2. *Let $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then f has a pole at z_0 if and only if $f(z) \to \infty$ as $z \to z_0$.*

Proof. We know that f has a pole if and only if

$$g = \begin{cases} \frac{1}{f} & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

is holomorphic near z_0. Therefore f has a pole if and only if $\frac{1}{f} \to 0$ as $z \to z_0$, which is equivalent to $f(z) \to \infty$. □

Now we will consider the final type of singularity: *essential singularities*. This is every singularity that is not a pole or a removable singularity. The canonical example is $f(z) = e^{1/z}\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

Theorem 2.4.3 (Casorati-Weierstrass)**.** *Let $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and z_0 be an essential singularity. Then $f(\Omega \setminus \{z_0\})$ is dense in $\mathbb{C}$. Equivalently, for all $\alpha \in \mathbb{C}$, there exists $z_n \in \Omega \setminus \{z_0\}$ such that $z_n \to z_0$ and $f(z_n) \to \alpha$.*

Proof. Assume there exists $\alpha \in \mathbb{C}$ such that $|f(z)-\alpha| > \delta$ for all $z \in \Omega$ and some $\delta > 0$. Then if we write $g(z) = \frac{1}{f(z)-\alpha}$, we see that $|g(z)| < \delta^{-1}$, so g is holomorphic on Ω. Therefore, we have $f(z) = \frac{1}{g(z)} + \alpha$. If $g(z_0) \neq 0$, then f has a removable singularity, and if $g(z_0) = 0$, then f has a pole. $\square$

Finally, we will consider Laurent series expansions.

Proposition 2.4.4. *Let* $f\colon \Omega \to \mathbb{C}$ *be holomorphic and* $A = \{z \mid r_1 < |z-z_0| < r_2\}$ *and assume* $|A| \subset \Omega$. *Then* $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ *for all* $z \in A$.

Proof. Consider the keyhole

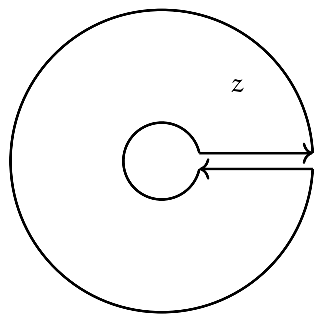

Figure 2.8: Contour γ_ε

Now note that $f(z) = \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(w)}{w-z}\,dw$, so as $\varepsilon \to 0$, we see that $\int_{\gamma_\varepsilon} \to \int_{C_{r_2}} - \int_{C_{r_1}}$. Now we obtain

$$\begin{aligned}
\int_{C_{r_2}} \frac{f(w)}{w-z}\,dw &= \int_{C_{r_2}} \frac{f(w)}{(w-z_0)-(z-z_0)}\,dw \\
&= \int_{C_{r_2}} \frac{1}{w-z_0} \frac{f(w)}{1-\frac{z-z_0}{w-w_0}} \\
&= \int_{C_{r_2}} \frac{1}{w-z_0} f(w) \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-w_0}\right)^n dw \\
&= \sum_{n=0}^{\infty} \left(\int_{C_{r_2}} \frac{1}{(w-z_0)^{n+1}} f(w)\,dw \right) \cdot (z-z_0)^n.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_{C_{r_1}} \frac{f(w)}{w-z}\,dw &= \int_{C_{r_1}} \frac{1}{z-z_0} \frac{f(w)}{\frac{w-z_0}{z-z_0}-1}\,dw \\
&= \int_{C_{r_1}} \frac{1}{z-z_0} (-f(w)) \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n dw \\
&= \sum_{n=0}^{\infty} \int_{C_{r_1}} -f(w)(w-z_0)^n (z-z_0)^{-(n+1)}.
\end{aligned}$$

Combining, we obtain the desired result. $\square$

In the special case where $r_1 = 0$, then if $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with $D \subset \Omega$, then we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on $D \setminus \{z_0\}$. Therefore the types of singularities correspond to

Removable: For all $n < 0$, $a_n = 0$.

Pole: There exists $k > 0$ such that $a_n = 0$ for all $n < -k$.

Essential: For all $N > 0$, there exists $n > N$ such that $a_{-n} \neq 0$.

Now we see that $e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$ has an essential singularity at 0.

Remark 2.4.5. The Laurent series expansion is unique.

Warning 2.4.6. We **cannot** write $a_n = \frac{f^{(n)}(z_0)}{n!}$.

2.5 Meromorphic Functions

First, we will define the Riemann sphere $\mathbb{P}^1 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Here, we consider stereographic projection $S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$. This is given by the formula $\varphi(x,y,z) = \frac{1}{1-z}(x+iy)$. This gives us a homeomorphism $S^2 \to \mathbb{CP}^1$, and the topology on $\mathbb{CP}^1$ is induced by this identification. This is a special case of the one-point compactification. In fact, we will see that $\mathbb{CP}^1$ has more structure: that of a *Riemann surface*, or a 1-dimensional complex manifold.

Definition 2.5.1. A *Riemann surface* X is a topological space X with an atlas of charts $X = \bigcup_\alpha U_\alpha$ with homeomorphisms $\varphi_\alpha\colon U_\alpha \to V_\alpha \subseteq \mathbb{C}$ such that the transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic. We also require that X is connected, Hausdorff, and second-countable.

Example 2.5.2. On $\mathbb{CP}^1$ we may consider the two charts $S^2 = U_1 \cup U_2$ with the two charts given by φ_1 being stereographic projection and φ_2 being stereographic projection from the south pole followed by conjugation. We can check that the transition map

$$\varphi_2 \circ \varphi_1^{-1}\left(\frac{x+iy}{1-z}\right) = \frac{x-iy}{1+z} = \left(\frac{x+iy}{1-z}\right)^{-1},$$

so $\varphi_2 \circ \varphi_{-1}(w) = \frac{1}{w}$. Therefore $\mathbb{P}^1$ can be given the structure of a **compact** Riemann surface with two charts $U_1 = \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$ and $U_2 = \mathbb{C} \cup \infty \setminus 0 \xrightarrow{z \mapsto 1/z} \mathbb{C}$.

Example 2.5.3. The second example of a Riemann surface is an elliptic curve. Choose $\lambda_1, \lambda_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. Then if we write $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, we can set $X = \mathbb{C}/\Lambda$. This has the quotient topology from $\mathbb{C}$. To define charts, let U_{z_0} be a small disk in $\mathbb{C}$ centered at z_0. Then under the map $q\colon \mathbb{C} \to X$, we have a homeomorphism $D \to q(D)$ and transition functions are given by $z \mapsto z + \lambda$ for some $\lambda \in \Lambda$.

Remark 2.5.4. It is also possible to define Riemann surfaces without using topology. We simply define the U_α as **sets** with bijections to open subsets of $\mathbb{C}$ and then specify holomorphic transition maps. Then the topology is given by the topology from $\mathbb{C}$, where each U_α is open. This allows for non-Hausdorff examples, like two copies of $\mathbb{C}$ glued by the identity map on $\mathbb{C} \setminus \{0\}$, called the *affine line with two origins*.[1]

Remark 2.5.5. If X, Y are Riemann surfaces, we can define holomorphic maps $F \colon X \to Y$. If we choose charts $p \in U \xrightarrow{\varphi} \mathbb{C}$ and $F(p) \in V \xrightarrow{\psi} \mathbb{C}$, then we require that $\psi F \varphi^{-1}$ is holomophic and that F is continuous.

Now the definition of a pole is equivalent to the map $\widetilde{f} \colon \Omega \to \mathbb{P}^1$ given by $\widetilde{f}(z) = f(z)$ when $z \neq z_0$ and $f(z_0) = \infty$ being holomorphic. Similarly, being meromorphic is equivalent to being a holomorphic map to $\mathbb{CP}^\infty$.

Definition 2.5.6. Let $f \colon \Omega \to \mathbb{C}$ be holomorphic and suppose there exists $R > 0$ such that $\{z \mid |z| > R\} \subset \Omega$. Then we can define $g \colon \left\{x \mid 0 < |z| < \frac{1}{R}\right\} \to \mathbb{C}$ by $g(z) = f(1/z)$. Then we say that f has a (removable singularity, pole of order k, essential singularity) at ∞ if g has the same thing at $z = 0$.

Example 2.5.7. If $f = \frac{P}{Q}$ is a rational function and $\deg P = n, \deg Q = m$, then f has a removable singularity at ∞ if $m \geqslant n$ and a pole of order $n - m$ if $m < n$. In particular, f is meromorphic on $\mathbb{C} \cup \infty$.

Example 2.5.8. We know that $f(z) = e^z$ has an essential singularity at ∞ because $g(z) = e^{z^{-1}}$ has an essential singularity at 0.

Theorem 2.5.9. *The meromorphic functions on $\mathbb{P}^1$ are the rational functions.*

Proof. Let $f \colon \mathbb{P}^1 \to \mathbb{P}^1$ be meromorphic. Note that the number of poles is finite because $\mathbb{P}^1 = S^2$ is compact. Then let the poles be $z_1, \dots, z_n$ and possibly ∞. Then at z_i, we expand f as a Laurent series and write

$$f_i(z) = \sum_{j=1}^{m_i} a_{ij}(z - z_i)^{-j} + g_i(z),$$

where g_i is holomorphic near z_i. By construction, $g = f - \sum_{i=1}^n f_i$ is holomorphic on $\mathbb{C}$. Now consider $h(z) = g(z^{-1})$ near $z = 0$. We have

$$h(z) = \sum_{j=1}^{m_\infty} a_{\infty j} z^{-j} + g_\infty(z),$$

so

$$g(z) = h(z^{-1}) = \sum_{j=1}^{m_\infty} a_{\infty j} z^{j} + g_\infty(z^{-1}).$$

Therefore If we write the first term as a polynomial $p(z)$, $g_\infty(z^{-1})$ is bounded as $z \to \infty$, so by Liouville it is constant. Finally, we have $f = c + p(z) + \sum f_i(z)$, which is rational. $\square$

Remark 2.5.10. Let $f = \frac{p}{q}$ be rational and suppose p, q are coprime and nonconstant. Then the number of zeros of f is equial to the number of poles of f, and both are given by $\max(\deg p, \deg q)$. This follows from the fundamental theorem of algebra and the calculuation of the behavior of f at ∞.

[1]This can be given a natural structure as a scheme. For more information about this, read a book about algebraic geometry.

More generally, if $f\colon X \to Y$ is a nonconstant holomorphic map between compact Riemann surfaces, we can define a well-defined *degree* of f by $\left|f^{-1}(q)\right| = \deg f$ counting multiplicity. We can now consider the automorphisms of $\mathbb{P}^1$. Clearly these are rational maps, and the maps given by

$$f(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, ad-bc \neq 0$$

have inverse $w \mapsto \frac{dz-b}{-cw+a}$. In fact, these are actually all automorphisms because any automorphism $f = \frac{p}{q}$ must satisfy $\deg f = \max(\deg p, \deg q) = 1$. Also, for all but finitely many $c \in \mathbb{C}$, all solutions of $f(z) = c$ have multiplicity 1. Therefore, we have $\operatorname{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ because we can identify $\mathbb{P}^1 = \mathbb{C}^2 \setminus \{0\}/\mathbb{C}^\times$ and then the point $(z_1 : z_2)$ corresponds to $\frac{z_1}{z_2}$. Then it is easy to see that these fractional linear transformations correspond to matrix multiplication.

With the prior discussion, we may now use the residue theorem at ∞.

Example 2.5.11 (Fall 2016 Qualifying Exam, Q9)**.** Consider $C = \{z \in \mathbb{C} \mid |z| = 2\}$ oriented counterclockwise and let $n \in \mathbb{Z}$ satisfy $n \geqslant 2$. We want to compute

$$I = \int_C \frac{z^{2n}\cos(1/z)}{1-z^n}.$$

Note that the residue theorem applies to essential singularities, so we may set $w = \frac{1}{z}, dz = \frac{-1}{w^2}\,dw$. Therefore we have

$$I = -\int_{C_{1/2}} \frac{\cos w}{w^{n+2}(w^n-1)} = 2\pi i \operatorname{Res}_0 \frac{\cos w}{w^{n+2}(w^n-1)} = \begin{cases} \frac{(-1)^{\frac{n+3}{2}}}{(n+1)!} & n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

2.6 Estimates for Holomorphic Functions

Theorem 2.6.1 (Argument principle)**.** *Let f be meromorphic on $\Omega \subset \mathbb{C}$ and γ be a simple closed curve such that γ and its interior are contained in Ω. Suppose f is holomorphic and nonzero on γ, oriented positively. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \#\{\textit{zeroes of } f \textit{ inside } \gamma\} - \#\{\textit{poles of } f \textit{ inside } \gamma\}.$$

Proof. Use the residue theorem. We have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z_0} \operatorname{Res}_{z_0} \frac{f'}{f}.$$

But then if z_0 is a zero or pole of order n, we have

$$\operatorname{Res}_{z_0} \frac{f'}{f} = \operatorname{Res}_{z_0}\left(\frac{n(z-z_0)^{n-1}}{(z-z_0)^n}\right) = n.$$

□

Remarks 2.6.2. The left hand-side of the formula is the winding number of $f \circ \gamma$ around $z = 0$. In addition, we can think of $\frac{f'}{f}\,dz$ as $d(\log f)$, but $\log z$ is not well-defined.

Theorem 2.6.3 (Rouché)**.** *Let $f, g\colon \Omega \to \mathbb{C}$ be holomorphic and γ be a simple closed curve with $\gamma + \operatorname{int}(\gamma) \subset \Omega$. Assume that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then the number of zeroes of f inside γ is equal to the number of zeroes of $f + g$ inside γ, counting multiplicity.*

Proof. Define for $t \in [0, 1]$ the function $f_t = f + tg\colon \Omega \to \mathbb{C}$. This is holomorphic on Ω and nonzero on γ. Then the function

$$n(t) := \frac{1}{2\pi i}\int_\gamma \frac{f_t'(z)}{f_t(z)}\,dz \in \mathbb{Z}$$

is continuous because the integrand depends continuously on t, but $\mathbb{Z}$ is discrete, so it must be constant. ☐

Example 2.6.4. Consider $f = z^5$, $g = 3z + 1$. Then if $|z| = 2$, $|f(z)| = 32$ while $|g(z)| \leqslant 3 \cdot 2 + 1 = 7$, so $f + g$ has 5 zeroes with $|z| < 2$. On the other hand, if $|z| = 1$, then $|g| \geqslant 2$ while $|f| = 1$, so $f + g$ has a single zero with $|z| < 1$.

Theorem 2.6.5 (Open mapping theorem)**.** *Let $f\colon \Omega \to \mathbb{C}$ be holomorphic and nonconstant on a connected Ω. If $U \subset \Omega$ is open, then $f(U)$ is open.*

Proof. Let $z_0 \in \Omega$ and write $w_0 = f(z_0)$. We need to show that for some $\varepsilon > 0$, the set

$$\{w \mid |w - w_0| < \varepsilon\} \subset f(\Omega).$$

Equivalently, we want to show that $f(z) - w = 0$ has a solution $z \in \Omega$ for $|w - w_0| < \varepsilon$. Choose $\delta > 0$ such that $\overline{D} = \{z \mid |z - z_0| \leqslant \delta\} \subset \Omega$. For δ small enough, $f(z) \neq w_0$ for $|z - z_0| = \delta$. Then we know $|f(z) - w_0| > \varepsilon$ on ∂D for some $\varepsilon > 0$, so for $|w - w_0| < \varepsilon$, Rouche's theorem tells us that the number of zeroes of $(f(z) - w_0) + (w_0 - w)$ is equal to the number of zeroes of $f(z) - w_0$ in D, and this is positive. ☐

Theorem 2.6.6 (Maximum Principle)**.** *Let $f\colon \Omega \to \mathbb{C}$ be holomorphic on some open set $\Omega \subset \mathbb{C}$. Then $|f|$ does **not** attan a maximum on Ω.*

Proof. Given $z_0 \in \Omega$, we know $f(z_0) \in \{w \mid |w - f(z_0)| < \varepsilon\} \subset f(\Omega) \subset \mathbb{C}$. Clearly there exists $w \in D$ such that $|w| > |f(z_0)|$. ☐

Here is another formulation. Suppose $f\colon \Omega \to \mathbb{C}$ is holomorphic and f extends continuously to $\overline{\Omega}$, which is bounded. Then the maximum of f on $\overline{\Omega}$ is obtained on the boundary, so we cannot obtain the maximum on Ω.

Remark 2.6.7. Suppose $f\colon \Omega \to \mathbb{C}$ is holomorphic and nonconstant on a connected Ω. Then the real part of f does not attain a maximum on Ω. Here, we simply apply the maximum principle to $g = e^f$.

Now recall that if $f = u + iv$ and f is holomorphic, then u, v are harmonic. This means that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Conversely, if u is a harmonic funciton on a disk $D \subset \mathbb{C} = \mathbb{R}^2$, there exists a harmonic v such that $f = u + iv$ is holomorphic on D. Therefore we obtain a maximum principle for harmonic functions. The physical intuition for this is that if $u\colon \overline{\Omega} \to \mathbb{R}$ is harmonic and we fix $u\big|_{\partial\Omega}$, then this corresponds to the steady state temperature of points in the region Ω. Then heat would flow away from a maximum, so the maximum cannot be attained on the interior.

Now recall from multivariable calculus that if $f\colon \mathbb{R}^2 \to \mathbb{R}$, we have critical points of f when

$$\left.\frac{\partial f}{\partial x}\right|_p = \left.\frac{\partial f}{\partial y}\right|_p = 0.$$

Then we may consider the matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} =: H.$$

This is called the *Hessian* of f. If we assume that $p = (0,0)$, then we can write

$$f(x,y) = f(0,0) + \frac{\partial^2 f}{\partial x^2} x^2 + 2\frac{\partial^2 f}{\partial x \partial y} xy + \frac{\partial^2}{\partial y^2}(0,0) y^2 + O(\|x,y\|^3).$$

If $\det H \neq 0$, then we have a nondegenerate quadratic form. After a linear change of coordinates, theere are three cases:

1. The form looks like $x^2 + y^2$ and $\det H > 0$. This is a minimum.
2. The form looks like $-(x^2 + y^2)$ and $\det G > 0$. This is a maximum.
3. The form looks like $x^2 - y^2$ and $\det H < 0$. This is a saddle point.

For any harmonic function, any nondegenerate critical point is a saddle point because

$$\det H = -\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \leqslant 0.$$

Example 2.6.8. Consider $f(z) = z^2$. Then $u = x^2 - y^2 = (x+y)(x-y)$ and $v = 2xy$. Clearly the origin is a saddle point for both. The graph of u looks like

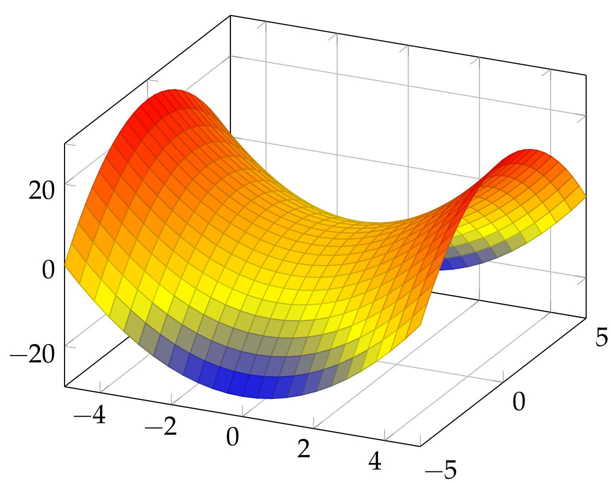

Figure 2.9: Graph of $z = x^2 - y^2$

Example 2.6.9. Consider $f(z) = z^3$. Then $u = x^3 - 3xy^2 = x(x - \sqrt{3}y)(x + \sqrt{3}y)$ is not nondegenerate at the origin. The graph looks like

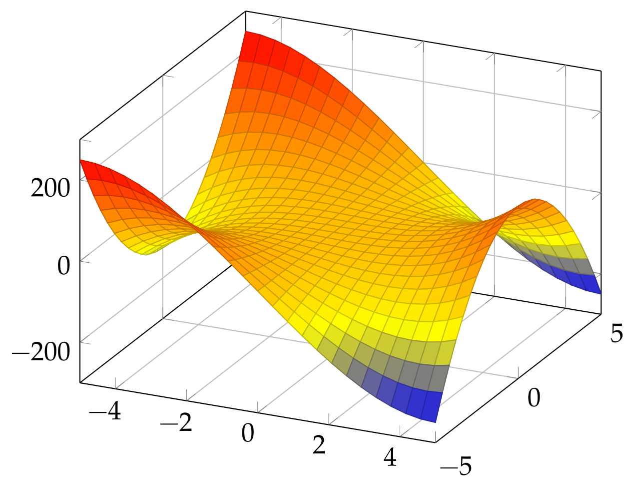

Figure 2.10: Graph of $z = x^3 - 3xy^2$

Now we can give another proof of the maximum principle using the mean-value property. Recall that

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z_0}\,dw\,.$$

Now we parameterize γ by $w = z_0 + re^{i\theta}$, so we obtain

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta\,,$$

and therefore $f(z_0)$ is the mean value of f on the circle γ. Therefore, if f is nonconstant, it obtains some larger value on the circle γ.

2.7 The Logarithm

Let $\Omega \subset \mathbb{C}$ and γ_0, γ_1 be two curves in Ω with the same endpoints α, β and parameterizations $z_0, z_1 \colon [a, b] \to \Omega$. Then γ_0, γ_1 are *homotopic* in Ω if there exists a continuous

$$F\colon [a,b] \times [0,1] \to \Omega \qquad F(s,0) = z_0(s), F(s,1) = z_1(s), F(a,t) = \alpha, F(b,t) = \beta.$$

Theorem 2.7.1. *Let f be holomorphic on Ω and γ_0, γ_1 curves in Ω. If γ_0, γ_1 are homotopic, then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz\,.$$

Proof. Choose a homotopy F. Then the image K of F is compact and $K \subset \Omega \subset \mathbb{C}$, so there exists $\varepsilon > 0$ such that for all $w \in K$ the disk $D(z, \varepsilon) \subset \Omega$. We need to show that the distance $d(p, q), p \in K, q \in \mathbb{C} \setminus \Omega$ is bounded below by $\varepsilon > 0$. Otherwise, there exists $p_n \in K$ and $q_n \in \mathbb{C}$ such that $d(p_n, q_n) \to 0$. But then K is compact, so $p_n \to p \in K$, so $q_n \to p$ as well. But then $\mathbb{C} \setminus \Omega$ is closed, and thus $p \in \mathbb{C} \setminus \Omega$, which is a contradiction.

F is uniformly continuous, so given $\varepsilon > 0$, there exists $\delta > 0$ such that $|F(s_1, t_1) - F(s_2, t_2)| < \varepsilon$ whenever $|(s_1, t_1) - (s_2, t_2)| < \delta$. If we fix t_1, t_2 such that $|t_1 - t_2| < \frac{1}{2}\delta$, then we can choose a subdivision $a = s_0 < s_1 < \cdots < s_n = b$, where $s_{i+1} - s_i < \frac{\delta}{2}$. Then we see that $\left|z_t(s) - z_{t'(s')}\right| < \varepsilon$ for all $(s,t), (s',t') \in [s_i, s_{i+1}] \times [t_1, t_2]$. But now we choose D to be a disk of radius ε such that $F([s_i, s_{i+1}] \times [t_1, t_2]) \subset D$. By the Cauchy formula for the disk, we see that

$$\int_{\text{perimeter}} f(z)\, dz = 0.$$

Summing over i, we see that $\int_{\gamma_{t_1}} - \int_{\gamma_{t_2}} = 0$. Subdividing in the t direction, we get $\int_{\gamma_0} = \int_{\gamma_1}$. □

Now let $\Omega \subset \mathbb{C}$ be open. We sau that Ω is *simply connected* if it is path connected and any two paths with the same endpoints are homotopic. Equivalently, Ω is path-connected and any loop γ is homotopic to the constant loop.

Example 2.7.2. Any convex set is homotopic by the straight line homotopy

$$z_t(s) = (1-t)z_0(s) + tz_1(s) \colon z_0 \Rightarrow z_1.$$

Example 2.7.3. The set $\mathbb{C} \setminus (-\infty, 0]$ is simply connected. Note that this set is homeomorphic to $\mathbb{R}_{>0} \times (-\pi, \pi)$ because any $z \in \Omega$ can be uniquely written as $z = re^{i\theta}$ for any $r > 0$ and $\theta \in (-\pi, \pi)$. Therefore, Ω is homeomorphic to a convex set and is thus simply connected.

Theorem 2.7.4. *Let $\Omega \subset \mathbb{C}$ be open and connected. Then Ω is simply connected if and only if $\mathbb{P}^1 \setminus \Omega$ is connected.*

Example 2.7.5. Consider $\mathbb{R} \times (0,1)$. Then the complement in $\mathbb{P}^1$ is connected because ∞ joins the two components of $\mathbb{C} \setminus (\mathbb{R} \times (0,1))$.

Theorem 2.7.6. *Let $\Omega \subset \mathbb{C}$ be simply connected and $f \colon \Omega \to \mathbb{C}$ be holomorphic. Then f has a primitive.*

Proof. Fix a basepoint $z_0 \in \Omega$. Then set $F(z) \coloneqq \int_{\gamma_z} f(w)\, dw$, where γ_z is a path from z_0 to z in Ω. Because Ω is simply-connected, this is well-defined. Now as in the case of the disk, we see that $F'(z) = f(z)$. □

Example 2.7.7. Consider $\Omega = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$. Then we see that

$$\int_{S^1} \frac{1}{z}\, dz = 2\pi i \neq 0,$$

and therefore $\frac{1}{z}$ does not have a primitive, so Ω is not simply-connected.

Now we may define the complex logarithm.

Theorem 2.7.8. *Let $\Omega \in \mathbb{C}$ be simply connected with $1 \in \Omega$ and $0 \notin \Omega$. Then there exists a unique holomorphic function $F = \log_\Omega \colon \Omega \to \mathbb{C}$ such that $e^{F(z)} = z$ for all $z \in \Omega$ and $F(1) = 0$.*

Proof. Let F be a primitive of $\frac{1}{z}$ on Ω normalized such that $F(1) = 0$. Then we show that $e^{F(z)} = z$. Here, we have

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} + z(-F'(z))e^{-F(z)} = 0,$$

so $ze^{-F(z)} = c$ for some constant $c \in \mathbb{C}$. Setting $z = 1$, we obtain $c = 1$, so $e^{F(z)} = z$.

Now we consider uniqueness of the logarithm. Let G be another such function. Then $G - F \colon \Omega \to 2\pi i\mathbb{Z}$, but Ω is connected, so $G - F$ is constant and thus evaluating at $z = 1$ gives us $G = F$. □

Example 2.7.9. Consider $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Then we defined earlier that for $z = re^{i\theta}$, $\log z = \log r + i\theta$.

More generally, we have the following result.

Theorem 2.7.10. *Let $\Omega \subset \mathbb{C}$ be simply connected and $f \colon \Omega \to \mathbb{C}$ be holomorphic and nowhere vanishing. Then there exists $g \colon \Omega \to \mathbb{C}$ such that $e^g = f$.*

Proof. Let g be a primitive of $\frac{f'}{f}$. Then we have

$$\frac{d}{dz}(f \cdot e^g) = f'e^{-g} - fg'e^{-g} = 0,$$

so we can simply adjust g by a constant until $fe^{-g} = 1$. $\square$

Now we may define $f^\alpha = e^{\alpha \log f}$ for any $\alpha \in \mathbb{C}$.

3

Global Theory

3.1 Conformal Mappings

Question 3.1.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open. Does there exist a holomorphic bijection $\Omega_1 \to \Omega_2$?*

Remark 3.1.2. A holomorphic bijection has holomorphic inverse.

Our goal will be to prove the following remarkable theorem.

Theorem 3.1.3 (Riemann Mapping Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open. Then there exists a holomorphic bijection $F\colon \Omega \to D$ if and only if Ω is simply connected and $\Omega \neq \mathbb{C}$. Here, D is the unit disk.*

Example 3.1.4. Let $\Omega = \mathcal{H} = \{z = x + iy \mid y > 0\} \subset \mathbb{C}$. Then there exists a holomorphic bijection $F\colon \mathcal{H} \to D$. Note that if $z \in \mathcal{H}$, then $|z - i| < |z + i|$. Therefore, we can set $F(z) = \frac{i-z}{i+z}$. To see that this is a bijection, we note that $\infty \mapsto -1$, $-i \mapsto \infty$. Therefore, $F\colon \mathbb{C} \setminus \{-i\} \to \mathbb{C} \setminus \{-1\}$ is a bijection, and thus it restricts to a bijection $\mathcal{H} \to D$ on the northern hemisphere.

Remark 3.1.5. Consider the map $\varphi\colon S^2 \to \mathbb{P}^1$. Then

$$\varphi^{-1}(D) = \Big\{(x, y, z) \in S^2 \mid z < 0\Big\} \qquad \varphi^{-1}(\mathcal{H}) = \Big\{(x, y, z) \in S^2 \mid y > 0\Big\}.$$

Then if r is the rotation by $\frac{\pi}{2}$ about the x-axis, the diagram

$$\begin{tikzcd} \{z < 0\} \arrow[r, "r", "\sim"'] \arrow[d, "\varphi", "\sim"'] & \{y > 0\} \arrow[d, "\varphi", "\sim"'] \\ D \arrow[r, "\widetilde{G}"] & \mathcal{H} \end{tikzcd}$$

commutes, where $\widetilde{G} = F^{-1} \circ \psi$, where $\psi(w) = iw$.

Here, rotation of S^2 corresponds to a holomorphic automorphism of $\mathbb{P}^1$ under φ. This is because stereographic projection preserves angles and rotations also preserve angles. Therefore $\widetilde{G} = \varphi \circ r \circ \varphi^{-1}$ will preserve angles, so it is holomorphic. This gives us a homomorphism $SO(3) \hookrightarrow \mathrm{Aut}(\mathbb{P}^1) = PGL_2(\mathbb{C})$. Recall that

$$SO(3) = \Big\{A \in GL_3(\mathbb{R}) \mid A^T A = I, \det A = 1\Big\},$$

The image of the homomorphism is $\mathrm{PSU}(2)$, where we have

$$\mathrm{SU}(2) = \left\{ B \in \mathrm{GL}_2(\mathbb{C}) \mid \overline{B}^T B = I, \det B = 1 \right\}.$$

In fact, any $B \in \mathrm{SU}(2)$ has the form $B = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$, so $\mathrm{SU}(2) \simeq S^3$. Then we have $\mathrm{PSU}(2) = \mathrm{SU}(2)/\pm I$. Now we have a chain of isomorphisms

$$\mathbb{RP}^3 = S^3/\pm 1 \cong \mathrm{SO}(3) \xrightarrow{\sim} \mathrm{PSU}(2) \xleftarrow{2:1} \mathrm{SU}(2) \cong S^3,$$

and this is somehow related to the spin of an electron.

Now here are some examples of conformal maps:

1. If $f\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic bijection, then there exists x, y such that $f(z) = az + b$.

2. Let $n \in \mathbb{N}$ and set $S = \left\{ z \in \mathbb{C} \mid 0 < \arg(z) < \frac{\pi}{n} \right\}$. Then $z \mapsto z^\alpha$ gives a holomorphic bijection $S \to \mathcal{H}$. If $\alpha > \frac{1}{2}$ is real, then we may consider $z \mapsto z^\alpha$.

3. Consider $\log z = \log r + i\theta$. This gives a holomorphic bijection $\mathbb{C} \setminus (-\infty, 0] \to \mathbb{R} \times i(-\pi, \pi)$. Restricting to $\mathcal{H}$, we have a holomorphic bijection $\mathcal{H} \to \mathbb{R} \times i(0, \pi)$.

Example 3.1.6. Consider the map $\mathbb{C} \setminus \{0\} \xrightarrow{F} \mathbb{C}$ given by $z \mapsto z + \frac{1}{z}$. Then if $z + \frac{1}{z} = w$, we have $z^2 - wz + 1 = 0$ and thus $z = \frac{w \pm \sqrt{w^2 - 4}}{2}$. Therefore F is 2-to-1 onto $\mathbb{C}$ and branched over $w^2 - 4 = 0$, or $w = \pm 2$. Then let z_1, z_2 be two roots of $z^2 - wz + 1 = 0$ for a fixed w. We know $z_1 z_2 = 1$, so exactly one of z_1, z_2 lies in $\mathcal{H}$ unless $z_1, z_2 \in \mathbb{R}$, which happens if and only if $w \in \mathbb{R}$ and $w^2 - 4 \geqslant 0$. Therefore F restricts to a bijection

$$\mathcal{H} \xrightarrow{\sim} \mathbb{C} \setminus (-\infty, -2] \cup [2, \infty).$$

We want to compute the preimage of $\mathcal{H}$. Then we have

$$z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2} = x\left(1 + \frac{1}{x^2 + y^2}\right) + iy\left(1 - \frac{1}{x^2 + y^2}\right).$$

For $y > 0$, we see that $1 - \frac{1}{x^2 + y^2} > 0$ if $x^2 + y^2 > 1$. Therefore we have $\Omega = \mathcal{H} \setminus \{|z| \leqslant 1\}$. By the reasoning, the map

$$\Omega = \{z \in \mathcal{H} \mid |z| < 1\} \to \mathcal{H} \qquad z \mapsto -\left(z + \frac{1}{z}\right)$$

is a holomorphic bijection.

Example 3.1.7. Set $\Omega = \{z \mid \Re(z) > 0\} \setminus [0, 1]$. Then the sequence

$$\Omega \xrightarrow{z \mapsto z^2} \mathbb{C} \setminus (-\infty, 1] \xrightarrow{z \mapsto z - 1} \mathbb{C} \setminus (-\infty, 0] \xrightarrow{z \mapsto i\sqrt{z}} \mathcal{H}$$

gives a holomorphic bijection $\Omega \to \mathcal{H}$.

Example 3.1.8. We have the following sequence taking a vertical corridor to $\mathcal{H}$.

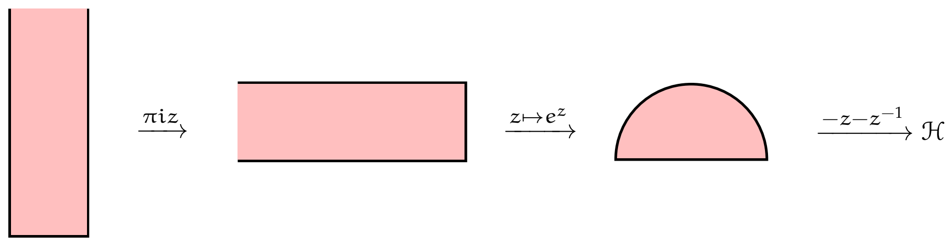

Figure 3.1: A sequence of holomorphic bijections

Now we will study fractional linear transformaitons in more detail. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be given by $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. Then we know that f is a holomorphic bijection and that $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$. Another important property is the following: given $z_1, z_2, z_3 \in \mathbb{P}^1$ and $w_1, w_2, w_3 \in \mathbb{P}^1$ distinct points, there exists a unique f such that $f(z_j) = w_j$ for $j = 1, 2, 3$.

To prove existence, we consider the case when $w_1, w_2, w_3 = 0, 1, \infty$. Then we simply set

$$f(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}.$$

To prove uniqueness, we need to do this for the case where $z_1, z_2, z_3 = w_1, w_2, w_3 = 0, 1, \infty$. First, fixing ∞ means that our transformation is given by $ax + b$. Fixing 0 means that $b = 0$, and fixing 1 gives us $a = 1$.

Now let $z_1, z_2, z_3, z_4 \in \mathbb{P}^1$. Then we define the *cross ratio*

$$\mathrm{CR}(z_1, z_2, z_3, z_4) \coloneqq \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}.$$

Then if z_1, z_2, z_3, z_4 are distinct, the cross ratio is a complex number. Note that if f is the unique Möbius transformation sending $z_2, z_3, z_4 \mapsto 1, 0, \infty$, then $\mathrm{CR}(z_1, z_2, z_3, z_4) = f(z_1)$. Therefore, fractional linear transformations preserve the cross ratio. Alternatively, we can compute using brute force.

Proposition 3.1.9. *Fractional linear transformations take circles and lines to circles and lines (for a nicer formulation, these are all circles on $\mathbb{P}^1 = S^2$, where lines are circles passing through ∞).*

Proof. We will prove that $\mathrm{CR} \in \mathbb{RP}^1$ if and only if z_1, z_2, z_3, z_4 lie on a circle or a line. To see this, write $\arg\left(\frac{z_1 - z_3}{z_1 - z_4}\right) = \alpha$ or $-(\pi - \alpha)$. Then $\arg(\mathrm{CR}) = 0$ if z_1, z_2 lie on the same side of the line connecting z_3, z_4 and π if they lie on opposite sides, so $\mathrm{CR} \in \mathbb{R}$.

Conversely, if we fix the circle passing through z_2, z_3, z_4, then $z_1 \in C$ if and only if $\arg(\mathrm{CR}) = 0$ or $\arg(\mathrm{CR}) = \pi$ if and only if $\mathrm{CR} \in \mathbb{RP}^1$. $\square$

For an alternative proof, we know the result is true for $z \mapsto az$ and $z \mapsto z + b$. Then we can check that the result holds for $z \mapsto \frac{1}{z}$, and finally we see that these transformations generate $\mathrm{PGL}_2(\mathbb{C})$.

Example 3.1.10. We may apply this to conformal mapping problems. Set

$$f(z) = \frac{z - 1}{z + 1} \frac{i + 1}{i - 1}.$$

Then $f(1) = 0, f(i) = 1, f(-i) = \infty$. We also know that $f(\partial D) = \partial\mathcal{H} = \mathbb{RP}^1$, and $f(D) = \mathcal{H}$ because $f(0) = i$.

Lemma 3.1.11. *Let $U, V \subset \mathbb{C}$ and $f\colon U \to V$ be holomorphic and injectivve. Then $f'(z) \neq 0$ for all $z \in U$ and $f^{-1}\colon f(U) \to U$ is holomorphic.*

Proof. Near z_0, write $f(z) = w_0 + (z - z_0)^m g(z)$, where $g(z_0) \neq 0$. In fact, we can write $f(z) = w_0 + (h(z))^m$ because we can locally define an m-th root of g. Then near z_0, we see that f is given as the composition

$$D \xrightarrow{h} \mathbb{C} \xrightarrow{z \mapsto w_0 + z^m} \mathbb{C} \qquad z_0 \mapsto 0 \mapsto w_0.$$

Then if $m \geqslant 2$, the map $z \mapsto z^m$ is not injective near 0, so f is not injective near z_0. Finally, f^{-1} is holomorphic by the inverse function theorem. $\square$

Theorem 3.1.12 (Inverse function theorem)**.** *Let $F\colon U \to \mathbb{R}^2$ and $p \in U$ such that $\det(DF(p)) \neq 0$. Then there exist open neighborhoods $p \in U' \subset U$ and $F(p) \in V \subset \mathbb{R}^2$ such that $F\colon U' \to V$ is a bijection with F^{-1} differentiable and $DF^{-1}(F(p)) = DF(p)^{-1}$.*

Lemma 3.1.13 (Schwarz)**.** *Let D be the unit disk and $f\colon D \to D$ be holomorphic such that $f(0) = 0$. Then*

1. *For all $z \in D$, $|f(z)| \leqslant |z|$;*
2. *If there exists $z_0 \in D$ such that $|f(z_0)| = |z_0|$, then f is a rotation.*
3. *$|f'(0)| \leqslant 1$ with equality if and only if f is a rotation.*

Proof. Consider the function $g(z) = \frac{f(z)}{z}$. Then if $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$, we know $a_0 = f(0) = 0$, so $g(z) = a_1 + a_2 z + a_3 z^2 + \cdots$ is holomorphic on D. Then we apply the maximum principle to g, and we note that $|g(z)| < \frac{1}{r}$ for $|z| = r < 1$, so $|g(z)| < \frac{1}{r}$ for $|z| \leqslant r$. Allowing $r \to 1$ from below, we see that $|g(z)| \leqslant 1$ for all $z \in D$ and therefore $|f|(z) \leqslant |z|$.

Now if $|f(z)| = |z|$ for some $z_0 \in D$, then $g(z)$ must be constant and therefore $f(z) = cz$ with $|c| = 1$. Finally, we note that $g(0) = f'(0)$ and then if $g(0) = 1$, we see that f is a rotation by the second part. $\square$

Now we are ready to consider automorphisms of the disk and upper half plane. In the case of the disk, we have two kinds of automorphisms:

1. Rotations $z \mapsto e^{i\theta} z$;
2. *Blaschke factors* $\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$, where $\alpha \in D$. We can check that $\psi_\alpha(\partial D) = \partial D$.

Proposition 3.1.14. *Every automorphism of the disk is of the form*

$$z \mapsto e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}$$

for some θ, α.

Proof. Given an automorphism $f\colon D \to D$, then the map $f \circ \psi_\alpha$ fixes the origin and is a rotation by the Schwarz lemma because it also fixes α. Therefore, $f = r \circ \psi_\alpha^{-1} = r \circ \psi_\alpha$, where r is some rotation. $\square$

Note that if $f\colon D \to D$ is an automorphism with $f(0) = 0$, then the Schwarz lemma implies that $|f(z)| \leqslant |z|$ and $\left|f^{-1}(z)\right| \leqslant |z|$ and therefore $|z| = |f(z)|$ and therefore f is a rotation.o

Now we want to study automorphisms of $\mathcal{H}$. We know that $\mathcal{H}, D$ are isomorphic, so $\operatorname{Aut} D = \operatorname{Aut} \mathcal{H}$.

Theorem 3.1.15. *We have $\operatorname{Aut}(\mathcal{H}) \cong \mathrm{PSL}_2(\mathbb{R})$.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$. Then we see that

$$\operatorname{Im}(f(x+iy)) = \frac{ay(cx+d) - cy(ax+b)}{(cx+d)^2 + y^2} = \frac{(ad-bc)y}{(cx+d)^2+y^2} > 0,$$

and thus we require $ad - bc > 0$. Therefore we have $\mathrm{PSL}_2(\mathbb{R}) \subset \operatorname{Aut}\mathcal{H}$.

Now we prove surjectivity. First, we note that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$. First, if $b \in \mathbb{R}$, $z \mapsto z + b$ preserves $\mathcal{H}$ and if $\lambda > 0$, $z \mapsto \lambda z$ preserves $\mathcal{H}$. Next, we consider the effect of rotation $\left(\begin{smallmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{smallmatrix}\right)$. But we can compute that

$$\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} -e^{i\varphi} & ie^{-i\varphi} \\ e^{i\varphi} & ie^{i\varphi} \end{pmatrix} = e^{i\varphi} \begin{pmatrix} e^{2i\varphi} & \\ & 1 \end{pmatrix} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}.$$

To finish, we show that given $f\colon \mathcal{H} \to \mathcal{H}$, then $f \in \mathrm{PSL}_2(\mathbb{R})$. Suppose $f(\alpha) = i$. Then there exists $g \in G$ such that $g(i) = \alpha$. Then $g \circ f$ fixes i, so $F \circ g \circ f \circ F^{-1}$ fixes 0 and is thus a rotation. Therefore there exists $h \in G$ such that $F \circ g \circ f \circ F^{-1} = F \circ h \circ F^{-1}$, so $g \circ f = h$ and $f = h \circ g^{-1}$. □

3.2 Riemann Mapping Theorem

Our goal in this section is to prove the Riemann mapping theorem.

Theorem 3.2.1. *Let $\Omega \subset \mathbb{C}$ be open and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions $f_n\colon \Omega \to \mathbb{C}$ which converges uniformly to f on every compact subset $K \subset \Omega$. Then f is holomorphic.*

Proof. Let $D = \{z \in \mathbb{C} \mid |z - z_0| < r\} \subset \Omega$. Then f_n being holomorphic implies that $\int_\gamma f_n(z)\,dz = 0$ for all closed curves γ in D. Because $f_n \to f$ uniformly, we have

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0.$$

Also we know that f is continuous. Now we define

$$F(z) = \int_{\gamma_z} f(w)\,dw \qquad F\colon \Omega \to \mathbb{C}.$$

Here, γ_z is a path from z_0 to z for a fixed basepoint z_0. Then F is well-defined and $F' = f$, so f is holomorphic. □

Lemma 3.2.2. *Let $\Omega \subset \mathbb{C}$ be open and $f_n\colon \Omega \to \mathbb{C}$ be holomorphic. Suppose that $f_n \to f$ uniformly on compact sets. Then $f_n' \to f'$ uniformly on compact sets.*

Proof. Let $K' = \{z \in \mathbb{C} \mid d(z, K) leq r\}$ for some $r \in \mathbb{R}$ such that $K' \subset \Omega$. Then we know $f_n \to f$ uniformly on K', and we will use this to show that $f_n' \to f'$ uniformly on K. Using the Cauchy integral formula to some holomorphic function h, we have

$$h'(z) = \frac{1}{2\pi i} \int_\gamma \frac{h(w)}{(w-z)^2}\,dw,$$

so then we can bound

$$\left|h'(z)\right| \leqslant \frac{1}{2\pi} 2\pi r \cdot \frac{1}{r^2} \sup_{w\in\gamma} |h(w)| = \frac{1}{r} \sup_{w\in\gamma} |h(w)|.$$

Applying this to $h = f_n - f$, we see that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(w) - f(w)| < \varepsilon$ on K' for all $n > N$, and this implies that $|f_n'(z) = f'(z)| < \frac{1}{r}\varepsilon$ on K. □

Now let $\Omega \subset \mathbb{C}$ be open and $\mathcal{F}$ be a set of holomorphic functions on Ω. We say that $\mathcal{F}$ is *normal* if every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on compact sets to some function $f\colon \Omega \to \mathbb{C}$. We say that $\mathcal{F}$ is *uniformly bounded on compact sets* if for all compact $K \subset \Omega$, there exists $M \in \mathbb{R}$ such that $|f(z)| \leqslant M$ for all $f \in \mathcal{F}, z \in K$. Finally, we say that $\mathcal{F}$ is *equicontinuous* on a compact set $K \subset \Omega$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all $z, w \in K$ such that $|z - w| < \delta$ and $f \in \mathcal{F}$.

Theorem 3.2.3 (Montel)**.** *Let $\mathcal{F}$ be a family of holomorphic functions on $\Omega \subset \mathbb{C}$ that is uniformly bounded on comact sets. Then*

1. *$\mathcal{F}$ is equicontinuous on compact sets;*
2. *$\mathcal{F}$ is normal.*

Remark 3.2.4. The first part uses holomorphicity, and the second part uses a general fact, the Arzelà-Ascoli theorem.

Example 3.2.5. Let $f_n(z) = z^n$. Then $\mathcal{F} = \{f_n\}$ is **not** equicontinuous on $K = [0,1]$. In fact, $|f_n(z) - f_n(1)| = |z^n - 1| \to 1$ as $n \to \infty$ for $z \in (0,1)$.

Example 3.2.6. Let $\mathcal{F} = \{f_n(z) = \sin(nz)\}$. Then $\mathcal{F}$ is **not** equicontinuous on $K = [0,1]$ because $f_n\left(\frac{\pi}{2n}\right) = 1$, but $f_n(0) = 0$.

Proof of Montel's theorem.

1. We will use the Cauchy integral formula to bound $|f(z) - f(w)|$. Let $K \subset \Omega$ be compact and choose $r > 0$ such that $K' = \{z \in \mathbb{C} \mid d(z, K) \leqslant 2r\} \subset \Omega$. Given $z, w \in K$ such that $|z - w| < r$, we will let γ be the circle with center z and radius $2r$. We know $\gamma \subset K'$. Now we have
$$f(z) - f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(u)}{u-z} - \frac{f(u)}{u-w}\,du = \frac{1}{2\pi i}\int_\gamma f(u)\left(\tfrac{1}{u-z} = \tfrac{1}{u-w}\,du\right).$$
Then we can bound
$$\left|\frac{1}{u-z} - \frac{1}{u-w}\right| = \left|\frac{z-w}{(u-z)(u-w)}\right| \leqslant \frac{|z-w|}{2r \cdot r} = \frac{1}{2r^2}|z-w|,$$
so
$$|f(z) - f(w)| \leqslant \frac{1}{2\pi} \cdot 2\pi \cdot 2r \cdot \frac{1}{2r^2}|z-w| \sup_{u \in \gamma}|f(u)| = \frac{1}{r}|z-w| \cdot M,$$
where $|f(u)| \leqslant M$ for all $u \in K, f \in \mathcal{F}$. Therefore $\mathcal{F}$ is equicontinuous on K.

2. Let $K \subset \Omega$ be compact and f_n be a sequence in $\mathcal{F}$. First let $(w_j)_{j=1}^\infty$ be a sequence in K which is dense. To construct such a sequence, recall that
$$K = \bigcup_{x \in K} D\left(x, \frac{1}{n}\right) \cap K.$$
By compactness, there exists a finite $S_n \subset K$ such that $K = \bigcup_{x \in S_n} D(x, 1/n) \cap K$, and now we can take $S = \bigcup_{n \geqslant 1} S_n$ and obtain a countable dense subset.

   Now we will use a "diagonal argument." If f_n is uniformly bounded, then the sequence $f_n(w_1)$ has a convergent subsequence $f_{n,1}(w_1)$. Continuing, there exists a subsequence $\{f_{n,2}\}$ of $\left\{f_{n,1}\right\}$ such that $f_{n,2}(w_2)$ converges. For each $m \in \mathbb{N}$, there exists a subsequence

$\{f_{n,m}\}$ of $\{f_n\}$ such that $f_{n,m}(w_k)$ converges for all $k \leqslant m$. Now take $g_n = f_{n,n}$. This is a subsequence of $\{f_n\}$ such that $g_n(w_k)$ converges for all $k \in \mathbb{N}$. If $z \in K$, then

$$|g_n(z) - g_m(z)| \leqslant \left|g_n(z) - g_n(w_j)\right| + \left|g_n(w_j) - g_m(w_j)\right| + \left|g_m(z) - g_m(w_j)\right|$$

by the triangle inequality.

Now carefully, we know that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|g_n(z) - g_n(w)| < \varepsilon$ whenever $|z - w| < \delta$ and $z, w \in K$. Because K is compact, $K \subseteq \bigcup_{j=1}^{J} D(w_j, \delta)$ for some $J \in \mathbb{N}$. Given $z \in K$, there exists w_j such that $\left|z - w_j\right| < \delta$, so $|g_n(z) - g_m(z)| < \varepsilon$. Now for n, m large, we have $|g_n(z) - g_m(z)| < 3\varepsilon$ because there exists $N \in \mathbb{N}$ such that $\left|g_n(w_j) - g_m(w_j)\right| < \varepsilon$ for $n, m \geqslant N$. Therefore $g_n(z)$ converges uniformly for all $z \in K$.

For all compact sets K, we need another diagonal argument. There exists an exhaustion

$$K_1 \subset K_2 \subset \cdots \subset \Omega$$

such that K_ℓ is contained in the interior of $K_{\ell+1}$ and for all $K \subset \Omega$ compact, $K \subset K_\ell$ for some ℓ. For example, set

$$K_\ell = \left\{z \in \Omega \mid d(z, \mathbb{C} \setminus \Omega) \geqslant \frac{1}{\ell}\right\} \cap \{z \in \mathbb{C} \mid |z| \leqslant \ell\}.$$

Finally, given a sequence $f_n \in \mathcal{F}$, we have a subsequence $g_{n,1}$ converging uniformly on K, a subsequence $g_{n,2}$ of $g_{n,1}$ converging uniformly on K_2, and now the sequence $h_n = g_{n,n}$ converges uniformly on all K_ℓ and hence on all K. □

We are now ready to prove the Riemann mapping theorem, and in fact we have a stronger statement.

Theorem 3.2.7 (Riemann mapping theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a nonempty proper simply-connected open subset of $\mathbb{C}$. Fix $z_0 \in \Omega$. Then there exists a unique holomorphic bijection $F\colon \Omega \to B$ such that $F(z_0) = 0, F'(z_0) \in \mathbb{R}_{>0}$.*

Remark 3.2.8. Uniqueness follows from the Schwarz lemma. If we have F_1, F_2 two such functions, then $G = F_2 \circ F_1^{-1}\colon D \to D$ has $G(0) = 0$ and $G'(0) \in \mathbb{R}_{>0}$, so $G(z) = e^{i\theta} z$. But then $G'(0) = e^{i\theta} \in \mathbb{R}_{>0}$, so $G = \mathrm{id}$.

Proof. Consider

$$\mathcal{F} = \left\{g\colon \Omega \hookrightarrow D \mid g(z_0) = 0, g'(z_0) \in \mathbb{R}_{>0}\right\}.$$

By construction, $\mathcal{F}$ is uniformly bounded and thus normal.

First, we show that $\mathcal{F}$ is nonempty. By assumption $\Omega \neq \mathbb{C}$, so let $a \in \mathbb{C} \setminus \Omega$. Then we can define $f\colon \Omega \to \mathbb{C}$ defined by $z \mapsto \log(z - a)$. Then $e^{f(z)} = z - a$. Note that f is injective and recall that $e^{w + 2\pi i k} = e^{w}$ for all $k \in \mathbb{Z}$. Take $w_0 \in f(\Omega)$. Then $f(\Omega)$ is open, so $D(w_0, \delta) \subset f(\Omega)$ for some $\delta > 0$. Then we see that $D(w_0 + 2\pi i, \delta) \cap f(\Omega) = \emptyset$, so we can take

$$g(z) = \frac{1}{f(z) - (w_0 + 2\pi i)},$$

and we have $|g| < \frac{1}{\delta}$. Because f is injective, g is injective. Composing with translation and rotation, we can assume that $g(z_0) = 0$ and $g'(z_0) \in \mathbb{R} > 0$. Composing with scaling, we may assume that $|g| < 1$, so $g \in \mathcal{F}$.

Now let g_n be a sequence in $\mathcal{F}$ such that $g_n'(z_0) \to \sup_{g\in\mathcal{F}} g'(z_0)$. By Montel's theorem, $g_n \to g$ uniformly on compact sets for some $g\colon \Omega \to \mathbb{C}$. then g is holomorphic and $g_n' \to g'$ uniformly on compact sets, so in particular $g'(z_0) = \sup_{h\in\mathcal{F}} h'(z_0)$. We will prove that g is a bijection. To prove injectivity, fix $z_1, z_2 \in \Omega$. Define $\widetilde{g}_n(z) = g_n(z) = g_n(z_1)$ and $\widetilde{g}(z) = g(z) - g(z_1)$. Then $\widetilde{g}_n(z) \neq 0$ for $z \neq z_1$. Given $z_2 \in \Omega \setminus \{z_1\}$, there exists $\delta > 0$ such that $\widetilde{g}(z) \neq 0$ for $0 < |z - z_2| \leqslant \delta$. Also assume $|z_1 - z_2| > \delta$. Let $\gamma = \{z \mid |z - z_2| = \delta\}$. Then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{\widetilde{g}'(z)}{\widetilde{g}(z)}\,dz = \lim_{n\to\infty} \frac{1}{2\pi i}\frac{\widetilde{g}_n'(z)}{\widetilde{g}_n(z)}\,dz = \lim_{n\to\infty} 0 = 0,$$

so g is injective.

We will prove surjectivity by contradiction. Suppose there exists $\alpha \in D \setminus g(\Omega)$. Define $U = \psi_\alpha(g(\Omega))$ and $U \subset D$ is open, simply connected, and $0 \notin U$. We can define $k\colon U \to D$ by $k(z) = \sqrt{z} = e^{\frac{1}{2}\log z}$. Now we can set $G = r_\theta \circ \psi_\beta \circ k \circ \psi_\alpha \circ g$, where $\beta = k(\alpha)$. Equivalently, we have $g = \psi_\alpha \circ \ell \circ \psi_\beta \circ r_{-\theta} \circ G$, and set $\psi_\alpha \circ \ell \circ \psi_\beta \circ r_{-\theta} = L$. Then $L\colon D \to D$ fixes 0 and is not a rotation, so $|L'(0)| < 1$. Therefore $|g'(z_0)| = |L'(0) \cdot G'(0)| < |G'(z_0)|$ by the chain rule, which is a contradiction. $\square$

3.3 Elliptic Functions

Recall that meromorphic functions on $\mathbb{P}^1$ are the rational functions. We would like to study meromorphic functions on the complex torus $X = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for $0 \neq \omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_! \notin \mathbb{R}$. Equivalently, we want to consider doubly periodic meromorphic functions on $\mathbb{C}$.

Lemma 3.3.1. *If f is holomorphic on X, then f is constant.*

Proof. Clearly, f is given by values on a parallelogram, so f is bounded and therefore constant. $\square$

Remark 3.3.2. If X is a compact Riemann surface, then all holomorphic functions $f\colon X \to \mathbb{C}$ are constant. This is because $|f|$ is bounded and attains its maximum on some chart, so it must be constant.

Lemma 3.3.3. *Let f be meromorphic on $\mathbb{C}/\Lambda$. Then f has at least 2 poles.*

Proof. Integrating around the boundary of the parallelogram, we see that

$$0 = \int_\gamma f(z)\,dz = 2\pi i \sum_p \operatorname{Res}_p f$$

because $\sum \operatorname{Res}_p f = 0$ by an analog of the residue theorem. $\square$

Remark 3.3.4. There is a suitably generalized version of the residue theorem for any compact Riemann surface. First, define a meromorphic 1-form ω. Choose charts $\varphi_i\colon U_i \simeq V_i$. On each V_i, we have a form $f_i(z)\,dz$ with $f_i(z)$ meromorphic. Then compatibility with transitions g_{ij} is given by

$$f_i(z) = f_j(g_{ij}(z))g_{ij}'(z)\,dz,$$

and thus we can define the integral of ω along a curve $\gamma \subset X$. Now the residue theorem says that if X is a compact Riemann surface and ω is a meromorphic 1-form on X, then $\sum_{p \text{ pole}} \operatorname{Res}_p \omega = 0$.

For example, on $X = \mathbb{P}^1$, we can consider $\omega = \frac{1}{z}\,dz$. Then on the second chart, we see that $\omega = w \cdot d\left(\frac{1}{w}\right) = -\frac{1}{w}\,dw$. Then we see that $\sum_p \operatorname{Res}_p \omega = 1 + (-1) = 0$.

To prove the residue theorem, triangulate X such that each triangle is contained in some U_i. Then $\sum_{\Delta_i} \int_{\partial\Delta_i} \omega = 0 = 2\pi i \sum_p \operatorname{Res}_p \omega$.

Corollary 3.3.5. *Let* f *be meromorphic on a compact Riemann surface* X*. Then the number of zeroes of* f *equals the number of poles of* f*.*

Proof. Given f, we may define the meromorphic form df. Then we can consider the meromorphic form $\omega = \frac{df}{f}$. Then by the residue theorem we see that

$$0 = \sum_p \operatorname{Res}_p \omega = \#\{\text{zeroes}\} - \#\{\text{poles}\}. \qquad \square$$

The geometric meaning is that if $f \colon X \to \mathbb{P}^1$ is holomorphic, then for all α, $f^{-1}(\alpha) = \deg f$, counting multiplicity.

Now we will define an explicit meromorphic function $\wp$ on $\mathbb{C}/\Lambda$ with a unique double pole at $0 \in X$. Equivalently, $\wp \colon \mathbb{C} \to \mathbb{P}^1$ has a double pole at each point in Λ. Define

$$\wp(z) \coloneqq \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}$$

because the first attempt $\wp(z) = \sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^2}$ doesn't converge.

Lemma 3.3.6. *The function* $\wp(z)$ *converges absolutely and uniformly on compact sets* $K \in \mathbb{C} \setminus \Lambda$ *and has a double pole at each* $\omega \in \Lambda$*.*

Proof. First note that

$$\left| \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{z^2 + 2z\omega}{(z+2)^2 \omega^2} \right| \leqslant C \cdot \frac{1}{|\omega|^3}$$

for $|\omega|$ large and z bounded. Now we show that $\sum_{0 \neq \omega \in \Lambda} \frac{1}{|\omega|^3}$ converges. First, we know that $\omega = x_1\omega_1 + x_2\omega_2$. Then for any $x_1, x_2 \in \mathbb{R}$, we know $|x_1\omega_1 + x_2\omega_2| \geqslant C\sqrt{x_1^2 + x_2^2}$. This is because the function $f(x_1, x_2) = |x_1\omega_1 + x_2\omega_2|$ satisfies $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$. Therefore if $\|x\| = 1$, we know f lands in the positive reals and has a positive minimum. Therefore, we have reduced to the case where $\omega_1 = 1, \omega_2 = i$. Now we want to show that

$$\sum_{0 \neq \mathbf{x} \in \mathbb{Z}^2} \frac{1}{\sqrt{(x_1^2 + x_2^2)}^3}$$

converges. Using the integral test, we see that

$$\begin{aligned} \sum_{0 \neq (x_1, x_2) \in \mathbb{Z}^2} \frac{1}{(x_1^2 + x_2^2)^{\alpha/2}} &\leqslant C + \int_{R} \frac{1}{(x_1^2 + x_2^2)^{\alpha/2}} \, dx_1 \, dx_2 \\ &= C + \int_0^{2\pi} \int_1^\infty \frac{1}{r^\alpha} r \, dr \, d\theta \\ &= C + 2\pi \int_1^\infty \frac{1}{r^{\alpha-1}} \, dr, \end{aligned}$$

and therefore it converges for $\alpha > 2$. $\square$

Now we need to check that $\wp$ is doubly periodic. Here, we simply note that

$$\wp'(z) = \frac{-2}{z^3} + \sum_{0 \neq \omega \in \Lambda} \frac{-2}{(z+\omega)^3} = \sum_{\omega \in \Lambda} \frac{-2}{(z+\omega)^3}$$

is doubly periodic, so $\wp(z+\omega) - \wp(z)$ is constant. Now checking $\omega = \omega_1$ and $z = -\frac{1}{2}\omega$, we see that $\wp\left(\frac{1}{2}\omega\right) - \wp\left(-\frac{1}{2}\omega\right) = 0$ because $\wp$ is even.

Theorem 3.3.7. *We have the differential equation*

$$(\wp')^2 = 4(\wp - \alpha_1)(\wp - \alpha_2)(\wp - \alpha_3)$$

where $\alpha_1, \alpha_2, \alpha_3 = \wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_1+\omega_2}{2}\right)$.

Proof. We will compute the zeroes and poles of the two sides of the equations and show that they agree. Then their quotient is holomorphic on $X = \mathbb{C}/\Lambda$ and thus constant, so we can compute the constant. Recall that $\wp$ has a pole of order 2 at 0 and $\wp'$ has a pole of order 3 at 0. Next, $\wp$ is even so $\wp'$ is odd. Then we know that

$$\wp'\left(\frac{\omega}{2}\right) = \wp'\left(\frac{\omega}{2} - \omega\right) = \wp'\left(-\frac{\omega}{2}\right) = -\wp'\left(\frac{\omega}{2}\right)$$

and thus $\wp'\left(\frac{\omega}{2}\right) = 0$. Then $\wp - \wp\left(\frac{\omega}{2}\right)$ has a zero of order 2 at $\frac{\omega}{2}$.

Now to compute the constant, we use the Laurent series at the origin. We know that

$$\wp = \frac{1}{z^2} + \cdots \qquad \wp' = \frac{-2}{z^3} + \cdots \qquad (\wp')^2 = \frac{4}{z^6} + \cdots \qquad \prod_{i=1}^{3}(\wp - \alpha_i) = \frac{1}{z^6} + \cdots,$$

and thus $(\wp')^2 = 4\prod_{i=1}^{3}(\wp - \alpha_i)$. □

Now we have the identity

$$\int \frac{dw}{\sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)}} = \int \frac{\wp'(z)\,dz}{\wp'(z)} = z = \wp^{-1}(w),$$

where $w = \wp(z)$ and $(\wp')^2 = \prod_{i=1}^{3}(\wp - \alpha_i)$.

Theorem 3.3.8. *Every meromorphic function on* $X = \mathbb{C}/\Lambda$ *is a rational function of* $\wp, \wp'$. *This means that the field of meromorphic funcions on* X *is given by*

$$\mathbb{C}(X) = k(\mathbb{C}[x,y]/(y^2 - 4(x-\alpha_1)(x-\alpha_2)(x-\alpha_3))) = \mathbb{C}(x)[y]/(y^2 - 4(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)).$$

Therefore we can write $f \in \mathbb{C}(x)$ *uniquely as* $a(\wp) + b(\wp)\wp'$ *for* $a, b \in \mathbb{C}(x)$.

Proof. First suppose that f is **even**. We will show that f is rational in $\wp$. We may assume that f has no poles on $\Lambda \subset \mathbb{C}$ up to replacing f by $\wp^{-m}f$. Now if we write down a function with the same zeroes and poles and is a rational function of $\wp$. Recall that $\wp(z) - \wp(a)$ has zeroes $\begin{cases} a, -a & a \neq -a \\ a \text{ (twice)} & a \in \frac{1}{2}\Lambda \end{cases}$. But then for our function f, because f is even, the zeroes and poles of f come in pairs $a, -a$ or $a \in \frac{1}{2}\Lambda$ has even multiplicity. Then f has the same poles as

$$\frac{\prod(\wp - \wp(a_i))}{\prod(\wp - \wp(b_j))}$$

for some $a_i, b_j \in \mathbb{C}$. Note that $\wp$ has a pole of multiplicity 2 at $0 \in X$ and no other poles and f has no zero or pole at $0 \in X$ by assumption. But we are fine because $\#\{a_i\} = \#\big\{b_j\big\}$.

In the general case, suppose f is meromorphic on X. Then write

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}.$$

Then if g is odd, $\frac{g}{\wp'}$ is even and thus rational in $\wp$, so we obtain the desired description. $\square$

Now we can give a geometric description of this. Note that $X = \mathbb{C}/\Lambda \xrightarrow{\wp} \mathbb{P}^1$ has degree 2. It has branch points at $a_1 = \wp\big(\frac{\omega_1}{2}, a_2 = \wp\big(\frac{\omega_2}{2}\big)\big), a_3 = \wp\big(\frac{\omega_1+\omega_2}{2}\big), \infty$. Now we have a map

$$X \xrightarrow{\wp,\wp'} \sim(y^2 = (x - a_1)(x - a_2)(x - a_3)) \subset \mathbb{P}^2.$$

Now we can make two branch cuts, then expand the branch cuts into circles, and obtain half of a torus. Gluing two halves together, we obtain the torus. There is an involution $(x, y) \mapsto (x, -y)$ that exchanges the two sheets of the torus. As a picture, it looks like

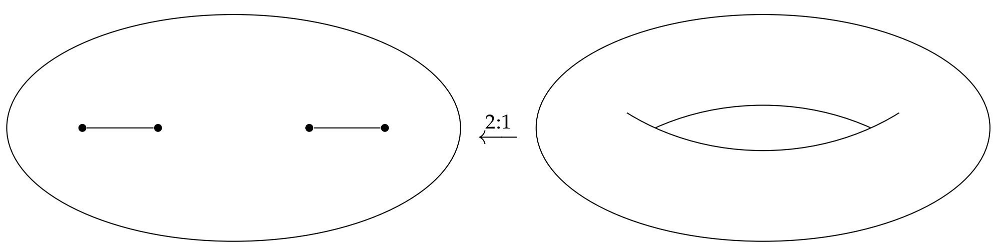

Figure 3.2: Branched cover of sphere by torus

3.4 Winding Numbers

Let $\gamma \subset \mathbb{C}$ be a closed path. For $a \in \mathbb{C} \setminus \gamma$, define the *winding number*

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - a}\, dz.$$

This is reasonable because if we consider the covering space $\mathbb{R} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then if we write $z(t) = a + r(t)e^{i\theta(t)}$, we have

$$\begin{aligned} n(\gamma, a) &= \frac{1}{2\pi i}\int_0^1 d(\log(z(t)-a)) \\ &= \frac{1}{2\pi i}\int_0^1 d(\log r(t)) + i\, d(\theta(t)) \\ &= \frac{1}{2\pi}(\theta(1)-\theta(0)). \end{aligned}$$

Theorem 3.4.1.

1. *For all a, γ, we have $n(\gamma, a) \in \mathbb{Z}$.*
2. *If a, b lie in the same component of $\mathbb{C} \setminus \gamma$, then $n(\gamma, a) = n(\gamma(b))$.*
3. *If a lies in the unbounded component of $\mathbb{C} \setminus \gamma$, then $n(\gamma, a) = 0$.*
4. *If γ, δ are homotopic in $\mathbb{C} \setminus \{a\}$, then $n(\gamma, a) = n(\delta, a)$.*

Proof.

1. Let $z\colon [0,1] \to \gamma \subset \mathbb{C}$. Then define
$$g(t) := \int_0^t \frac{z'(s)}{z(s)-a}\, ds\,.$$
Then we see that $\left((z(t)-a)e^{-g(t)}\right)' = z'(t)e^{-g(t)} - (z(t)-a)g'(t)e^{-g(t)} = 0$. But then $e^{g(1)} = e^{g(0)} \equiv c \cdot (z(0)-a)$, so $g(1)-g(0) \in 2\pi i\mathbb{Z}$. Therefore $n(\gamma, a) \in \mathbb{Z}$.

2. We know that the winding number $\mathbb{C} \setminus \gamma \to \mathbb{Z} \subset \mathbb{C}$ is continuous, so it must be locally constant.

3. Note that
$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\, dz$$
so as $a \to \infty$, $n(\gamma, a) \to 0$, but it must be constant, so $n(\gamma, a) = 0$.

4. Note that $\frac{1}{z-a}$ is holomorphic on $\mathbb{C} \setminus \{a\}$. Then the desired result follows from Theorem 2.7.1. □

We can compute winding numbers using *Alexander numbering*.

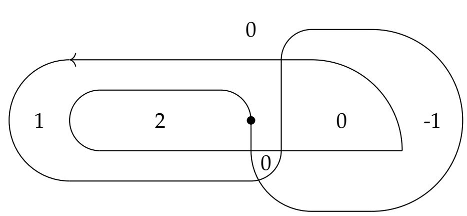

Figure 3.3: Winding numbers

We know the winding number vanishes on the unbounded component, and given $\alpha \uparrow \beta$, we have $n(\gamma, \alpha) = n(\gamma, \beta) + 1$ because in the diagram

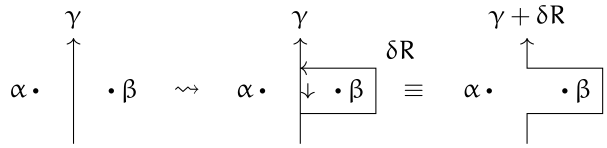

Figure 3.4: Pictoral proof of Alexander numbering

we have $n(\gamma, \alpha) = n(\gamma + \delta R, \alpha) = n(\gamma + \delta R, \beta) = n(\gamma, \beta) + 1$.

Theorem 3.4.2 (Cauchy). *Let $f\colon \Omega \to \mathbb{C}$ and $\gamma \subset \Omega$ be a closed curve such that $n(\gamma, a) \neq 0$ for all $a \in \mathbb{C} \setminus \Omega$. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Corollary 3.4.3 (General Residue Theorem). *Let $\Omega \subset \mathbb{C}$ be open and $S \subset \Omega$ be a discrete set. Suppose $f\colon \Omega \setminus S \to \mathbb{C}$ is holomorphic and $\gamma \subset \Omega \setminus S$ is a closed curve. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{a \in S} n(\gamma, a) \operatorname{Res}_a f.$$

Remark 3.4.4. The assumption that S is discrete implies that there are only finitely many $a \in S$ with nonzero winding number.

Proof. We will reduce to Cauchy. Set $\delta \coloneqq \gamma - \sum_{a \in S} n(\gamma, a) \cdot \gamma_a$, where γ_a is a small circle around a. By construction, $n(\delta, a) = 0$ for all $a \in \mathbb{C} \setminus (\Omega \setminus S)$ and therefore the ordinary Cauchy theorem implies

$$\int_\delta f(z)\,dz = \int_\gamma f(z)\,dz - \sum_{a \in S} n(\gamma, a) \int_{\gamma_a} f(z)\,dz = 0,$$

as desired. □

Before we prove Cauchy, we need the following lemma:

Lemma 3.4.5. *Let $\Omega \subset \mathbb{C}$ and $\gamma\colon [a, b] \to \Omega$ be a path. Then there exists a rectangular path $\eta\colon [a, b] \to \Omega$ such that there exists a subdivision $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma(a_i) = \eta(a_i)$ for all i and there exist disks $D_i \subset \Omega$ such that $\gamma([a_i, a_{i+1}]), \eta([a_i, a_{i+1}]) \subset D_i$.*

Proof. There exists $\varepsilon > 0$ such that if $z \in \gamma$, then $D(z, \varepsilon) \subset \Omega$. Then there exists $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < \varepsilon$ for $|s - t| < \delta$. Subdivide $a = a_0 < a_1 < \cdots < a_n = b$ such that $|a_{i+1} - a_i| < \delta$ and set $D_i = D(\gamma(a_i), \varepsilon) \subset \Omega$. Now we can build the rectangular path inside each D_i. □

Proof of Cauchy. We may assume that γ is a rectangular path. Then there exist rectangles R_i such that $\gamma = \sum m_i \partial R_i$ for some $m_i \in \mathbb{Z}$. To see this, just draw a fine enough rectangular grid until every segment of γ is aprt of the grid.

Now we need to show that if $n(\gamma, a_i) \neq 0$, then $R_i \subset \Omega$, where a_i is in the interior of R_i. But this is because $R_i^\circ \subset A$ for some connected component A of $C \setminus \gamma$. By assumption, $A \subset \Omega$, so $R_i \subset \overline{A} \subset \Omega$.

Now we show that $\gamma = \sum n(\gamma, a_i)\partial R_i$, where $a_i \in R_i^\circ$. We will show that if $\delta \coloneqq \gamma - \sum n(\gamma, a_i)\partial R_i$, then $\delta = 0$. First, we know that $n(\delta, a) = 0$ for all $a \in \mathbb{C} \setminus \delta$ (simply check in each rectangle). Now if $\delta \neq 0$, then δ contains some component $m\sigma$, so $\delta = m\sigma + \delta'$ for some $m \neq 0$ and $\sigma \not\subset \delta'$. But then if σ splits α, α', we have $n(\delta, \alpha) = n(\delta, \alpha') + m$, a contradiction. ☐

Theorem 3.4.6. *Let $\Omega \subset \mathbb{C}$ be open and connected. Then the following are equivalent:*

1. *Ω is simply connected.*
2. *$\mathbb{P}^1 \setminus \Omega$ is connected.*
3. *For all closed curves γ in Ω, then $n(\gamma, a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$.*

Proof.

2 implies 3: If $\mathbb{P}^1 \setminus \Omega$ is connected, then $\mathbb{C} \setminus \Omega$ is contained in the unbounded component of $\mathbb{C} \setminus \gamma$, and thus $n(\gamma, a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$.

3 implies 2: Suppose $\mathbb{P}^1 \setminus \Omega$ is not connected. Then we can write $\mathbb{P}^1 \setminus \Omega = A \cup B$ for closed and disjoint A, B where $B \in \infty$. Now we need to produce $\gamma \subset \Omega$ such that $n(\gamma, a) \neq 0$ for some $a \in \mathbb{C} \setminus \Omega$. There exists $\delta > 0$ such that $d(A, B) > \delta$. Then take the square grid with side length less than $\frac{\delta}{\sqrt{2}}$ and now take

$$\gamma = \sum_{R \cap A \neq \emptyset} \partial R.$$

Observe that $\gamma \cap A = \emptyset$ and $\gamma \cap A = \emptyset$, so $\gamma \subset \Omega$. But then we see that $n(\gamma, a) = 1$ for all $a \in A \cap R^\circ$, where R is a rectangle in the grid.

1 implies 3: Recall that $n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz$. If Ω is simply connected, then $\int_\gamma f(z)\, dz = 0$ for all $\gamma \subset \Omega$ and $f\colon \Omega \to \mathbb{C}$. Now apply this to $f = \frac{1}{z-a}$ for $a \notin \Omega$.

3 implies 1: Recall that there exists a holomorphic bijection $F\colon \Omega \xrightarrow{\sim} D$ if Ω is simply conencted. However, the simply connected assumption was only used to define $\log(z-a)$, $\sqrt{z-a}$ for $a \in \Omega$. Then $\log(z-a)$ is defined as the primitive of $\frac{1}{z-a}$. Now we know that if f is holomorphic on Ω, then $\int_\gamma f(z)\, dz = 0$ for all $\gamma \subset \Omega$ closed curves, so f has a primitive g. But then we obtain a holomorphic bijection $F\colon \Omega \to D$, and thus Ω is simply connected. ☐

Remark 3.4.7. In general, $n(\gamma, a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$ does not imply that γ is homotopic to a constant path. For example, if $\Omega = \mathbb{C} \setminus \{a, b\}$ and γ is not homotopic and u, v generate $\pi_1(\Omega) = \mathbb{Z} * \mathbb{Z}$, then the loop $\gamma = uvu^{-1}v^{-1}$ satisfies $n(\gamma, a) = n(\gamma, b) = 0$.

Now we would like to apply the general Cauchy's theorem in practice. Let $\Omega \subset \mathbb{C}$ be open and connected and suppose $\mathbb{P}^1 \setminus \Omega = A_1 \cup \cdots \cup A_N \cup A_\infty \ni \infty$ has finitely many components. Then there exist curves γ_j in Ω such that if we choose $a_i \in A_i$, then $n(\gamma_i, a_j) = \delta_{ij}$. Now if $\gamma \subset \Omega$, it has the same winding numbers around $a \in \Omega$ as $\sum n(\gamma, a_i)\gamma_i$, so by Cauchy, we have

$$\int_{\gamma - \sum n(\gamma, a_i)\gamma_i} f(z)\, dz = 0$$

for $f\colon \Omega \to \mathbb{C}$ holomorphic. Therefore

$$\int_\gamma f(z)\, dz = \sum n(\gamma, a_i) \int_{\gamma_i} f(z)\, dz.$$

In particular, f has a primitive if and only if $\int_{\gamma_i} f(z)\, dz = 0$ for $i = 1, \ldots, N$.

Example 3.4.8. Consider $\Omega = \{z \in \mathbb{C} \mid |z| > 4\}$ and $f(z) = \frac{z}{(z-1)(z-2)(z-3)}$. Now we see that if γ is a closed curve in Ω, then

$$\begin{aligned}\int_{\gamma_1} f(z)\,dz &= n(\gamma,0)\int_{\gamma_1} f(z)\,dz \\ &= n(\gamma,0)2\pi i \sum_{a=1,2,3} \operatorname{Res}_a f \\ &= n(\gamma,0)2\pi i\left(\frac{1}{(1-2)(1-3)} + \frac{2}{(2-1)(2-3)} + \frac{3}{(3-1)(3-2)}\right) \\ &= n(\gamma,0)\left(\frac{1}{2} - 2 + \frac{3}{2}\right) = 0,\end{aligned}$$

so f does have a primitive.

Example 3.4.9. We want to compute the integral

$$\int_{|z|=2} \sqrt{z^2-1}\,dz\,.$$

If we consider $\Omega = \mathbb{C} \setminus [-1,1]$, then we may consider $\gamma_{\varepsilon,\delta}$ given by

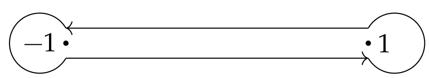

Figure 3.5: Contour $\gamma_{\varepsilon,\delta}$

with radius ε and width δ. Then $\int_\gamma = \int_{\gamma_{\varepsilon,\delta}}$. Then we have

$$\begin{aligned}\lim_{\varepsilon,\delta\to 0}\int_{\gamma_{\varepsilon,\delta}} \sqrt{z^2-1}\,dz &= \int_{-1}^{1} i\cdot\sqrt{1-x^2}\,dx + \int_{-1}^{1} -i\sqrt{1-x^2}\,dx \\ &= -2i\int_{-1}^{1}\sqrt{1-x^2}\,dx = -\pi i.\end{aligned}$$

Alternatively, we may use the transformation $z = \frac{1}{2}$ to obtain

$$\int_{|z|=2}\sqrt{z^2-1}\,dz = \int_{|w|=\frac{1}{2}} \frac{\sqrt{1-w^2}}{w^3}\,dw = 2\pi i \operatorname{Res}_0 \frac{\sqrt{1-w^2}}{w^3} = 2\pi i\cdot\frac{-1}{2} = -\pi i.$$

Example 3.4.10. Consider the contour γ given by

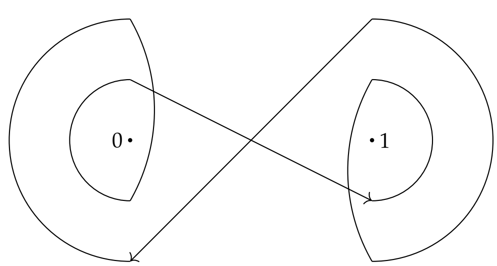

Figure 3.6: Contour γ

Then we have

$$\begin{aligned}\int_\gamma \frac{e^z-1}{z^2(z-1)}\,dz &= 2\pi i(n(\gamma,0)\operatorname{Res}_0 f + n(\gamma,1)\operatorname{Res}_1 f)\\ &= 2\pi i(-2(-1)+2(e-1))\\ &= 2\pi i e.\end{aligned}$$

3.5 Harmonic Functions

We say $u\colon \Omega \to \mathbb{R}$ is *harmonic* if it has continuous second partial derivatives and satisfies **Laplace's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Harmonic functions have applications in electricity and magnetism, fluid dynamics, gravity, and heat conduction.

Lemma 3.5.1. *Let $f\colon \Omega \to \mathbb{C}$ be holomorphic and write $f(x+iy) = u(x,y) + iv(x,y)$. Then u, v are harmonic.*

Proof. By the Cauchy-Riemann equations, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = 0$$

and similar for v. ☐

There is a converse to this.

Theorem 3.5.2. *Let $\Omega \subset \mathbb{C}$ be simply connected and $u\colon \Omega \to \mathbb{R}$ be harmonic. Then there exists a holomorphic $f\colon \Omega \to \mathbb{C}$ such that $u = \operatorname{Re}(f)$. Moreover, f is unique up to $f \rightsquigarrow f + ic$ for $c \in \mathbb{R}$.*

Proof. Define $g \coloneqq \frac{\partial u}{\partial u} - i\frac{\partial u}{\partial y}$ and integrate. Then we see that g is holomorphic because the components have continuous first derivatives and the Cauchy-Riemann equations are satisfied because u is harmonic and by symmetry of mixed partials.

Then because Ω is simply connected, there exists a primitive $f\colon \Omega \to \mathbb{C}$ such that $f' = g$. Now write $f = \widetilde{u} + i\widetilde{v}$ and note that

$$\begin{aligned} f' = \frac{\partial \widetilde{u}}{\partial x} + i\frac{\partial \widetilde{v}}{\partial x} &= \frac{\partial \widetilde{u}}{\partial x} - i\frac{\partial \widetilde{u}}{\partial y}\\ = g &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.\end{aligned}$$

Therefore $\widetilde{u} = u + a$ for some $a \in \mathbb{R}$, so up to $f \rightsquigarrow f - a$, we have $\operatorname{Re}(f) = u$.

To show uniqueness, note that if f_1, f_2 are two such functions, then $\operatorname{Re}(f_1 - f_2) = 0$ and thus $f_1 - f_2 \in i\mathbb{R}$. But this implies that Ω is sent to something that is not open so $f_1 - f_2$ is a constant. ☐

Theorem 3.5.3. *Let $u\colon \Omega \to \mathbb{R}$ be harmonic. Then for all $z \in \Omega, r \in \mathbb{R}_{>0}$ such that $\overline{D}(z,r) \subset \Omega$, we have*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\,d\theta\,.$$

Proof. There exists $D = D(z, r') \subset \Omega$ for $r' > r$ and f holomorphic on D such that $u\big|_D = \mathrm{Re}(f)$. Applying the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z,r)} \frac{f(w)}{z-w}\,dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta\,.$$

Now the desired result follows by taking real parts. □

Theorem 3.5.4 (Maximum principle for harmonic functions)**.** *Let u be harmonic on a connected $\Omega \subset \mathbb{C}$. Then if u has a maximum in Ω, then u is constant.*

Proof. If Ω is simply connected, then we know u is the real part of some holomorphic f, so using the maximum principle for $g = e^f$, we obtain the desired result. In general, suppose u has a maximum c at a point $z_0 \in \Omega$. Then for $z_0 \in D \subset \Omega$ we know u is constant on D. Now define $U = \{z \in \Omega \mid u \equiv c \text{ near } z\}$. By definition, U is open, but U is also closed in Ω, so $U = \Omega$. □

Example 3.5.5. The function $u(x,y) = \log\left(\sqrt{x^2+y^2}\right) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is harmonic.

Corollary 3.5.6. *Let $\Omega \subset \mathbb{C} = \mathbb{R}^2$ be bounded. Suppose $u_1, u_2 \colon \overline{\Omega} \to \mathbb{R}$ are continuous on $\overline{\Omega}$ and harmonic on Ω. Further suppose that $u_1 = u_2$ on $\partial\Omega$. Then $u_1 = u_2$.*

Proof. Write $u = u_1 - u_2$. We know $u = 0$ on $\partial\Omega$ and is harmonic on Ω and continuous on $\overline{\Omega}$. By the maximum principle, if u has a max in Ω then it is constant, but we know u attains a maximum on $\overline{\Omega}$ Therefore $u \leqslant 0$. Applying the same reasoning to $-u$, we see that $u \geqslant 0$. □

Now we would like to study the Dirichlet problem. Given $\Omega \subset \mathbb{C} = \mathbb{R}^2$ and $g \colon \partial\Omega \to \mathbb{R}$ continuous, we want to find $u \colon \overline{\Omega} \to \mathbb{R}$ continuous on $\overline{\Omega}$ and harmonic on Ω such that $u\big|_{\partial\Omega} = g$.

Example 3.5.7. Let $\Omega = A = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$ and set $u(z) = 0$ if $|z| = R_1$ and $u(z) = 1$ when $|z| = R_2$. Then the Laplace equation is a linear PDE and we expect $u = u(r) = u(\sqrt{x^2+y^2})$. In fact, we can set

$$u = \frac{\log\sqrt{x^2+y^2} - \log R_1}{\log R_2 - \log R_1}.$$

Now the generalized Dirichlet problem is as follows: Suppose Ω is simply connected with piecewise smooth boundary. Then the Riemann mapping theorem gives us a holomrophic bijection $F \colon \Omega \to D$. Then this extends to a homeomorphism $\overline{F} \colon \overline{\Omega} \to \overline{D}$. Now fix a finite set $S \subset \partial\Omega$ and continuous $g \colon \partial\Omega \setminus S \to \mathbb{R}$.

Problem 3.5.8 (Generalized Dirichlet problem)**.** *Find a continuous function $u \colon \overline{\Omega} \setminus S \to \mathbb{R}$ that is harmonic on Ω such that $u\big|_{\partial\Omega\setminus S} = g$. Note that if such a u exists, then it is unique.*

Example 3.5.9. Let $\Omega = \mathcal{H}$, $S = 0$, and set $g(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases}$

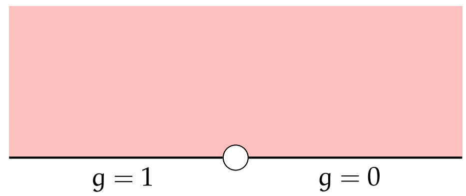

Figure 3.7: Generalized Dirichlet problem on upper half plane.

Then we take $u = \frac{\arg(z)}{\pi}$.

Example 3.5.10. Consider the generalized Dirichlet problem given by

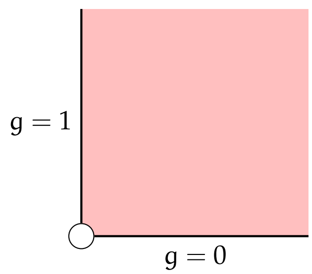

Figure 3.8: Generalized Dirichlet problem on first quadrant.

Then we may take $u = \frac{2}{\pi}\arg(z)$.

Lemma 3.5.11. *Let* $f\colon \Omega_2 \to \Omega_1$ *be holomorphic and* $u_1\colon \Omega_1 \to \mathbb{R}$ *be harmonic. Then* $u_2 = u_1 \circ f$ *is harmonic.*

Proof. Locally, we can write $u_1 = \mathrm{Re}(g)$ for some holomorphic g. Then $u_2 = \mathrm{Re}(g \circ f)$ must be harmonic. $\square$

Example 3.5.12. Consider $\Omega = \{z \mid |z| < 1\}$ and $S = \{1, i\}$.

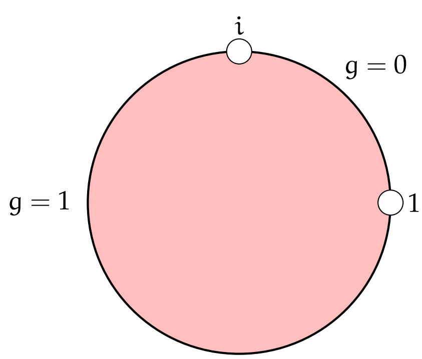

Figure 3.9: Generalized Dirichlet problem on disk

Then a Möbius transformation $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ taking $D \to \mathcal{H}$ and $1, i, -1 \mapsto \infty, 0, 1$ is given by

$$f(z) = \frac{z-i}{z-1}\frac{-1-1}{-1-i} = \frac{z-i}{z-1}\frac{2}{1+i},$$

so we may take

$$\begin{aligned} u(z) &= (1 - \frac{1}{\pi}\arg\left(\frac{z-i}{z-1}\frac{2}{1+i}\right)) \\ &= 1 - \frac{1}{\pi}(\arg(1-i) + \arg(z-i) - \arg(z-1)) \\ &= \frac{5}{4} - \frac{1}{\pi}(\arg(z-i) - \arg(z-1)). \end{aligned}$$

Now we will give a general solution of the Dirichlet problem for the disk. Recall the mean value property

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$$

for $0 < r < 1$. Assume that u extends continuously to $\overline{D} \setminus S$ and is bounded. Then as $r \to 0$, we see that $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$. Now we use the Blaschke factor

$$\psi_z\colon D \xrightarrow{\sim} D \qquad w \mapsto \frac{z-w}{1-\bar{z}w}.$$

Then if $\tilde{u} = u \circ \psi_z$, we have

$$\begin{aligned} u(z) &= \tilde{u}(0) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{i\phi})\,d\phi \\ &= \frac{1}{2\pi}\int_{|v|=1} \tilde{u}(v)\,d(\arg v) \\ &= \frac{1}{2\pi}\int_{|v|=1} \tilde{u}(v)\frac{1}{i}\frac{dv}{v} \\ &= \frac{1}{2\pi}\int_{|w|=1} u(w)\frac{1-|z|^2}{|w-z|^2}\frac{1}{i}\frac{dw}{w} \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\phi})\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi \end{aligned}$$

Because we have

$$\begin{aligned} \frac{dv}{v} &= \frac{d(z-w)}{z-w} - \frac{d(1-\bar{z}w)}{1-\bar{z}w} \\ &= dw\left(\frac{-1}{z-2} + \frac{\bar{z}}{z-\bar{z}w}\right) \\ &= dw\left(\frac{-1+\overline{zw}+z\bar{z}-w\bar{z}}{(z-w)(1-\bar{z}w)}\right) \\ &= \frac{dw}{w}\left(\frac{-1+|z|^2}{(z-w)(\overline{w}-\bar{z})}\right) \\ &= \frac{dw}{w}\frac{1-|z|^2}{|w-z|^2}. \end{aligned}$$

Now if Ω is simply connected, we have $\overline{F}\colon \overline{\Omega} \xrightarrow{\sim} \overline{D}$ that restricts to a holomorphic bijection $F\colon \Omega \xrightarrow{\sim} D$. Then if $\widetilde{u}\colon D \to \mathbb{R}$ is a solution on D, $\widetilde{u} \circ \overline{F}$ is a solution on Ω.