Math 703: Manifolds

Taught by Michael Sullivan; Notes by Patrick Lei

University of Massachusetts, Amherst

Fall 2019

Abstract

Topics to be covered: smooth manifolds, smooth maps, tangent vectors, vector fields, vector bundles (in particular, tangent and cotangent bundles), submersions,immersions and embeddings, sub-manifolds, Lie groups and Lie group actions, Whitney's theorems and transversality, tensors and tensor fields, differential forms, orientations and integration on manifolds, The De Rham Cohomology, integral curves and flows, Lie derivatives, The Frobenius Theorem.

Contents

1 Lecture 1 (Sep 4)

1.1 Overview We will be following Lee's book on smooth manifolds. Because this is no longer an exam class, there will be homework and a take home final. Prereqs are calculus, point-set topology and a little bit of π_1. Homework 0 will be to go to `math.stonybrook.edu/Videos/IMS/DifferentialTopology` or search for "John Milnor 1965 Hedrick Lectures."[1][2]

We will give an overview. In topology, we have the notion of continuity with equivalence being either homeomorphism or homotopy equivalence. Then we specialize to differential topology by introducing smoothness. Inside differential topology, we have Riemannian geometry ($O(n)$), complex geometry ($GL_n(\mathbb{C})$), and symplectic geometry ($Sp(2n)$). The intersection of these three things is Kähler geometry ($U(n)$).

We will begin with some examples and nonexamples of manifolds.

Example 1. Some examples of manifolds are circles, spheres, tori, and $\mathbb{R}^n$.

Example 2. Some non-examples of manifolds are two intersecting lines, a sphere with a line attached, and the graph of $y = |x|$.

We may also consider functions from a manifold to $\mathbb{R}$ and the consider the level sets. Then the preimages of generic values are manifolds, while at critical points, the preimages are not manifolds.[3] In addition, manifolds can be intersected transversally to form new manifolds.

1.2 Basic Notions and Examples

Definition 3. A *topological n-manifold* M is a second-countable Hausdorff topological space M that is locally Euclidean of dimension n.

Definition 4. Two charts $(U, \varphi), (V, \psi)$ are smoothly compatible if the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Definition 5. A smooth atlas on M is a collection of smoothly compatible charts $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ that cover M.

Definition 6. A smooth structure on M is a maximal smooth atlas.

Lemma 7. *Every smooth atlas on M is contained in a unique maximal smooth atlas. Two smooth atlases determine the same smooth structure iff their union is a smooth atlas.*

[1] Mike says he prepared this lecture and then watched these videos and thought "I should have taught it this way."
[2] Milnor was also the person who invented differential topology and won the Fields Medal.
[3] He just did Morse theory without saying it.

Example 8. The simplest example of a manifold is $\mathbb{R}^n$ with the standard smooth structure.[4] We can also consider the set of matrices $M_{m\times n}(\mathbb{R})$ identified with R^{mn}.

Remark 9. Hausdorffness and second-countability are preserved under both subspace and product topologies.

Example 10. If $N \subseteq M$ is open, then N is a smooth n-manifold. For example, we can consider $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$.

Example 11. If M_1, M_2 are smooth of dimensions n_1, n_2, then $M_1 \times M_2$ is a smooth (n_1+n_2)-manifold.

Example 12. Consider $S^2 \subset \mathbb{R}^3$. The standard smooth structure is defined using stereographic projection to $\mathbb{R}^2$ by

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1 \pm x_3}, \frac{x_2}{1 \pm x_3}\right).$$

Homework 0.1 is to check that the transition is smooth.

More generally, we can define the standard structure on $S^n \subset \mathbb{R}^{n+1}$. However, there may be exotic smooth structures on S^n. For example, there are exotic spheres of dimension 7, 13, 14, 15, 16, and many higher dimensions. This question is open for $n = 4$.

Example 13. The torus $T^2 \simeq S^1 \times S^1$ has a unique smooth structure.

Example 14. $\mathbb{RP}^2$ is the space of lines through the origin in $\mathbb{R}^3$. We will attempt to build a natural smooth structure.

Lemma 15. *Given a set M and a collection $\{U_\alpha\}$ of subsets with injections $\varphi_\alpha : U_\alpha \hookrightarrow \mathbb{R}^n$ with*

1. *For all α, $\varphi_\alpha(U_\alpha)$ is open;*
2. *$\varphi_\alpha(U_\alpha \cap U_\beta)$ is open;*
3. *The transitions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms;*
4. *M is covered by countably many U_α;*

Then M has a topology with basis $\varphi_\alpha^{-1}(V)$ for all $V \subset \mathbb{R}^n$ open. Moreover, if the topology is Hausdorff, then M has a unique smooth structure where $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas.

Continuing Example 14, we will take the sets $U_\alpha = (x_\alpha \neq 0)$ to be the standard Euclidean charts. Clearly the φ_α are injective (in fact they are bijective). Next, the intersections are both open. Third, the transitions are given by

$$\varphi_2 \circ \varphi_1^{-1} : (x_1, x_2) \mapsto \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right),$$

so they are diffeomorphisms. Next time we will check that $\mathbb{RP}^n$ is Hausdorff.

[4] This is misleading because there are uncontably many exotic $\mathbb{R}^4$.

2 LECTURE 2 (SEP 9)

The first homework has been posted. It is due in 14 days. The problems from the book are 1.1, 1.5, 1.7, 2.1, 2.4, 2.10, and 2.14. In addition, prove that diffeomorphism is an equivalence relation and construct a smooth structure on the square.

Today we will complete Example 14 and show that $\mathbb{RP}^2$ is Hausdorff. First assume that $l_1, l_2 \in \mathbb{RP}^2$ are in the same Euclidean patch. Then $\mathbb{R}^2$ is Hausdorff, so there are disjoint neighborhoods $V_i \ni l_i$. Now suppose $l_1 \in U_1 \setminus U_2, l_2 \in U_2 \setminus U_1$. Thus $l_1 = [1 : 0 : u_1], l_2 = [0 : 1 : u_2]$. Then we can write $\varphi_i(l_i) = (0, u_i) \in \mathbb{R}^2$ for some $u_i \neq 0$. Then set $V_i = B_\varepsilon(0, u_i)$.

Finally, we show that for small ε, the $\varphi_i^{-1}(V_i)$ are disjoint. If they intersect, then we obtain $l = [1 : y_1^1 : y_2^1 + u_1] = [y_1^2 : 1 : y_2^2 + u_2]$, which implies that $y_1^1 y_1^2 = 1$, contradicting our assumption on the size of ε.

Definition 16. A topological manifold M is a *complex n-manifold* if M admits a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}$ to $\mathbb{C}^n$. Here holomorphic is taken to mean $J \cdot Df = Df \cdot J$, where J is a matrix corresponding to multiplication by I.[5]

Theorem 17. *Lemma 15 holds in the holomorphic setting.*

Example 18. $\mathbb{CP}^n$ is the set of (complex) lines in $\mathbb{C}^{n+1}$ through the origin. Homework 1 will show that this is a complex n-manifold.

2.1 Morphisms

We will now construct morphisms of smooth manifolds.

Definition 19. Let M be a smooth manifold. $f : M \to \mathbb{R}$ is a *smooth function* if for all $p \in M$ there exists a chart (U, φ) with $p \in U$ such that $f \circ \varphi^{-1}$ is smooth.

Definition 20. Let M, N be smooth manifolds. $f : M \to N$ is a *smooth map* if for all $p \in M$ there exist charts (U, φ) with $p \in U$ and (V, ψ) with $f(p) \in V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth.

Remark 21. Smoothness is independent of the choice of charts.

Definition 22. Let $F : M \to N$ be a smooth map. F is a *diffeomorphism* if F^{-1} exists and is smooth.

Definition 23. $F : M \to N$ is a *local diffeomorphism* if for all $p \in M$ there exists an open $U \ni p$ such that $F(U)$ is open and $F|_U$ is a diffeomorphism to $F(U)$.

Proposition 24. *The following are true:*

1. *Smooth implies continuous, but the converse is false;*
2. *Smooth maps make* **Diff** *into a category;*
3. *The set of smooth functions $C^\infty(M)$ is a commutative ring;*
4. *Smooth maps $M \to N$ pull back smooth functions $C^\infty(N) \to C^\infty(M)$.*[6]

Example 25. Consider the following basic examples:

1. If $N \subset M$ is open and M is a smooth manifold, then the inclusion $\iota : N \hookrightarrow M$ is smooth;

[5] Note that in general this only gives us an almost complex structure.
[6] This is entirely analogous to the case of algebraic geometry.

2. **Diff** has products. In addition, the inclusions $m \mapsto (m, 0)$ are smooth.

Example 26. We will see that the inclusion $S^2 \hookrightarrow \mathbb{R}^3$ is smooth. To do this, compute with the coordinate charts. Also, the inverses of the coordinate charts are given by

$$(y_1, y_2) \mapsto \frac{1}{y_1^2 + y_2^2 + 1}(2y_1, 2y_2, \pm(1 - y_1^2 - y_2^2)).$$

Example 27. We show that the projection $\mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ is smooth. In the charts, we have

$$(x_1, x_2, x_3) \mapsto \left[1 : \frac{x_2}{x_1} : \frac{x_3}{x_1}\right],$$

which are smooth.

Example 28. The Hopf fibration $S^3 \to \mathbb{CP}^1$ is smooth.[7]

Example 29. The smooth composition of Examples 26 and 27 is a local diffeomorphism. In fact, this is a 2-to-1 cover of $\mathbb{RP}^2$ and demonstrates that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

3 Lecture 3 (Sep 11)

We began class with Mike getting to know everyone. There are people enrolled in the class who were not here, but there was apparently a good mix of algebra, analysis, and geometry.

3.1 Example 29 Continued We will show that the map $S^2 \to \mathbb{RP}^2$ in Example 29 is a local diffeomorphism. Assume $l = F(p)$ is contained in the first Euclidean chart. Then

$$p = \pm\frac{1}{\sqrt{1 + x_1^2 + x_2^2}}(1, x_1, x_2).$$

Then for some small $\varepsilon > 0$ let $W = B_\varepsilon(y_1, y_2)$ and set $V = \varphi_1^{-1}(W)$. Then note that the preimage of any point consists of two antipodal points. Thus for sufficiently small ε, $F_{-1}(V)$ is a disjoint union of two open sets. Then this is easy to see that $F|_U$ is injective. Next we prove that the inverse of $F|_U$ is smooth. We can do this by computing the inverse explicitly in charts.

Now recall the definition of a covering map from topology. We may replace **Top** with **Diff**, obtaining the notion of a *smooth covering map.*

Example 30. $S^2 \to \mathbb{RP}^n$ is a smooth cover.

Proposition 31. *If M is a connected smooth n-manifold and $\pi : \widetilde{M} \to M$ is a topological covering map, then $\widetilde{M}$ has a unique smooth structure such that π is a smooth cover.*

Now note that any manifold is locally contractible, so it is locally connected and locally simply connected. Therefore any smooth connected manifold has a smooth universal cover.

[7] The existence of this morphism shows that $\pi_3(S^2) \neq 0$.

Definition 32. A Lie group G is a group object in **Diff**. More concretely, it is a group which is also a smooth manifold such that multiplication and inverse are smooth.

Example 33. Some examples of Lie groups are:

1. $G = \mathbb{R}^n$ is the simplest Lie group;
2. $G = GL_n(\mathbb{C})$;
3. $G = S^1 \hookrightarrow C^* = GL_1(C)$;
4. $G = S^3 \hookrightarrow \mathbb{H} \setminus \{0\}$.

Theorem 34. *Let G be a connected Lie group. Then there exists a smooth universal cover $\pi : \widetilde{G} \to G$ which is a morphism of Lie groups.*

Remark 35. Lee has a chapter on Lie groups, but we will probably not get to them in this course.

3.2 Partitions of Unity

Definition 36. Let M be a topological space and $\mathcal{X} = \{U_\alpha\}$ be any open cover of M. A *partition of unity* subordinate to $\mathcal{X}$ is a collection of continuous functions $\{f_\alpha\}$ such that:

1. $0 \leqslant f_\alpha \leqslant 1$;
2. $\operatorname{supp} f_\alpha \subset U_\alpha$.[8]
3. For all $x \in M$, there exists $U \ni x$ such that only finitely many f_α have support intersecting U.
4. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$.

Theorem 37. *Let M be a smooth manifold and $\mathcal{X} = \{X_\alpha\}$ be an open cover of M. Then there exists a partition of unity $\{f_\alpha\}$ subordinate to $\mathcal{X}$.*[9]

Before we prove Theorem 37, we need some preliminary notions.

Lemma 38. *There exists an $h \in C^\infty(\mathbb{R})$ such that*

$$h(t) = \begin{cases} 1 & t \leqslant 1 \\ 0 & t \geqslant 2 \\ h(t) \in (0,1) & 1 < t < 2 \end{cases}$$

Sketch of Proof. First note that

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leqslant 0 \end{cases}$$

is smooth. Then we can build h. □

Lemma 39. *The bump function $H : \mathbb{R}^n \to \mathbb{R}$ given by $x \mapsto h(|x|)$ is smooth.*

[8] These do not need to be compactly supported, but in practice they will be.

[9] This is like Seifert-Van Kampen and Mayer-Vietoris in algebraic topology, which allow us to compute things locally.

Definition 40. Suppose X is a topological space. Given an open cover $\mathcal{U}$ of X, another open cover $\mathcal{V}$ of X is a *refinement* of $\mathcal{U}$ if for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$.

Definition 41. X is *paracompact* if every cover of X admits a locally finite refinement.

Definition 42. Let X be a smooth manifold. Let $\{W_i\}$ be an open cover of M. Then W is *regular* if the following holds:

1. The cover $\{W_i\}$ is countable and locally finite;
2. Each W_i is the domain of a smooth coordinate map φ_i such that $\varphi_i(W_i) = B_3(0) \subset \mathbb{R}^n$.[10]
3. Let $U_i = \varphi_i^{-1}(B_1(0))$. Then the $\{U_i\}$ cover M.

Proposition 43. *Let M be a smooth manifold. Then every open cover of M has a regular refinement.In particular, M is paracompact.*[11]

Proof of Theorem 37. First we will build a partition of unity subordinate to a regular refinement $\{W_i\}$ of our cover $\mathcal{X}$. Let $U_i = \varphi_i^{-1}(B_1(0))$ and $V_i = \varphi_i^{-1}(B_2(0))$. Recall that $W_i = \varphi_i^{-1}(B_3(0))$. Now define

$$f_i = \begin{cases} H \circ \varphi_i & x \in W_i \\ 0 & c \in M \setminus W_i \end{cases}$$

and set

$$g_i(x) = \frac{f_i(x)}{\sum_j f_j(x)}.$$

This is well-defined because the U_i cover M. Also, $0 \leqslant g_i(x) \leqslant 1$. In addition, $\sum_i g_i(x) = 1$ and $\operatorname{supp} g_i \subset W_i$. Thus $\{g_i\}$ is a partition of unity subordinate to $\{W_i\}$.

We now need to construct $\{f_\alpha\}$. Because $\{W_i\}$ is a refinement of $\mathcal{X}$, then for all i there exists $\alpha = \rho(i)$ such that $W_i \subset X_\alpha$. Then for all α, define $f_\alpha = \sum_{i \in \rho^{-1}(\alpha)} g_i$. Then it is clear that conditions $1, 4$ of being a partition of unity hold.

It is also easy to see that $\operatorname{supp} f_\alpha \subset X_\alpha$. Fixing $x \in \operatorname{supp} f_\alpha$, define $\{y_n\}$ such that $y_n \to x$ and $f_\alpha(y_n) = 0$. Because

$$f_\alpha = \sum_{i \in \rho^{-1}(\alpha)},$$

for each n there exists i_n such that $g_{i_n}(y_n) \neq 0$. Because $\{W_i\}$ is locally finite, there exists a neighborhood $U \ni x$ such that $I_x = \{i \in I \mid W_i \cap U \neq \emptyset\}$ is finite. Then there exists $i \in I_x$ such that $\{n \mid i_n = i\}$ is infinite. $y_n \in \operatorname{supp} g_i \subset \overline{V}_i$.

Finally, we show that $\{\operatorname{supp} f_\alpha\}$ is locally finite. Because W_i is locally finite, there exists $U \ni x$ such that I_x as defined above is finite. Then for all α such that $U \cap \operatorname{supp} f_\alpha \neq \emptyset$, there exists $y \in U$ such that $f_\alpha(y) \neq 0$. This implies $g_i(y) \neq 0$ for some $i \in \rho^{-1}(\alpha)$. Thus $y \in U \cap W_i$, so $i \in I_x$. Thus $\alpha = \rho(i) \subset \rho(I_x)$. $\square$

[10] This is specifically for the proof. We are staying in the integers for the number theorist in the room.

[11] I asked if we needed the condition of countable refinement, then Connor said that refinements of partitions add points, and Mike made an analogy to refining flour.

4 LECTURE 4 (SEP 16)

We began class by finishing the proof of Theorem 37. Everything there has been added to the notes from last time. Today we will discuss tangent vectors.[12]

4.1 Tangent Spaces

Definition 44. Let M be a smooth manifold and $p \in M$. Then an $\mathbb{R}$-linear map $X : C^\infty(M) \to \mathbb{R}$ is called a *tangent vector at* p if for all $f, g \in C^\infty(M)$, $X(fg) = f(p)X(g) + g(p)X(f)$.

Notation 45. The set of all tangent vectors at p is denoted T_pM. It is easy to see that this is an $\mathbb{R}$-vector space.[13]

Definition 46. Let $F : M \to N$ be smooth. The *differential of* F *at* p $F_* : T_pM \to T_{F(p)}N$ is given by

$$(F_*X)(f) = X(f \circ F)$$

for all $f \in C^\infty(N)$.[14] Usually, this is denoted by dF_p.

Proposition 47. *The following are true:*

1. $(G \circ F)_* = G_* \circ F_*$;
2. *If* $F : M \to N$ *is a local diffeomorphism, then* F_* *is an isomorphism.*

Proposition 48 ("Localization"). *Let* M *be a smooth manifold and* $U \subset M$ *open with inclusion* $i : U \to M$. *Then for all* $p \in U$, $i_* : T_pU \to T_pM$ *is an isomorphism.*

Lemma 49. *Fix* $p \in M$ *and* $f, g \in C^\infty(M)$. *If there exists a neighborhood* $B \ni p$ *such that* $f|_B = g|_B$, *then* $X(f) = X(g)$ *for all* $X \in T_pM$.

Proof. Let $\mathcal{X} = \{B, M \setminus \{p\}\}$ and ψ_1, ψ_2 be a partition of unity subordinate to $\mathcal{X}$. Let $h = f - g$. Then $\psi_2 = 1$ on $M \setminus B$. Thus $h = h\psi_2$ on $M \setminus B$. Because $h = 0$ on B, then $h = h\psi_2$ on B. Finally, $\psi_2(p) = 0$. Thus $X(h) = X(h\psi_2) = h(p)X(\psi_2) + \psi_2(p)X(h) = 0$. □

Lemma 50. *Consider* $A \subset U \subset M$, *where* A *is closed and* U *is open. Then there exists an extension map* $C^\infty(U) \to C^\infty(M)$ *given by* $f \mapsto \widetilde{f}$ *such that* $\widetilde{f}|_A = f$ *and* $\operatorname{supp} \widetilde{f} \subset U$.

Proof. Let ψ_1, ψ_2 be subordinate to $\{U, X \setminus A\}$. Define $\widetilde{f} = \psi_1 f$. □

Proof of Proposition 3.7. Fix a ball $B \ni p$ such that $\overline{B} \subset U$. Suppose $i_*X = 0$. Let $\widetilde{f} \in C^\infty(M)$ be the extension of f by Lemma 50. Then $X(f) = X(\widetilde{f}) = X(\widetilde{f} \circ i) = i_*X(f) = 0$.

Now we show that i_* is surjective. For $Y \in T_pM$, define $X \in T_pU$ by $X(f) = Y(\widetilde{f})$. Checking that X is linear and satisfies the Leibniz rule is straightforward, and it is easy to see that $i_*X = Y$. □

[12] There are many ways to define this. For example, we can define the tangent space as the stalk of the tangent sheaf. Alternatively, we can make a construction analogous to the Zariski tangent space from algebraic geometry. Finally, we can embed M into $\mathbb{R}^n$ by Whitney and use the classical notion of tangent space.

[13] Later we will define the tangent bundle.

[14] This is also called the pushforward induced by F.

Proposition 51. *Fix $p \in \mathbb{R}^n$. Define $D : \mathbb{R}^n \to T_p\mathbb{R}^n$ by $v \mapsto D_v$ where D_v is the directional derivative. Then D is an isomorphism. If $x^1, \dots, x^n$ are coordinates of $\mathbb{R}^n$ and $v = (v_1, \dots, v_n)$, then $D_v f = v \cdot (\nabla f)$.*

Definition 52. Now consider some chart $\varphi : U \to \mathbb{R}^n$ and the projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$. Let $x^i = \pi_i \circ \varphi$. Then $\{x^i\}$ are called *local coordinates* of U.

5 Lecture 5 (Sep 18)

Last time, we defined local coordinates on $\mathbb{R}^n$ at the end of class. Note that a chart φ can be written as $\varphi(q) = (x^1(q), \dots, x^n(q))$. We may define local coordinates on an arbitrary manifold analogously.

5.1 Working in Coordinates

Fix $p \in U \subset M$ and let $f \in C^\infty(U)$. Denote $\widehat{p} = \varphi(p) \in \mathbb{R}^n$ and $\widehat{f} = f \circ \varphi^{-1}$. Because $\varphi(U) \subset \mathbb{R}^n$, we can use multivariable calculus to define

$$\left.\frac{\partial \widehat{f}}{\partial x^i}\right|_{\widehat{p}} \in C^\infty(U).$$

This allows us to define the directional derivative on M as

$$\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \coloneqq \varphi_*^{-1}\left(\left.\frac{\partial}{\partial x_i}\right|_{\widehat{p}}\right)).$$

Proposition 53. *The set of directional derivatives with respect to the local coordinates is a basis for T_pM. In particular, $T_pM \simeq \mathbb{R}^n$.*

This allows us to compute tangent vectors by pushforward:

$$\begin{aligned}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(f) &= \varphi_*^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)(f) \\ &= \left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}(f \circ \varphi^{-1}) \\ &= \left.\frac{\partial \widehat{f}}{\partial x^i}\right|_{\widehat{p}}.\end{aligned}$$

Now we may define the Jacobian of a smooth map. Let $F : M \to N$ be a smooth map with $p \in M, q = F(p) \in N$. Define charts $U \ni p, V \ni q$ with charts φ, ψ with local coordinates $x^1, \dots, x^m$ and $y^1, \dots, y^n$. We will write F_* as a matrix with respect to the standard bases.

Let $\widehat{F}^j = y^j \circ F \circ \varphi^{-1}$, so the induced map $\varphi(U) \to \psi(V)$ is given by $(\widehat{F}^1, \ldots, \widehat{F}^n)$. Then

$$\begin{aligned}\left(F_* \frac{\partial}{\partial x^i}\bigg|_p\right)(y^j) &= F_* \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\right)(\pi^j \circ \psi) \\ &= \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\right)(\pi^j \circ \psi \circ F \circ \varphi^{-1}) \\ &= \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\right)(\pi^j \circ \widehat{F}) \\ &= \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\right)(\widehat{F}^j) \\ &= \frac{\partial \widehat{F}}{\partial x^i}\bigg|_{\widehat{p}}.\end{aligned}$$

From multivariable calculus, we get that

$$F_* \left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{j=1}^{n} \left(\frac{\partial \widehat{F}^j}{\partial x^i}\bigg|_{\widehat{p}}\right)\left(\frac{\partial}{\partial y^j}\bigg|_q\right).$$

Writing this as a matrix, we get exactly the Jacobian.

Example 54. We will compute the pushforward of the identity on two different charts. In particular, $F_* = \mathrm{id}$. Then we see that

$$\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^{n} \left(\frac{\partial(\psi \circ \varphi^{-1})^j}{\partial x^i}\right) \frac{\partial}{\partial y^j}\bigg|_p.$$

Example 55. Suppose $F : M \to \mathbb{R}$. Then $F_* : T_pM \to F_{F(p)}\mathbb{R} \simeq \mathbb{R}$. Thus for all $F \in C^\infty(U)$, we can consider $dF_p = F_* \in (T_pM)^*$.[15] Note that $dF_p(X) = X(F)$.

In local coordinates, observe that

$$\left(\frac{\partial}{\partial x^i}\bigg|_p\right)(x^j) = \delta_{ij}.$$

Thus $\{dx_p^i\}$ is the dual basis of $(T_pM)^*$.

Example 56. Let $\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to N$ such that $\gamma(0) = q \in N$. Then γ is called a smooth curve through q. Note that $T_0(-\varepsilon, \varepsilon) = T_0\mathbb{R} = \mathbb{R}^1 = \mathbb{R}\{(\partial/\partial t)|_0\}$. We denote the tangent vector

$$\gamma_* \left(\frac{\partial}{\partial t}\bigg|_0\right) \in T_{\gamma(0)}N$$

by $\gamma'(0)$ and call it the tangent vector of γ at the point q.

[15] This is called the cotangent space for obvious reasons.

5.2 Alternative Approaches to Tangent Spaces We present alternative ways to define the tangent space. The first way is to use equivalence classes of smooth curves:

Fix $p \in M$. Let $\mathcal{C}_p = \{\gamma : I \to M \mid \gamma(0) = p\}$. Then let $\gamma_1 \sim \gamma_2$ if they share a tangent vector (after pushing forward to $\mathbb{R}^n$ and using the calculus notion of tangent vector.)

The second way, which will not be discussed, is as the stalk of the tangent sheaf.[16]

5.3 Vector Fields.

Definition 57. Let M be a smooth manifold. The *tangent bundle* of M is

$$TM = \bigsqcup_{p \in M} T_p M$$

as a set.

Theorem 58. *The following are true:*

1. *If* $\dim M = n$, *then* TM *is naturally a smooth* $(2n)$*-manifold.*
2. *Moreover, the projection* $\pi : TM \to M$ *is a smooth map.*
3. *For any smooth* $F : M \to N$, *there is a smooth* $F_* : TM \to TN$, *defined in the obvious way.*

Proof of (1). Let $U \subset M$. Then consider $\pi^{-1}(U) = TU$. Now let $\{(U_\alpha, \varphi_\alpha)$ be a smooth atlas of M. Then we will define local trivializations:

Let $x^1, \ldots, x^n$ be local coordinates and consider the standard basis of $T_p M$. Thus we can write $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}\big|_p$. Then we can define $\widetilde{\varphi}_\alpha(p, X) = (\varphi_\alpha(p), X^1, \ldots, X^n)$. It is easy to see that the conditions of Lemma 15 hold. Applying Lemma 15, we have a smooth structure. □

6 Lecture 6 (Sep 23)

Mike cannot be here on Wednesday, December 11, so we will need to make up the lecture. We will begin at 2 PM on three days that are to be determined.

6.1 Vector Fields, Continued

Definition 59. A (continuous, smooth) section $M \to TM$ of the tangent bundle is called a *(continuous, smooth) vector field*.

Remark 60. An open question is to find a necessary and sufficient condition on manifolds M, N such that $TM \simeq TN$.

Lemma 61 (Smoothness Criteria)**.** *A section* X *of the tangent bundle is smooth if and only if one of the following two conditions holds:*

1. *The representative on charts* $\widehat{X} = \widetilde{\varphi} \circ X \circ \varphi^{-1}$ *is smooth.*

[16] This is closer to the algebraic point of view.

2. *For all open* $U \subset M$ *and* $f \in C^\infty(U)$*, the function* $Xf : p \to X_p(f)$ *is smooth.*

Notation 62. We will denote the space of global sections of the tangent bundle by $\mathcal{X}(M)$.

We can check that $\mathcal{X}(m)$ is a nonempty module over $C^\infty(M)$.[17]

Definition 63. A map $Y : C^\infty(M) \to C^\infty(M)$ is called a *derivation* if it satisfies the Liebniz rule.

Proposition 64. $\mathcal{Y}$ *is a derivation if and only if* $\mathcal{Y}(f) = Yf$ *for some* $Y \in \mathcal{X}(M)$.

Now let $F : M \to N$ be a smooth map. If F is a diffeomorphism, for $X \in \mathcal{X}(M)$ define $Y \in \mathcal{X}(N)$ by

$$Y := F_* \circ C \circ F^{-1}.$$

Thus $Y \circ F = F^* \circ X$. However, if F is not a diffeomorphism, such a Y need not exist.

Example 65. Consider the figure-eight $S^1 \to \mathbb{R}^n$. Then we cannot push forward a vector field onto the double point. Also, how do we push forward the vector field onto points that are not mapped onto?

Definition 66. Let $F : M \to N$ be a smooth map and $X \in \mathcal{X}(M), Y \in \mathcal{X}(N)$. Then X, Y are F-*related* if $F_* \circ X = Y \circ F$,

Example 67. Consider the vector field $\partial_t \in \mathcal{X}(\mathbb{R})$. Then if $F : \mathbb{R} \to \mathbb{R}^2$ is the covering map of the circle, an F-related vector field on $\mathbb{R}^2$ is $Y = x\partial_y - y\partial_x$.

6.2 Lie Algebras Note that if X, Y are vector fields, then $X \cdot Y$ is not necessarily a vector field. However, their Lie bracket is a vector field.

Example 68. Let $\varphi : U \subset M \to \mathbb{R}^n$ be a chart with coordinates $x^1, \ldots, x^n$. Then we have the relations generating the Weyl algebra on the Lie bracket of the standard basis vectors. Proof boils down to symmetry of mixed partials after symbol pushing to get us into $\mathbb{R}^n$.

Note that $\mathcal{X}(M)$ is a Lie algebra.[18]

Proposition 69. $[fX, gY] = fg[X, Y] + (fXg)y - (gYf)X$. *Also, if* X_i *is* F*-related to* Y_i *for* $i = 1, 2$*, then* $[X_1, X_2]$ *is* F*-related to* $[Y_1, Y_2]$.

We can express the Lie bracket locally. If $X = \sum x^i \partial_i$, then

$$[X, Y] = \sum_i \sum_j (X^j \partial_j Y^i - Y^j \partial_j X^i)\partial_i = \sum [X, Y]^i \partial_i.$$

Recall the definition of a Lie algebra.[19]

Example 70. Some examples of Lie algebras are:

1. $\mathfrak{g} = \mathcal{X}(M)$ with the Lie bracket defined above.

[17] In fact, we can extend this to define the tangent sheaf.
[18] Apparently the thing we are supposed to study is ∞-algebraic structures. Mike wishes there was a course here on higher algebra taught by Ivan or Owen.
[19] This can be internalized to any category enriched over **Ab**.

2. $\mathfrak{g} = \mathbb{R}^n$ with zero Lie bracket.[20]
3. $\mathfrak{g} = M_n(\mathbb{R})$ with the commutator. In fact, this generalizes to any associative $\mathbb{R}$-algebra.
4. In Homework 2 we will find the 2-dim and 3-dim Lie algebras.
5. The category of Lie algebras has products.
6. Suppose the vector space $\mathfrak{g}$ has basis $X_1, \dots, X_n$. Define

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

   Then Jacobi holds if and only if for all s,

$$\sum_{\ell=1}^{n} (c_{jk}^{\ell} c_{i\ell}^{s} + c_{ki}^{\ell} c_{j\ell}^{s} + c_{ij}^{\ell} c_{k\ell}^{2}).$$

Let G be a Lie group. Then define the left translation $L_g : G \to G$ by $h \mapsto gh$. This is a diffeomorphism with inverse $(L_g)^{-1} = L_{g^{-1}}$.

Definition 71. $X \in \mathfrak{X}(G)$ is *left-invariant* if $(L_g)^*(X) = X$ for all $g \in G$.

Define $\mathrm{Lie}(G)$ be the set of left-invariant vector fields. We can check that $\mathrm{Lie}(G)$ is a Lie algebra with the usual Lie bracket.

Theorem 72. *Let G be a Lie group with $e \in G$ the identity. Define $\varepsilon : \mathrm{Lie}(G) \to T_e G$ by $X \mapsto X_e$. Then ε is an isomorphism of vector spaces. In particular,* $\dim \mathrm{Lie}(G) = \dim G$.

Theorem 73. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. Then suppose $F : G \to H$ is a morphism of Lie groups. Then for $X \in \mathfrak{g} = T_e G$, there exists a unique $Y \in T_e H$ which is F-related to X. This defines a Lie algebra homomorphism $F_* : \mathfrak{g} \to \mathfrak{h}$.*

Corollary 74. *If G is a Lie subgroup of H, then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{h}$.*

Corollary 75. *If $\pi : G \to H$ is a smooth cover of Lie groups, then $\mathfrak{g} \simeq \mathfrak{h}$.*

Example 76. Let $G = GL_n(\mathbb{R}) \subsetneq M_n(\mathbb{R})$. Then $T_{I_n} G = T_{I_n} M = M_n(\mathbb{R})$.

Proposition 77. *Under the identification $\varepsilon : T_{I_n} G \to \mathrm{Lie}(G)$, the Lie bracket of* $\mathrm{Lie}(G)$ *is sent to the commutator of matrices.*

7 Lecture 7 (Sep 25)

Today we will begin discussion of vector bundles.

[20] This is called *abelian*.

7.1 Vector Bundles

Definition 78. Let X be a topological space. A *real vector bundle of rank* n *over* X is a morphism $\pi : E \to X$ such that

1. For all $p \in X$, the fiber over p is an n-dimensional real vector space.
2. For all $p \in X$, there exists $U \ni P$ and a homeomorphism $\Phi : \pi^{-1}U \to U \times \mathbb{R}^n$ such that $\pi_1 \circ \Phi = \pi$ and such that for all $q \in U$, $\Phi|_{E_q}$ is an isomorphism of vector spaces.

Example 79. The tangent bundle of a smooth manifold is a vector bundle.

Definition 80. A *complex vector bundle* is as in the above definition, but $\mathbb{R}$ is replaced with $\mathbb{C}$.

Suppose $E \to X$ is a vector bundle. Suppose $\{U_\alpha\}$ is an open cover with trivializations Φ_α. Then we have the following diagram:

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times \mathbb{R}^n \arrow[dr, "\pi"'] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[l, "\Phi_\beta"'] \arrow[r, "\Phi_\alpha"] \arrow[d, "\pi"] & (U_\alpha \cap U_\beta) \times \mathbb{R}^n \arrow[dl, "\pi"] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}.$$

Then we consider the map $\Phi_\beta \circ \Phi_\alpha^{-1}$. Over every point q, we obtain some $\tau_{\beta\alpha}(q) \in GL_n(\mathbb{R})$. Thus the transitions maps of E are smooth maps $U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$.

Proposition 81. $\tau_{\gamma\beta}\tau_{\beta\alpha} = \tau_{\gamma\alpha}$.

Example 82. On the tangent bundle, the transitions are simply the Jacobians.

7.2 Pullback Bundles

Lemma 83. *Let X be a topological space, E a set, and $\pi : E \to X$ surjective. Suppose the fibers of π are vector spaces of dimension n and that there exists a cover $\{U_\alpha\}$ of X such that:*

1. *For all α there exists a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ commuting with projections and $\Phi_\alpha|_{E_q}$ is an isomorphism.*
2. *For all α, β, $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ is continuous.*

Then $E \to X$ is a topological vector bundle.

Remark 84. Lemma 83 holds in the smooth category.

Example 85. The trivial bundle $M \times \mathbb{R}^k$ is a smooth vector bundle.

Example 86. Let $E = \{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in x\}$. This is the *tautological line bundle over* $\mathbb{RP}^n$. We will choose the standard affine charts. The trivializations will be the obvious ones (projection of v to the ith coordinate.) It is easy to see that the transitions are simply rescaling, so the transitions satisfy the conditions of Lemma 83.

Example 87. Consider $E = \{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in x\}$. This is the tautulogical bundle over $\mathbb{CP}^n$.[21]

[21] This generalizes to any field. Also, this is a holomorphic line bundle.

Suppose $E \to M$ is a vector bundle and $F : N \to M$ is a smooth map. Then we will give the set $E \times_M N$ the structure of a vector bundle over N.[22]

Theorem 88. *Vector bundles can be pulled back.*

Proof. Let $\{U_\alpha\}$ be an open cover of M with trivializations Φ_α. Now pull everything back to N in the natural way. It is a mechanical exercise to verify that everything works and that we can use Lemma 83. □

Example 89. Consider the embedding $\mathbb{RP}^k \to \mathbb{RP}^n$. Then pulling back the tautological bundle from $\mathbb{RP}^n$ gives the tautological bundle on $\mathbb{RP}^k$.

Example 90. Let M be the Klein bottle. Let $N = S^1$ and F sends the circle to a loop around the cylinder part of the Klein bottle. We know that $TM \to M$ is not trivial, but its pullback to S^1 is trivial.

8 Lecture 8 (Sep 30)

Last time we discussed vector bundles. Some examples are the tautological bundles over projective space. We also defined pullbacks of vector bundles.

8.1 Sections and Frames Let $E \to M$ be a vector bundle with $U \subset M$ open. Then a local section of E over U is a smooth $\sigma \in \mathcal{O}(E)|_U$. A global section is a section $\sigma \in \Gamma(E)$.

Definition 91. Let $E \to M$ be a smooth vector bundle of rank n. Then an ordered tuple $\{\sigma_1, \ldots, \sigma_n\}$ of local sections of E over $U \subset M$ open is called a *local frame* if the values at each $p \in U$ form a basis of E_p. If $U = M$, then we have a *global frame*.

Example 92. A section of the tangent bundle is the same thing as a vector field. In addition, every vector bundle carries a zero section.

Example 93. The tangent bundle $TM \to M$ carries standard local coordinate frames on the trivialiazations.

Proposition 94. *Let $E \to M$ be a vector bundle. Then there exists a bijection between local (resp. global) frames and local (resp. global) trivializations.*

Proof. Given a local frame, then we do the obvious thing to locally trivialize the vector bundle. In the other direction, define each frame to be given by the coordinate functions. □

Thus a vector bundle is trivial if and only if it has a global frame.

Definition 95. A manifold is *parallelizable* if $TM \to M$ is trivial.

Example 96. S^1 is parallelizable. One argument is that all Lie groups are parallelizable. In this spirit, S^3 is also parallelizable. However, S^2 is not parallelizable (Brouwer's fixed point theorem).

Proposition 97. *Every Lie group is parallelizable.*

[22]Compare to base change in algebraic geometry.

Proof. Take a frame for T_eG and then apply the $\mathfrak{g}$-action to get a left-invariant frame. $\square$

Example 98. The tautological line bundle is not trivial.

Example 99. The Mobius strip is a nontrivial vector bundle on S^1.

Theorem 100. *Let M be a smooth manifold with open cover $\{U_\alpha\}$. Given smooth $\tau_{\beta\alpha}$ such that $\tau_{\gamma\beta}\tau_{\beta\alpha} = \tau_{\gamma\alpha}$, then there exists a smooth rank-n vector bundle over M with these transition functions.*

8.2 Induced Bundles Let $G \subset GL_n(\mathbb{R})$ be a Lie subgroup and let ρ be a Lie group homomorphism. Then suppose $E \to M$ is a smooth rank n vector bundle with transition functions $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to G \subset GL_n(\mathbb{R})$. Then postcomposing with $\rho : G \to GL_n(\mathbb{R})$ induces a vector bundle $\widetilde{E}$, called the *induced bundle of E by ρ*.

Example 101. The dual bundle is induced by taking the inverse of the transpose of each transition function.

9 Lecture 9 (Oct 2)

9.1 Examples of Dual Bundles

Example 102. Let $E = TM$. Then the dual is $E^* = T^*M$, which is the cotangent bundle of M. This is the main example of a symplectic manifold. Also, we call a local trivialization of TM a frame and a local trivialization of T^*M a coframe.

Physically, we can think of M as position, TM as position and velocity, and T^*M as position and momentum. In particular, the laws of mechanics can be phrased in both TM and T^*M.

Example 103. Consider the projection $\mathbb{RP}^m \to \mathbb{RP}^{n-1}$ given by projection onto the first n coordinates.[23]. Now remove the point where π is not regular. We claim E^* is the tautological line bundle on $\mathbb{RP}^n$.[24]

Example 104. Let E_1, E_2 be vector bundles of rank n_1, n_2. Then define $E = E_1 \oplus E_2$ is a vetor bundle of rank $n_1 + n + 2$ with transitions given by the obvious morphism $GL_{n_1} \times GL_{n_2} \to GL_{n_1+n_2}$.

9.2 Subbundles and Quotient Bundles

Definition 105. Let $E \to M$ be a rank n vector bundle. Then $E' \subset E$ is a *subbundle* of rank k if

1. For all $p \in M$, E'_p is a k-dimensional subspace of E_p.
2. For all $p \in M$, there exists a local frame of E over some neighborhood $U \ni p$ such that for all $q \in U$, $\sigma_1(q), \dots, \sigma_k(q)$ are a basis for E'_q.

Proposition 106. *For any smooth vector bundle E with subbundle E' we can cover M by U_α such that*

1. *Both E, E' are trivial U_α.*

[23] This is a standard example in algebraic geometry

[24] In particular, $E = \mathfrak{O}(1)$.

2. *The transition functions are block upper-triangular of the form* $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.

Now define

$$\rho_1 : \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto A, \rho_2 : \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto C.$$

Definition 107. The bundle induced from E by ρ_2 is called the *quotient bundle*.

9.3 Cotangent Bundle We have the cotangent bundle $T^*M \to M$ with coframe $\{dx^i\}$. A section of the cotangent bundle is called a *covector field*.[25]

Example 108. Let M be a smooth manifold and $f \in C^\infty(M)$. The differential df of f is defined by $df_p(X_p) = X_p f$. By the cotangent version of Lemma 61, this is a smooth covector field.

We may ask whether all smooth covector fields arise this way, and the answer is no, because of $H^1(M)$. We may read on our own how to integrate covector fields over paths.

Differentials do satisfy the usual calculus rules. In particular, $df = 0$ if and only if f is constant.

Pushforwards of tangent vectors are replaced by pullbacks here. If $F : M \to N$ is smooth, then the pullback is given by $F^*\omega(X) = \omega(F_*X)$ pointwise. Thenwe can check that this pulls smooth covector fields to smooth covector fields.

9.4 Submersions, Immersions, Embeddings .

Definition 109. Let $F : M \to N$ be smooth. Then we define the *rank* of F at p to be the rank of $(F_*)_p$.

- F is a *submersion* if $(F_*)_p$ is surjective for all $p \in M$.
- F is an *immersion* if $(F_*)_p$ is injective for all $p \in M$.
- F is a *smooth embedding* if F is an immersion that is a homeomorphism onto its image.

Example 110. The map $\mathbb{A}^1 \to (y^2 = x^3 - x^2) \subset \mathbb{A}^2$ is an immersion but not an embedding.

Remark 111. An injective immersion must be a smooth embedding if it is proper; in particular when M is compact.

Theorem 112 (Inverse Function Theorem)**. Euclidean Version:** *First, we have the Euclidean version: Suppose* $U, V \subset \mathbb{R}^n$ *and* $F : U \to V$ *a smooth map. If the Jacobian* $DF(p)$ *is nonsingular, then there exists a connected neighborhood* $U_0 \subset U$ *of* p *and* $V_0 \subset V \ni F(p)$ *such that* $F|_{U_0}$ *is a diffeomorphism onto* V_0.

Manifold Version: *Let* $F : M \to N$ *be smooth. If* $(F_*)_p$ *is a bijection, then* F *is a local diffeomorphism near* p.

Theorem 113 (Rank Theorem)**. Euclidean Version:** *Let* $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ *and* $F : U \to V$ *be a smooth map of constant rank* k. *Then for all* $p \in U$, *there exist charts* (U_0, φ) *of* p *and* (V_0, ψ) *of* $F(p)$ *such that the coordinates satisfy*

[25] Later, this will be called a 1-form.

$$\begin{tikzcd}
(x_1,\dots,x_k,x_{k+1},\dots,x_m) \arrow[rr, "\psi\circ F\circ\varphi^{-1}"] \arrow[dr, "\pi"] & & (x_1,\dots,x_k,0,\dots,0) \\
& (x_1,\dots,x_k) \arrow[ur, hook] &
\end{tikzcd}.$$

Manifold Version: *Let $F: M \to N$ be a smooth map of constant rank k. Then for all $p \in M$, there exist local coordinates $x_1, \dots, x_m$ at p and $y_1, \dots, y_n$ at $F(p)$ such that the coordinate representation of F is given by $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$.*

We may read about the Implicit Function Theorem on our own.

10 Lecture 10 (Oct 07)

Theorem 114. *Let $F : M \to N$ be smooth of constant rank. Then*

1. *If F is surjective, then F is a submersion.*
2. *If F is injective, then F is an immersion.*
3. *If F is bijective, then F is a diffeomorphism.*

Example 115. Define $f : S^1 \times S^1 \to \mathbb{R}^3$ given by

$$(q_1, q_2, q_3, q_4) \mapsto (q_1(2 + q_3), q_2(2 + q_3), q_4).$$

This embeds $S^1 \times S^1$ into $\mathbb{R}^3$.

10.1 Submanifolds

Definition 116. Let M be a smooth manifold of dimension n and $S \subset M$. We say S is an *embedded submanifold of dimension* $k \leqslant n$ if for all $p \in S$, there exists a smooth chart $U \ni p$ of M such that

$$S \cap U = \{(x_1, \dots, x_n) \in U \mid x_{k+1} = \cdots = x_n = 0\}.$$

Here (U, φ) is called a *slice chart* for S, $x_1, \dots, x_n$ are the *slice coordinates* for S, and $n - k$ is the *codimension*.

Theorem 117. *Let $S \subset M$ be an embedded submanifold of dimension k. With the subspace topology, S is a topological manifold of dimension k and has a unique smooth structure such that $S \hookrightarrow M$ is a smooth embedding.*

Theorem 118. *The image of a smooth embedding is an embedded submanifold.*

Example 119. The torus is a submanifold of $\mathbb{R}^3$.

Example 120. Smooth embeddings of S^1 in $\mathbb{R}^3$ are knots and their study is called knot theory. In general, we can study $S^k \hookrightarrow \mathbb{R}^n$ and the story varies by codimension.

Proof of Theorem 117. First we show that S is a topological manifold. It is easy to check that S is Hausdorff and second-countable in the subspace topology. To see that S is locally Euclidean, we simply project the slice charts down to $\mathbb{R}^k$.

Now we need to give S a smooth structure. We will simply use the slice charts from M with their transition functions restricted. Now restricting a smooth map to a coordinate subspace is smooth, so $S \hookrightarrow M$ is an immersion and topological embedding.

Finally, we show that the smooth structure is unique. What we want to show is that ψ, θ are two charts from different atlases. Then $\psi\theta^{-1}$ is a homeomorphism and is smooth. We can also see that $\psi\theta^{-1}$ is an immersion. Because the domain and target have the same dimension, this is a local diffeomorphism. □

Definition 121. Let $F : M \to N$ be a smooth map. Then a point $c \in N$ is a *regular value* of F if for all $p \in F^{-1}(C)$, $(F_*)_p$ is surjective. Otherwise, c is a *critical value* of F.

Corollary 122. *For any regular value* c *of a smooth map* $F : M \to N$, *the preimage* $F^{-1}(c)$ *is an embedded submanifold of* M *of codimension* $\dim N$. *Moreover,* $T_p(F^{-1}(c)) = \ker(F_*)_p$.

Example 123. Consider a smooth map $F : \mathbb{R}^n \to \mathbb{R}$ given by

$$(x_0, \ldots, x_n) \mapsto \sum_{k=0}^{n} x_k^2.$$

Then $1 \in \mathbb{R}$ is a regular value of F and $F^{-1}(1) = S^n$ is an embedded submanifold. Then $T_x S^n = \{y \mid y \perp x\}$.

Example 124. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ given by

$$(x, y, z) \mapsto (\sqrt{x^2 + y^2} - 2)^2 + z^2.$$

Example 125. Let $f : M \to N$ be a smooth map. Then consider the *graph* of f in $M \times N$. Then define $F : M \to M \times N$ given by $F(p) = (p, f(p))$. This is a smooth embedding and $\Gamma(f)$ is the image of F, so it is a submanifold of $M \times N$.

Proposition 126. *Let* $F : M \to N$ *be smooth. Suppose* $S \subset N$ *is an embedded submanifold and* $F(M) \subset S$. *Then the range restriction* $F_S : M \to S$ *is smooth.*

Remark 127. There is a similar result for domain restriction.

Example 128. Consider multiplication of quaternions on $\mathbb{H} = \mathbb{R}^4$. Then multiplication restricted to S^3 is smooth.

11 LECTURE 11 (OCT 09)

11.1 Lie Subgroups

Definition 129. A *Lie subgroup* of a lie group G is a subgroup of G with a smooth structure making it an immersed submanifold of G.

Proposition 130. *If* G *is a Lie group and* H *is any subgroup which also is an embedded submanifold, then* H *is an closed Lie subgroup.*

To prove this, we need to check that multiplication and inversion are smooth on H and that H is closed.

Example 131. $SL_n(\mathbb{R})$ is a closed Lie subgroup of $GL_n(\mathbb{R})$ of codimension 1, and the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ is the set of trace 0 matrices.

To show this, we simply show that 1 is a regular value of the determinant function. The easy way to see this is that det is a Lie group homomorphism with derivative tr.

Example 132. In general, SL_n is a closed algebraic subgroup of GL_n over any field k.

Example 133. GL_n acts on $\mathbb{P}^{n-1}$ and the group of actions is called PGL_n. This has dimension 3 over k.

Example 134. The orthogonal group $O(n)$ is a Lie group with Lie algebra $\mathfrak{o}(n)$ consisting of matrices that satisfy $A^T + A = 0$.

11.2 Lie Group Actions

Let M be a smooth manifold and G be a Lie group.

Definition 135. A *left-action of* G *on* M is a smooth map $Theta : G \times M \to M$ that satisfies the usual group action axioms.

Recall that an action is transitive if for some p, $G.p = M$. The stabilizer is called the *isotropy group*, and the action is free if every point has trivial stabilizer.

Example 136. The simplest example of a Lie group action is the trivial action, where every element of G acts by the identity.

Example 137. If M is a vector space V, then a G-action is called a representation.

Example 138. Consider the action of G on itself by conjugation. Then differentiate the action to obtain a Lie group homomorphism $G \to GL(T_eG)$, which is called the adjoint representation of G.

Definition 139. An action $G \times M \to M$ is called *proper* if the map $G \times M \to M \times M$ given by $(g, p) \mapsto (g.p, p)$ is proper.

Proposition 140. *Let* $g.K$ *be the image of* K *under the action of* G*, and let* $G_K = \{g \in G \mid g.K \cap K \neq \emptyset\}$*. Then the action of* G *is proper if for all compact* $K \subset M$*,* G_K *is compact.*

Proposition 141. *The action of* G *on* M *is proper if for all convergent subsequences* $\{p_i\}$ *of* M *and any sequence* $\{g_i\}$ *of* G *such that* $\{g_i.p_i\}$ *is convergent, then there exists a subsequence of* g_i *converging in* G*.*

Remark 142.

1. For a proper action, G_K is compact when $K = \{p\}$.
2. Compact Lie group actions are proper.
3. For a discrete group, the action is proper only if G_p is finite for any $p \in M$. Thus there exists an invariant neighborhood of p.

12 Lecture 12 (Oct 15)

Recall that last time we discussed Lie group actions. Today we will relate them to manifolds.

12.1 Equivariance

Theorem 143. *Suppose a Lie group* G *has a proper free action on a manifold* M*. Then the orbit space is a topological manifold of dimension* $\dim M - \dim G$ *and there exists a unique smooth structure on* M/G *such that the quotient map* $M \to M/G$ *is a submersion.*

Example 144. Recall that the Hopf fibration is the quotient map $S^3 \to S^3/S^1 = \mathbb{CP}^1$.

Example 145. Consider the smooth $\mathbb{Z}/2\mathbb{Z}$-action on S^n by ± 1. Then the quotient $S^n/(\mathbb{Z}/2\mathbb{Z})$ is $\mathbb{RP}^n$.

Definition 146. Suppose M, N are smooth G-manifolds. Then a smooth map $F : M \to N$ is G-*equivariant* if it induces a natural transformation of functors $G \to$ **Diff**.

Theorem 147. *Suppose* $F : M \to N$ *is an equivariant map between* G*-manifolds. Then suppose* G *acts transitively on* M*. Then* F *has constant rank and* $F^{-1}(c) \subset M$ *is an embedded submanifold.*

Proof. Fix $p_0 \in M$. Then for all $p \in M$ there exists $g \in G$ such that $p = g \cdot p$. Then by equivariance, the diagram

$$\begin{tikzcd} T_{p_0}M \arrow[r, "F_*"] \arrow[d, "g"] & T_{F(p_0)}N \arrow[d, "g"] \\ T_pM \arrow[r, "F_*"] & T_{F(p)}N \end{tikzcd}$$

commutes. Thus F must be of constant rank. □

Corollary 148. *Let* G *have a smooth, free, proper action on* M*. Then the orbit* $G.p$ *is an embedded submanifold for all* $p \in M$*.*

Proposition 149. *Let* $F : G \to H$ *be a Lie group homomorphism. Then* $\ker F$ *is an embedded Lie subgroup of* G*.*

Proof. Consider the action of G on itself and its induced action on H. Then use Theorem 147, □

Definition 150. A smooth manifold M is called a *homogeneous space* if it admits a smooth transitive Lie group action for some G.

Theorem 151. *Let* M *be a homogeneous space with a transitive Lie group action of* G*. Fix any point* $p \in M$*. Then the map* $G/G_p \to M$ *is an equivariant diffeomorphism. Moreover,* G_p *is an embedded submanifold of* M*.*

Alternatively, let G be a Lie group and $H \subset G$ be a closed Lie group. Then consider the right H-action on G. Then this action is smooth and proper, and the quotient space G/H is a smooth manifold of dimension $\dim G - \dim H$. Then it is easy to see that the G-action on G/H is transitive, so we can define a homogeneous space to be G/H.

Proof of Theorem 151. The F is clearly well-defined by basic properties of groups. Then note that

$$F(g'gH) = (g'g)p = g'F(gH).$$

Now set $G_p = F^{-1}(p)$, which is an embedded submanifold of G. Finally, it is easy to see that F is a bijection. Because F has constant rank, then F is a diffeomorphism. □

Example 152. Consider $M = S^n$ and $G = O(n+1)$. Then M has a natural transitive G-action. Let $p = (0, 0, \ldots, 0, 1)$ be the north pole. Then $G_p = O(n)$, so $S^n = O(n+1)/O(n)$.

Example 153. Let $M = G_2(\mathbb{R}^4)$. Then consider the natural transitive action of $G = GL_2(\mathbb{R})$. Let p be the plane spanned by e_1, e_2. Then G_p is the set of matrices of the block form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.

Therefore $G_2(\mathbb{R}^4) = G/G_p$.

12.2 Whitney's Embedding and Approximation Theorems

Theorem 154 (Whitney Embedding Theorem)**.** *Let M be a compact smooth n-manifold. Then there exists a smooth embedding of M into $\mathbb{R}^{2n+1}$ and there exists a smooth immersion of M into $\mathbb{R}^{2n}$.*

Theorem 155 (Whitney's Approximation Theorem)**.** *Let $f : M \to R^k$ be a smooth map with $k \geqslant 2n+1$. Then for all $\varepsilon > 0$ there exists a smooth embedding $\widetilde{f}$ such that $\|f(p) - \widetilde{f}(p)\|_\infty < \varepsilon$.*

Proof of Theorem 154. First we need to embed M into $\mathbb{R}^r$ for some very large dimension r. By Proposition 43, there exists a finite regular cover $\{W_i\}$ of M. Now we build our regular partition of unity $\lambda_i : W_i \to \mathbb{R}$. Now define $f : M \to R^{m(n+1)}$ given by

$$x \mapsto (f_1, \ldots, f_m),$$

where $f_i(x) = (\lambda_i \varphi_i(x) \lambda_i(x))$. We show that f is a smooth embedding. It is easy to see that every pushforward is injective and that f is injective. Because M is compact, f is a smooth embedding.

Now given an embedding $M \to \mathbb{R}^r$ with $r > 2n+1$, we will attempt to embed M into $\mathbb{R}^{r-1}$. For $v \in S^{r-1}$, define $\pi_V : \mathbb{R}^r \to \mathbb{R}^{r-1}$ be the projection parallel to v. We want to choose $v \in S^{r-1}$ such that $\pi_V : M \to \mathbb{R}^{r-1}$ is an embedding. We need two conditions on v:

1. $v \neq (p-q)/\|p-q\|$ for all $p, q \in M$.
2. $v \neq w/\|w\|$ for $w \in TM \subset T\mathbb{R}^n$.

For the first condition, define the smooth map $F_1 : M \times M \setminus \Delta \to S^{r-1}$. We need $v \in S^{r-1} \setminus \mathrm{Im}(R_1)$. For the second condition, let $T_1M = \{w \in TM \mid |w| = 1$. Then T_1M is a smooth $(2n-1)$-manifold, so let $F_2 : T_1M \to S^{r-1}$ given by projection in the second coordinate. Thus we need to choose $v \in S^{r-1} \setminus \mathrm{Im}(F_2)$.

Now we need the following result:

Theorem 156. *Let* M, N *be smooth manifolds with* $\dim M < \dim N$ *and let* $F : M \to N$ *be a smooth map. Then the subset* $N \setminus \mathrm{Im}(F)$ *is dense in* N*. In fact, the image of* F *has measure* 0*.*

Therefore the desired ν exists. Running this procedure until $r = 2n + 1$, we have the desired embedding. Doing it again, we obtain an immersion $M \to \mathbb{R}^{2n}$. □

13 Lecture 13 (Oct 16)

Last time we stated Theorem 156, which says that the image of a smooth map $M \to N$ from a lower-dimensional manifold is measure zero. Recall the definition of measure from analysis. Here, we will consider the Lebesgue measure.

Remark 157.

1. Being measure 0 is a local property.
2. $\mathbb{Q}^n \subset \mathbb{R}^n$ has measure 0,
3. If $k < n$, then $\mathbb{R}^k \subset \mathbb{R}^n$ has measure 0.

Theorem 158 (Sard)**.** *Let* $F : M \to N$ *be a smooth map. Then the set of critical values has measure zero.*

13.1 Tensors Let V be a real vector space, and define $T^k(V) = (V^*)^{\otimes k}$, called the space of *covariant* k*-tensors*. Equivalently, this is the space of multilinear maps $V^{\otimes k} \to \mathbb{R}$.

Example 159. The determinant is an alternating multilinear map $\bigwedge^n V \to \mathbb{R}$.

Define $T_\ell(V) = V^{\otimes \ell}$ to be the space of *contravariant* ℓ*-tnesors*. Similarly, we can define $T^k_\ell V = T^k(V) \otimes T_\ell(V)$, the space of *mixed vectors of type* (k, ℓ).

Now we will pass to the vector bundles. Let $E \to M$ be a real vector bundle of rank n. We may define the associated tensor bundles $T^k_\ell(E)$ analogously to the case of vector spaces.

Let $\{e_i\}$ be a basis of V with dual basis $\{\varepsilon_i\}$. Then we can form a basis $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \otimes e_{j_1} \otimes \cdots \otimes e_{j_\ell}\}$ of $T^k_\ell(V)$.

Example 160. Given a vector bundle $E \to M$, we can produce a smooth vector bundle $E \otimes E \to M$.

To do this, we use the construction lemma. To trivialize, we simply use the standard basis. The transition functions are simply given by the tensor powers of the original transitions. Finally, it is not hard to show that the transition functions are smooth.

Definition 161. Let M be a smooth n-manifold. Then define $T^k M$ to be the *bundle of covariant* k*-tensors*. Analogously, define $T_\ell M$ and $T^k_\ell M$.

Remark 162. $T^1 M = T^* M$, $T_1 M = TM$, ad $T_0 M = T^0 M = T^0_0 M = M \times \mathbb{R}$.

Definition 163. A (k, ℓ)*-tensor field* is a smooth section of $T^k_\ell M$.

Example 164. $H^0(M, T_0 M) = H^0(M, T^0 M) = H^0(M, T^0_0 M) = C^\infty(M)$.

In local coordinates, we can write a tensor field as

$$\sigma = \sum \sigma^J_I dx^{\otimes I} \otimes \partial_x^{\otimes I}.$$

Proposition 165. *Consider an $\mathbb{R}$-linear map $\phi : \mathfrak{X}(M)^{\otimes k} \to C^\infty(M)$. Suppose that ϕ is $C^\infty(M)$-linear. Then there exists a unique $\sigma \in H^0(M, T^k M)$ such that $\sigma_p(x_1, \dots, x_k) = \phi(x_1, \dots, x_k)(p)$.*

Definition 166. Let $F : M \to N$ be smooth. Then there is a pullback $F^* : H^0(N, T^k N) \to H^0(M, T^k M)$ defined analogously to the pullback of 1-forms.

Remark 167. If $k = 0$, then this is just the pullback of smooth functions.

If $k = 1$, this is given by

$$\begin{aligned} (F^* \, df))_p(X) &= df_{F(p)}(F_{*,p}X) \\ &= F_{*,p}(X)(f) \\ &= X_p(f \circ F) \\ &= d(f \circ F)_p(X). \end{aligned}$$

Therefore $F^*(df) = d(F^* f)$.

In local coordinates, the computations become very cumbersome. Fortunately, in actual mathematics, people rarely work with more than four indices.

Proposition 168. *Let $\xi \in H^0(N, T^k N), \eta \in H^0(N, T^\ell N)$. Then $F^*(\xi \otimes \eta) = F^*(\xi) \otimes F^*(\eta)$. In addition, $(G \circ F)^* = F^* \circ G^*$ and $F^*(f\sigma) = (f \circ F) F^* \sigma$.*

13.2 Riemannian Metrics

Definition 169. A smooth covariant 2-tensor field $g \in H^0(M, T^2 M)$ is a *Riemannian metric* if for all $p \in M$, g_p is a symmetric, positive definite, nondegenerate bilinear form. The pair (M, g) is called a *Riemannian manifold*.

In local coordinates, we write $g = \sum g_{ij} dx^i \otimes dx^j$. Then g is a Riemannian metric if (g_{ij}) is a positive-definite symmetric matrix.

Example 170. The standard example is the Euclidean metric on $\mathbb{R}^n$. A non-example from physics is the Lorentz spacetime metric, which has signature $(3, 1)$.

14 LECTURE 14 (OCT 21)

We continue our example of Riemannian manifolds from last time.

14.1 Riemannian Manifolds Continued

Definition 171. Let (M, g) be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a smooth curve. Define the *length* of γ to be

$$L_g(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t)) \, dt.$$

Definition 172. Let M, g be a connected Riemannian manifold. Define the *distance* between p, q to be the infimum over all paths γ from p to q of $L(\gamma)$.

Proposition 173. *Distance is well-defined.*

Proposition 174. *Every manifold is Riemannian.*

Proof. Let $(U_\alpha, \varphi_\alpha)$ cover M. Now let f_α be a partition of unity subordinate to U_α. Define g_α to be the standard Riemannian metric on $\mathbb{R}^n$. Then let $g = \sum f_\alpha g_\alpha$. It is easy to see that this is a metric. $\square$

Theorem 175. TM *and* T^*M *are isomorphic vector bundles.*

Proof. Choose a metric g on M. Define $\widetilde{g} : TM \to T^*M$ by

$$X \mapsto g_p(X, -).$$

This is the standard isomorphism $V \to V^*$ given by a nondegenerate bilinear form. This is a fiberwise isomorphism, so we simply need to check that this is smooth, which can be done locally, and is easy. $\square$

Definition 176. Let (M, g) be a Riemannian manifold. Then consider a local frame $\{e_i\}$ on $U \subset M$ is an *orthonormal frame* with respect to g if $g(e_i, e_j) = \delta_{ij}$.

Definition 177. Let $G \subset GL_n(\mathbb{R})$ be a Lie subgroup. Then a (smooth) rank-n vector bundle E *admits a* G*-reduction to a* G*-bundle* if there exists a trivialization of E such that the corresponding transition functions have image lying in G. With such a reduction, E is called a G-bundle.

Example 178. If E is a trivial bundle. Then E admits a reduction to a $\{e\}$-bundle.

Example 179. If E is an $O(n)$-bundle, then E is self-dual. This follows from the definition of $O(n)$.

14.2 Almost Complex Structures Recall that $\text{Hom}(V, W) = V^* \otimes W$. Thus $H^0(M, T^1_1 M) = H^0(M, \text{End}(TM))$.

Definition 180. Let $J \in H^0(M, T^1_1 M$. J is called an *almost complex structure* on M if for all $p \in M$, $J_p : T_p M \to T_p M$ satisfies $J_p^2 = -\text{Id}_p$.

Proposition 181. *If* M *admits an almost complex structure, then* M *has even dimension. Also,* TM *admits a reduction to a* $GL_n(\mathbb{C})$*-bundle.*

Now from linear algebra, if $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies $J^2 = -\text{Id}$, then there exists a basis $\{x_i, y_i\}$ of $\mathbb{R}^{2n}$ such that J acts like multiplication by $i \in \mathbb{C}$. For vector bundles, this applies to local trivializations and loca frames. Now consider the complex conjuation of J, allowing us to form the complex vector bundle $\overline{E}_J \simeq E_J^*$.

14.3 Differential Forms Differential forms form a basis for the link between analysis and topology, which culminates in the Atiyah-Singer index theorem. We begin with some multilinear algebra, then build forms over a manifold.

14.3.1 Multilinear Algebra Let V be a n-dimensional real vector space and consider the symmetric group S_k. Recall the notion of the sign of a permutation σ.

Definition 182. $T \in T^k(V)$ is alternating if for all $X_1, \dots, X_k \in V$, $T(X_1, \dots, X_n) = {}^{\sigma}T(X_1, \dots, X_n)$.

Denote by $\bigwedge^k V$ to be the subspace of alternating elements of $T^k(V)$.

Definition 183. Define the alternating projection by

$$T \mapsto \frac{1}{k!} \sum_{\sigma} \operatorname{sign}(\sigma)\ {}^{\sigma}T.$$

A basis for $\bigwedge^2 V$ is simply $e^i \otimes e^j - e^j \otimes e^i$. This space has dimension $\binom{n}{2}$. Also, the top exterior power has dimension 1 and basis element $T_0 = \sum_{\sigma} \varepsilon^{\otimes \sigma} \operatorname{sign}(\sigma)$.

We can verify that T_0 acts by the determinant and that any linear map $V \to V$ pulls back to $\bigwedge^k V$. On the top exterior power, the pullback is multiplication by the determinant.

15 Lecture 15 (Oct 23)

Notation 184. For $I = (i_1, \dots, i_k)$, define $\varepsilon^I = k!\mathrm{Alt}(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k})$. We call this the elementary alternating tensor.

Recall the standard basis for $\bigwedge^k V$. Also, note that

$$\varepsilon^I(X_1, \dots, X_k) = \det \begin{pmatrix} \varepsilon^{i_1}(X_1) & \cdots & \varepsilon^{i_1}(X_k) \\ \vdots & & \vdots \\ \varepsilon^{i_k}(X_1) & \cdots & \varepsilon^{i_k}(X_k) \end{pmatrix}.$$

Definition 185. For $\omega \in \bigwedge^k(V), \eta \in \bigwedge^\ell(V)$, define the *wedge product* by

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(\omega \otimes]eta).$$

Proposition 186. *1. $\varepsilon^I \otimes \varepsilon^J = \varepsilon^{IJ}$ if I, J are distinct.*

2. *The wedge product is bilinear.*
3. *The wedge product is associative.*
4. *The wedge product is graded-commutative.*[26]
5. *If $\omega^1, \dots, \omega^k \in V^*$, then $(\omega_1 \wedge \cdots \wedge \omega^k)(X_1, \dots, X_k) = \det(\omega^j(X_i))$.*

Definition 187. The exterior algebra of V is given by

$$\bigwedge^{*}(V) = \bigoplus_{k=0}^{n} \bigwedge^{k}(V).$$

[26] This suggests a cohomology theory.

Note that the exterior algebra is graded-commutative.

Definition 188. The *contraction* or interior multiplication on $\bigwedge^*(V)$ is defined by

$$i_X(\omega)(x_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}).$$

Lemma 189. *1.* $i_X^2 = 0$.

2. $i_{aX+bY} = ai_X + bi_Y$.
3. *For all* $\omega \in \bigwedge^k(V), \eta \in \bigwedge^\ell(V)$, $i_X(\omega \wedge \eta) = i_X(\omega) \wedge \eta + (-1)^k \omega \wedge (i_X \eta)$.

Example 190. For $I = (i_1, \ldots, i_k)$, then $i_{e_\ell} \varepsilon^I = 0$ if $\ell \notin I$, and $(-1)^{s+\ell} \varepsilon^{i_1} \wedge \cdots \wedge \widehat{\varepsilon^{i_s}} \wedge \cdots \wedge \varepsilon^{i_k}$ if $\ell = i_s$.

Example 191. Let $\omega \in \bigwedge^2(V)$. Ten write $\omega = \sum w_{ik} \varepsilon^i \wedge \varepsilon^j$. Now form an antisymmetric matrix from the ω_{ij}. Then define $\widetilde{\omega} : V \to V^*$ by $X \mapsto i_X \omega$. Writing $X = \sum X^i e_i$ and computing, we find that $\widetilde{\omega}$ is an isomorphism if $\det(w_{ij}) = 0$.

Definition 192. If $\omega \in \bigwedge^2(V)$ is nondegenerate, then ω is a *symplectic form* on V.

Lemma 193. *Suppose (V, ω) is symplectic. Then there exists a basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$. This is called a symplectic basis.*

Note the standard symplectic form is $\omega = \sum dx^i \wedge dy_i$.

Proof of Lemma 193. First, for e_1, choose f_1 such that $\omega(e_1, f_1) = 1$.Let W be the span of e_1, f_1. Then take the orthogonal complement $W^\perp$.

First we show taht $W \oplus W^\perp = V$. To see that the intersection is trivial. Then given $v \in V$,

Next we show that ω is a symplectic form restricted to $W^\perp$. This is easy assuming the above and using nondegeneracy of ω. ☐

Definition 194. Let (V, ω) be a symplectic space. A subspace $W \subset V$ is

- *Symplectic* if $W \cap W^\perp = \{0\}$;
- *Isotropic* if $W \subset W^\perp$;
- *Coisitropic* if $W \supset W^\perp$;
- *Lagrangian* if $W = W^\perp$.

Exercise 195 (Hw4 Problem 12-9)**.** For each $W \subset (V, \omega)$ of the above type, there exists a symplectic basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ such that

1. If W is symplectic, then $e_1, \ldots, e_k, f_1, \ldots, f_k$ is a basis for W.
2. If W is isotropic, then W is the span of $e_1, \ldots, e_k$.
3. If W is coisotropic, then W is the span of $e_1, \ldots, e_n, f_1, \ldots, f_k$.
4. If W is Lagrangian, then W is the span of $e_1, \ldots, e_n$.

There are two generalizations. First, we relax nondegeneracy of ω and obtain vectors $g_1, \dots, g_k$ that kill all of the e_i, f_i. The g_i measure the degeneracy of ω.

If (V, ω) is a symplectic space and g is a metric, then there exists a symplectic basis which is orthonormal with respect to g.

16 LECTURE 16 (OCT 28)

Today we will discuss manifold versions of the constructions we performed last time.

16.1 Differential Forms on Manifolds Define $\bigwedge^k M$ to be the k-th exterior power of the tangent bundle. This is a smooth vector bundle of rank $\binom{n}{k}$.

Definition 196. A smooth section of $\bigwedge^k M$ is called a *differential* k*-form*.

We will denote the space[27] of differential k-forms by $\Omega^k(M)$. This is a subspace of $H^0(M, T^kM)$. We may define the wedge product and contraction of differential forms as we did for vector spaces. This is a graded derivation. In addition, we may define local versions of all of these notions.

Definition 197. The pullback of differential k-forms is defined by $F^*\omega(X) = \omega(F_*X)$.

Lemma 198. *Let* $F : M \to N$ *be a smooth map. Then*

1. *The pullback is linear.*
2. *The pullback respects the wedge product.*[28]
3. *If* $X \in H^0(M, TM), Y \in H^0(N, TN)$ *are* F*-related, then* $i_X(F^*\omega) = F^*(i_Y\omega)$.
4. *In any smooth chart,* $F^*(\sum_I \omega_I dy^{\wedge I}) = \sum_I (\omega_I \circ F) d(y^{\wedge I} \circ F)$.

Proposition 199. *If* M, N *have equal determinant, then* F^* *is simply the determinant of the Jacobian.*

Definition 200. The *exterior algebra*[29] $\Omega^*(M)$ is defined by $\bigoplus_k \Omega^k(M)$.

16.2 Exterior Differentiation

Theorem 201. *Let* M *be a smooth manifold. Then there exists a unique* $\mathbb{R}$*-linear map* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *such that:*

1. *For* $f \in C^\infty(M) = \Omega^0(M)$, df *is the differential of* f *as defined before.*
2. d *is a graded derivation.*
3. $d^2 = 0$.

Moreover, these imply:

(a) *In local coordinates,* $d\left(\sum_I \omega_I dx^{\wedge I}\right) = \sum_I d\omega_I \wedge dx^{\wedge I}$.

[27] sheaf
[28] Cohomology, anyone?
[29] complex

(b) *If* $\omega = \widetilde{\omega}$ *on open* $U \subset M$ *then* $d\omega = d\widetilde{\omega}$.

(c) *For* $U \subset M$ *open,* $d(\omega|_U) = (d\omega)|_U$.

Lemma 202. *Let* $F : M \to N$ *be a smooth map. Then* $d(F^*\omega) = F^*(d\omega)$.

The proof involves computing in a local chart using Lemma 198.

Proposition 203. *Let* $\omega \in \Omega^1, X, Y \in H^0(M, TM)$. *Then* $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$. *More generally, for* $\omega \in \Omega^k(M)$ *and* $X_1, \ldots, X_{k+1}$ *vector fields, we have*

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_i (-1)^{i-1} X_i \omega(X_1, \ldots, X_{k+1}) + \sum_{i \leqslant j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, X_{k+1}).$$

Example 204. Let X, Y be vector fields. Then if $[X, Y] = 0$, α, β are dual to X, Y, and $\alpha_p(Y_p) = 0 = \beta_p(X_p)$, then $d\alpha = 0 = d\beta$.

Definition 205. $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$. It is *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

Note that exact forms are closed. Thus we may define a cohomology theory, called *de Rham cohomology*, and denoted $H^k_{dR}(M)$.

Example 206. All de Rham cohomology groups are trivial on $\mathbb{R}^n$. However, $H^1(S^1)$ is not trivial.

Definition 207. $\omega \in \Omega^2(M)$ is called a *symplectic form* if it is closed and nondegenerate.

The pair (M, ω) is called a *symplectic manifold*.

Definition 208. Let (M, ω) be a symplectic manifold of dimension $2N$. Let $Q \subset M$ be a submanifold of dimension N. Let $i : Q \hookrightarrow M$ be the smooth embedding. Then Q is a *Lagrangian submanifold* if $i^*\omega = 0$.

Example 209. Let (M, ω) be $\mathbb{R}^{2n}$ with the standard symplectic form. Then let Q be given by setting all y^i to be constant.

Definition 210. A diffeomorphism $F : (M, \omega) \to (M', \omega')$ is a *symplectomorphicm* if $F^*\omega' = \omega$.

Exercise 211 (Homework)**.** Suppose $F : M \to M$ is a symplectomorphism. Then let $N = M \times M$. Let $\omega_N = \pi_1^*\omega - \pi_2^*\omega$. Then

1. N is a symplectic manifold.
2. The graph of F is a Lagrangian submanifold of N.

Example 212. Suppose M is smooth. Then T^*M has a canonical symplectic form. We will define $\tau \in \Omega^1(T^*M)$ as follows:

For all $p \in M, \nu \in T_p^*M$, let $\tau_{(p,\nu)} = \pi^*(p, \nu) \circ V$. τ is called the tautological 1-form for T^*M. Then we define the canonical symplectic form to be $\omega = -d\tau$.

17 Lecture 17 (Oct 30)

We will have our makeup class on November 11th at 1PM.

Definition 213. Two submanifolds $N_1, N_2 \subset M$ are *transverse* if for all $p \in N_1 \cap N_2$, T_pN_1, T_pN_2 span T_pM.

Exercise 214 (Homework)**.** Let $S \subset T^*M$. Then S is Lagrangian and is transverse to each fiber T_p^*M and intersects each fiber at one point if and only if S is the image of a closed 1-form.

Example 215. For $K \subset \mathbb{R}^3$ a knot, define $L_K = \{\xi \in T^*\mathbb{R}^3 \mid \xi(v) = 0 \text{ for all } v \in T_pK, p \in K\}$. Then L_K is Lagrangian.

17.1 Orientations

We first define orientations for vector spaces and then for manifolds.

Definition 216. An *orientation* for an $\mathbb{R}$-vector space V is an equivalence class of ordered bases of V. Here $(e_1, \dots, e_n) \simeq (f_1, \dots, f_n)$ if the change of basis matrix A has positive determinant.

Lemma 217. *Alternatively, an orientation is a choice of a nonzero element $\Omega \in \bigwedge^n(V)$.*

The proof of this is a basic fact about the top exterior power of a vector space.

Definition 218. Given a choice of orientation $[(e_1, \dots, e_n)]$, we say $(e_1', \dots, e_n')$ is *positively oriented* if it is in the same equivalence class. Otherwise, we call it *negatively oriented*.

Definition 219. Given an orientation $[\Omega]$, we say that Ω is *positively oriented* if $\Omega(e_1, \dots, e_n) > 0$.

Now let M be a smooth manifold.

Definition 220. An *orientation* on M is an equivalence class of non-vanishing continuous sections of the canonical bundle K_M. Here $\Omega, \Omega' \in H^0(M, K_M)$ are equivalence if $\Omega = \lambda\Omega'$ where $\lambda \in C^0(M)$ is a everywhere positive continuous function.

Definition 221. M is *orientable* if it admits an orientation. A global frame $(e_1, \dots, e_n)$ is *positively oriented* if $(e_1|_p, \dots, e_n|_p)$ is positively oriented with respect to Ω_p.

Definition 222. A choice of $\Omega \in [\Omega]$ is called a *volume form*.[30]

Remark 223. $S^2, \mathbb{T}^2$ are orientable, but $\mathbb{RP}^2$ is not. Also, all parallelizable manifolds are orientable. Finally, if M is connected and orientable, there are only two orientations of M.

Definition 224. A collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ is *consistently oriented* if $\det J(\varphi_\beta\varphi_\alpha^{-1}) > 0$ for all α, β.

Exercise 225 (Lee, Exercise 13.3)**.** M is orientable if and only if it has a consistently orientable collection of charts.

Definition 226. A local diffeomorphism $F : M \to N$ between smooth oriented manifolds is *orientation preserving* (resp *reversing*) if $[F^*\Omega_N] = [\Omega_M]$ (resp. $-[\Omega_M]$).

Proposition 227. *Let M have dimension at least 2 and $N \subset M$ be a hypersurface. Suppose there exists a continuous section $S : N \to TM|_N$ such that for all $p \in N$, $S(p) \notin T_pN$. Then for all orientations Ω_M on M, there exists an induced orientation Ω_S on N determined by S.*

Remark 228. S and $-S$ induce the two orientations on N if N is connected. In addition, define $\nu_N = i^*(TM)/TN$ the *normal bundle* of N in M. Then S descends to a section of N. If M is

[30] More on this later in the course.

orientable, then Ω_M induces an orientation on N if and only if ν_N is trivial. In this case, we call N *co-orientable.*

Example 229. Consider $S^n \subset \mathbb{R}^{n+1}$. Choose the standard orientation on $\mathbb{R}^{n+1}$. If S^n is embedded as the unit sphere, let $S(p) = p \in \mathbb{R}^{n+1}$. then Ω_S is an orientation for S^n.

Example 230. $\mathbb{RP}^2$ is not orientable. To show this, define the antipodal map $A : S^2 \to S^2$. This gives a smooth covering $\pi : S^2 \to \mathbb{RP}^2$. Now let $S : S^2 \to T\mathbb{R}^3|_{S_2}$ be as in the previous example and let Ω_S be the orientation defined above. It is easy to see that A is orientation reversing.

Now suppose $\mathbb{RP}^2$ is orientable with volume form Ω. Then let $\widetilde{\Omega} = \pi^*\Omega$. Then $\pi \circ A = \pi$, which implies that $A^*\widetilde{\Omega} = -\widetilde{\Omega}$, which gives us a contradiction.

Exercise 231 (Homework)**.** $\mathbb{RP}^n$ is orientable if and only if n is odd.

Example 232. Let M be orientable and $f \in C^\infty(M)$. Then $f^{-1}(r)$ is an orientable hypersurface for all regular values r of f.

18 Lecture 18 (Nov 4)

The makeup class has been moved to November 11 at 10 AM with bagels, cream cheese, and lox.

18.1 Orientation Coverings

Lemma 233. *Let M be a smooth manifold of dimension n. Then R^+ acts on $\bigwedge^n M$ by fiberwise multiplication. This action is smooth, free, and proper.*

Definition 234. The quotient space $\widetilde{M}$ is called the *orientation covering of* M.

Theorem 235. *Let M be a smooth connected n-manifold. Then we have a smooth surjective map $\widetilde{M} \to M$. Then:*

- *M is orientable if and only if there exists a global section.*
- *M is not orientable if and only if $\widetilde{M}$ is connected.*

Theorem 236. *Let M be a smooth connected n-manifold which is not oriented. Then there exists a unique double cover $\widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. Moreover, $\widetilde{M}$ is diffeomorphic to the orientation covering.*

Proposition 237. *Suppose M is a connected oriented smooth manifold and Γ is a discrete group acting smoothly, freely, and properly on M. We say the action is orientation-preserving if each diffeomorphism $\gamma \in \Gamma$ is orientation preserving. Then M/Γ is orientable if and only if Γ is orientation-preserving.*

Corollary 238. *If $\pi_1(M)$ has no subgroups of index 2, then M is orientable. More generally, there is a bijection between nontrivial line bundles over M and morphisms $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$.*

The idea of the proof is to construct the first Stiefel-Whitney class[31] of E.

Proposition 239. *Let (M, g) be an orientable Riemannian manifold of dimension n. Then there exists a unique orientation form Ω such that for all oriented local orthonormal frames, we have $\Omega(e_1, \ldots, e_n) = 1$. Ω us called a volume form for (M, g).*

[31] This is a characteristic class like the Chern class.

Lemma 240. *Let (M, g) be as above. Write $g = (g_{ij})$ in a local positively oriented coordinate chart. Then locally, the volume form is given by*

$$\sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

Proof. Let $\Omega = f\, dx^1 \wedge \cdots \wedge dx^n$ with $f > 0$. Let $(e_1, \dots, e_n)$ and $\varepsilon^1, \dots, \varepsilon^n$ be a positively oriented orthonormal frame and its coframe. The rest of the proof is computing the determinant of the change of basis matrix A from e_j to the δ_i coordinates. □

18.2 Manifolds with Boundary

Define $\mathbb{H}^n$ to be the closed upper halfspace in $\mathbb{R}^n$.

Definition 241. M is an n-dimensional *smooth manifold with boundary*[32] if

1. M is Housdoeff and second-countable;
2. M has a cover $\{U_\alpha, \varphi_\alpha\}$ where $\varphi\alpha$ is a homeomorphism onto its image in $\mathbb{H}^n$ or $\mathbb{R}^n$;
3. The transition functions are smooth.

The notion of a manifold with boundary extends to a manifold with corners, which are modeled on intersections of closed half-spaces. For simplicity, we will stick to boundaries.

We may also define smooth maps between manifolds with boundaries and also T_pM the space of derivations with local representations.[33] We can also construct the cotangent bundle, tensor fields, and differential forms as before. In addition, we can define pushforwards and pullbacks.

Definition 242. We may define the interior and the boundary of a manifold with boundary M.

Proposition 243. *M is the disjoint union of the interior and the boundary. In addition, the boundary has a unique smooth structure such that $\partial M \to M$ is a smooth embedding.*

Definition 244. Let $p \in \partial M$, $N \in T_pM$ such that $N \notin T_p(\partial M)$. Then N is *inward pointing* if there exists $\varepsilon > 0$ and smooth path $\gamma : [0, \varepsilon] \to M$ such that $\gamma'(0) = N$.

Lemma 245.

1. *If N is inward pointing then for all charts, $dx^i(N) > 0$.*
2. *If there exists a chart such that $dx^n(N) > 0$, then N is inward-pointing.*

Lemma 246. *There exists a smooth outward pointing vector field N along ∂M.*

Corollary 247. *The normal bundle of ∂M is trivial.*

19 LECTURE 19 (NOV 6)

Proof of Lemma 246. Take charts with $\varphi_\alpha : U_\alpha \to \mathbb{H}^n$. Then let $V_\alpha = U_\alpha \setminus \partial M$. This is an atlas for the boundary.

Now let $\{f_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. Then we define $N(p) = \sum_\alpha f_\alpha(p)N_\alpha(p)$, and it is easy to check that this is inward pointing. □

[32] For some notions of stratification, this admits a stratification.
[33] All tangent spaces here still have the same dimension.

Proposition 248. *Let M be an oriented smooth manifold with orientation Ω_M and suppose M has nonempty boundary. Define $\Omega_{\partial M} = i_{-N}\Omega_M$. Then this is a well-defined orientation for the boundary.*

Proof. Let N' be another inward pointing normal vector field. Thus $N' = fN + T$, where f is a positive smooth function and T is a vector field on ∂M. Then we simply compute $i_{-N'}\Omega_M(p)$ and see that the orientation is well-defined. □

Example 249. Let $M = \mathbb{H}^n$ and take the standard orientation form $dx^1 \wedge \cdots \wedge dx^n$. Then we have

$$i_{\partial_n}(dx^1 \wedge \cdots \wedge dx^n) = (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.$$

Lemma 250. *Let M be connected and let S be a connected submanifold of codimension 1. Suppose $M \setminus S = M_1 \sqcup M_2$ is disconnected. Then $\overline{M}_i$ are manifolds with boundary S. Also, $\nu_S \to S$ is trivial.*

Example 251. Let $M = S^2, S = S^1$ Then $S^2 \setminus S^1$ is not connected, so the normal bundle is trivial.

Example 252. If $M = \mathbb{T}^2, S = S^1$, then ν_S is trivial but $M \setminus S$ is not trivial.

Example 253. If $M = \mathbb{RP}^2, S = \mathbb{RP}^1$, then ν_S is the tautological bundle. Therefore $\mathbb{RP}^2 \setminus \mathbb{RP}^1 = \mathbb{R}^2$ is connected.

Theorem 254 (Tubular Neighborhood Theorem)**.** *If ∂M is compact, then there exists a neighborhood of ∂M in M diffeomorphic to $\partial M \times [0, \varepsilon)$.*

19.1 Integration Let $U \subset \mathbb{R}^n$ with orientation $dx^1 \wedge \cdots \wedge dx^n$. Suppose $\omega \in \Omega^n(U)$ with compact support. Thus we have $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some compactly supported f. Then D is an oriented compact manifold with boundary, which we call the *domain of integration*.

Then we define

$$\int_U \omega := \int_D f dx_1 \cdots dx_n$$

as the usual Riemann (or Lebesgue) integral. Similarly, we can define integrals on subsets of $\mathbb{H}^n$.

Proposition 255. *Let D, E be domains of integration in $\mathbb{R}^n$ with $\omega \in \Omega^n(E)$. Let $F : D \to E$ be smooth such that F is a diffeomorphism on the interior of D that is orientation preserving. Then*

$$\int_E \omega = \int_D F^*\omega.$$

The proof of this fact is a change of basis calculation in multivariate calculus.

To turn all of this into a global story, we use a partition of unity. Here, M is an oriented smooth n-manifold. Let $\omega \in \Omega^n(M)$ with compact support. Choose a positively oriented atlas U_α and f_α be a partition of unity subordinate to U_α. Then we define

$$\int_M \omega = \sum_\alpha \varphi_\alpha(U_\alpha)(\varphi_\alpha^{-1})^* f_\alpha \omega.$$

Lemma 256. *The integral is independent of the chart and the partition of unity.*

Proposition 257. *Integration is linear and orientation-sensitive. In addition, integration is invariant under diffeomorphism.*

For a 0-dimensional manifold, a 0-form is simply a function to $\mathbb{R}$, and an orientation is an assignment of a sign to each point. Then the integral is simply a finite sum.

Theorem 258 (Stokes)**.** *Let* M *be an oriented smooth manifold of positive dimension. Then suppose* $\omega \in \Omega^{n-1}(M)$ *has compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. It suffices to prove this locally. We have a chart V with $\varphi : V \to \mathbb{H}^n$ a positively oriented chart. Then write

$$\omega = \sum_{i=1}^{n} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Then we have

$$d\omega = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^k} dx^j \wedge dx^1 \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Therefore we have

$$\begin{aligned}
\int d\omega &= \int_{-R}^{R} \cdots \int_{-R}^{R} \int_0^R \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} \, dx^1 \cdots \, dx^n \\
&= (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(0) \, dx^1 \cdots \, dx^{n-1} \\
&= \int_{\varphi(V) \cap \partial \mathbb{H}^n} \omega.
\end{aligned}$$

□

20 LECTURE 20 (NOV 11)

We will complete the global proof of Stokes' theorem. We simply cover the support of ω by finitely many positively oriented charts and let f_α be a partition of unity subordinate to V_α. Then we simply have

$$\begin{aligned}\int \omega &= \int \sum f_\alpha \omega \\ &= \sum \int f_\alpha \omega \\ &= \sum \int d(f_\alpha)\omega \\ &= \sum \int df_\alpha \wedge \omega + f_\alpha d\omega \\ &= \int d\omega.\end{aligned}$$

Now we take a detour into geometry and analysis,

Definition 259. Let (M, g) be an oriented Riemannian manifold with dimension n. A *Riemannian volume form* is $dV_g = \xi^1 \wedge \cdots \wedge \xi^n$, where ξ^i form a positively oriented orthonormal coframe. The *volume* of M is $\int_M dV_g$. More generally, if f is a compactly supported continuous function on M, define the *integral of* f *over* M to be

$$\int_M f = \int_M f \, dV_g.$$

Remark 260. The volume form generalizes to densities, which in turn generalize to measures.

Lemma 261. *Let* (M, g) *and* f *be continuous and compactly supported. Then if* $f \geqslant 0$, $\int_M f \geqslant 0$. *In addition,* $f = 0$ *if and only if* $f \geqslant 0$ *and* $\int_M f = 0$.

Now let $\widetilde{g}$ be an induced Riemannian metric on ∂M and let N be an outward-pointing unit normal vector field along ∂M. Now define the volume form on ∂M by $d\widetilde{V_g} = i_N(dV_g)$.

Note that for any $X \in H^0(M, TM)$, we have $X = \langle N, X\rangle_g N + T$. This allows to write

$$i_X dV_g = \langle N, X\rangle_g d\widetilde{V}_g.$$

Definition 262. For all vector fields X, define the divergence $\operatorname{div}(X)$ by $\operatorname{div}(X) dV_g = d(i_X dV_g)$.

Theorem 263 (Divergence Theorem)**.** *Let* (M, g) *be a Riemannian manifold with outward pointing unit normal vector field* N. *Then*

$$\int_M \operatorname{div}(X) dV_g = \int_{\partial M} \langle X, N\rangle_g d\widetilde{V}_g.$$

This is simply a special case of Stokes' theorem. Also, the Divergence theorem is called Gauss theorem in $\mathbb{R}^3$.

Example 264. Consider $\mathbb{R}^n$ with the usual metric. Then the volume form is simply $dx_1 \wedge \cdots \wedge dx_n$. Then define $X = \sum x^i \partial_{x^i}$. Then we can verify that $\operatorname{div}(X) = n$.

20.1 De Rham Cohomology

Note here we do not need to know algebraic topology for this. Let M be a smooth n-manifold. Recall the de Rham complex

$$0 \to \Omega^0(M) \to \cdots \to \Omega^n(M) \to 0.$$

This is known as a *(co)chain complex*. For any such complex, we may define *(co)cycles* Z^p as $\ker(d)$ and *(co)boundaries* $B^p = \operatorname{Im}(d)$. Then we may define the *(co)homology* as $H^p = Z^p/B^p$. The cohomology of the de Rham complex is known as *de Rham cohomology*, denoted by $H^*_{dR}(M)$.

Proposition 265. *Let $G : M \to N$ be smooth. Then this induces a graded map $G^* : H^*_{dR}(N) \to H^*_{dR}(M)$. Moreover, cohomology is a functor from $\mathbf{Diff}^{op} \to \mathbf{Ab}$.*

Proof. Note that we have a map between the de Rham complexes of M, N that commutes with d. This induces a map on the cohomology. $\square$

Definition 266. $F, G : M \to N$ are *smoothly homotopic* if there exists a smooth homotopy $H : M \times [0, 1] \to N$ between them.

Lemma 267. *For all p, there exists a linear $h : \Omega^p(M \times I) \to \Omega^{p-1}(M)$ such that $h \circ d + d \circ h = i_1^* - i_0^*$.*

Remark 268. This is known as a *chain homotopy*.[34]

Proof. Define $h\omega = \int_0^1 i_{\partial_t}\omega$. $\square$

Theorem 269. *If $F, G : M \to N$ are smoothly homotopic, then $F^* = G^*$.*

Proof. Let ω be a closed form. Then

$$G^*\omega - F^*\omega = d(h \circ H^*\omega).$$

$\square$

Definition 270. Any two spaces M, N are *homotopy equivalent* if there exist maps $F : M \to N, G : N \to M$ such that $F \circ G$ and $G \circ F$ are homotopic to the identity.

Remark 271. Homotopy equivalence is the weakest equivalence relation on manifolds.[35]

Corollary 272. *If M, N are homotopy equivalent, then they have the same de Rham cohomology.*

Proof. Use the Whitney approximation theorem to construct a smooth homotopy from a continuous homotopy. Then use Theorem 269. $\square$

[34] This is the wrong notion of homotopy equivalence for chain complexes. The right notion leads to the derived category.
[35] For more general spaces, there is weak homotopy equivalence.

21 Lecture 21 (Nov 13)

We will compute some examples with de Rham cohomology. We note that:

1. If $M = M_1 \sqcup M_2$, then $H^*(M) = H^*(M_1) \times H^*(M_2)$.
2. If M is connected, then $H^0(M) = \mathbb{R}$.
3. If M is contractible, then $H^*(M) = H^0(M) = \mathbb{R}$.
4. If M is simply connected, then $H^1(M) = 0$.

Theorem 273. *Let G be a finite group acting freely on M. Recall that the quotient π to M/G is smooth. Then π^* has image the left-invariant cohomology classes and is injective.*

Proof. It is easy to see that $g^*\pi^* = \pi^*$, so π^* maps to the invariant elements. Now let ω be a closed form such that $\pi^*\omega = 0$ in cohomology. Therefore we have $\pi^*\omega = d\eta$. Then we consider

$$\widetilde{\eta} = \frac{1}{|G|} \sum_{g \in G} g^*\eta.$$

This will be completed later. □

Corollary 274. *If $\pi_i(M)$ is finite, then $H^1(M) = 0$.*

21.1 Some Homological Algebra

Here we will introduce the Mayer-Vietoris sequence.

Definition 275. A *(co)chain map* $F : A^* \to B^*$ is a graded map $A^k \to B^k$ that commutes with d.

It is easy to check that this induces a map on cohomology.

Definition 276. A *short exact sequence* is a sequence $0 \to A^* \to B^* \to C^* \to 0$ that is exact at every level.

Lemma 277 (Snake Lemma)**.** *A short exact sequence of (co)chain complex induces a long exact sequence of cohomology.*

Proof of this is given by a standard diagram chasing argument.

Remark 278. The connecting morphism $\delta : H^p(C) \to H^{p-1}(A)$ is useful to know.

Theorem 279 (Mayer-Vietoris Sequence)**.** *Consider a smooth manifold $M = U \cup V$. then the following is a short exact sequence:*

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0.$$

Proof. At M, we see that U, V form an open cover of M. At the middle term, all the terms are given by restriction, so the kernel contains the image. Then, we note that if $(i^* - j^*)(\eta, \eta') = 0$, then η, η' agree on $U \cap V$. Then because Ω^p is a sheaf, we have the desired result.

Finally, at $U \cap V$, let $\omega \in \Omega^p(U \cap V)$. Let φ, ψ be a partition of unity subordinate to U, V. Then define $\eta = \psi\omega$ on $U \cap V$ and 0 elsewhere. In addition, let $\eta' = -\varphi\omega$ on $U \cap V$ and 0 elsewhere. Then we have $(i^* - j^*)(\eta, \eta') = \omega$. □

Corollary 280. *Given $\omega \in Z^p(U \cap V)$, let $\eta = \psi\omega$ as in the above proof. Extend by 0 elsewhere. Then $\delta(\omega) = d\eta$.*

Theorem 281. *$H^p(S^n) = \mathbb{R}$ if $p = 0, n$ and 0 otherwise.*

Proof. Note we have $0 \to H^0(S^n) \to H^0(\mathbb{R}^n) \oplus H^0(\mathbb{R}^n) \to H^0(S^{n-1}) \to H^1(S^n) \to H^1(\mathbb{R}^n) \oplus H^1(\mathbb{R}^n) = 0$. Thus $H^1(S^n) = 0$. In addition, because $\mathbb{R}^n$ is contractible, we have $H^p(S^{n-1}) = H^{p-1}(S^n)$ for $0 < p < n$. Finally, we have $H^n(S^n) = \cdots = H^1(S^1) = \mathbb{R}$. $\square$

22 LECTURE 22 (NOV 18)

Mike gave the proof of Theorem 273. The idea is that π^* is injective and surjective at the level of differential forms by definition. Then we compute.

Corollary 282. $H^p_{dR} = \begin{cases} 0 & 0 < p < n \\ \mathbb{R} & p = 0 \\ 0 & p = n = 2m \\ \mathbb{R} & p = n = 2m+1 \end{cases}.$

Proof. The antipodal map on S^n is orientation preserving if and only if n is odd. $\square$

More generally, we have:

Theorem 283. *Let M be a compact connected smooth manifold. Then if M is not orientable, the top cohomology class vanishes, and if M is orientable, $H^n(M) \simeq \mathbb{R}$.*

Proof. Let $[\Omega_0]$ be an orientation of M. Let $b := \int_M \Omega_0 \neq 0$. Thus we have a surjective map to $\mathbb{R}$ given by integration. Now suppose $\int_M \omega = 0$. Choose a finite cover of M by $U_i \simeq \mathbb{R}^n$ and order the U_i so that $U_k \cap U_{k-1} \neq 0$. Now we proceed by induction.

For the base case, we note that $\int_{\mathbb{R}^n} \omega = 0$, so $\omega = d\eta$. Then for the inductive step, choose η with support contained in $M_k \cap U_{k+1}$ with $\int_{M_{k+1}} \eta = 1$. Then φ, ψ is a partition of unity subordinate to M_k, U_{k+1}. Let $c = \int_{M_{k+1}} \varphi_\omega$. Then we have $\int_{M_{k+1}} \varphi\omega - c\eta = 0$. By induction, there exists α such that $d\alpha = \varphi\omega - c\eta$. Similarly, we can find β such that $d\beta = \psi\omega$, and we see that $d(\alpha + \beta) = \omega$. $\square$

Recall the notion of the connected sum of topological spaces. For smooth manifolds, the connected sum is a smooth manifold, and $H^p(M_1 \# M_2) = H^p(M_1) \times H^p(M_2)$.

Proposition 284. *Let M be a compact connected orientable manifold with dimension at least 3. Fix $q \in M$ and $0 \leqslant k < n$. Then the inclusion map $M \setminus \{q\} \to M$ induces an isomorphism $H^k(M) \to H^k(M \setminus \{q\})$. If $k = n$, the induced map is 0.*

Proof. Let M_1 be an open ball and $M_2 = M \setminus B$. Then for $0 < p < n$, because balls are contractible, we have the desired result. For $k = 0$, M without a point is connected. Finally, the case of $k = n$ is a simple application of Mayer-Vietoris. $\square$

Example 285. Consider the Klein bottle. We can cut K into U, V, which are most homotopy equivalent to S^1. Then their intersection is a disjoint union of two S^1. By Mayer-Vietoris, we can see that the cohomology of the Klein bottle is $\mathbb{R}$ if $p = 0, 1$ and 0 if $p = 2$.

23 Lecture 23 (Nov 20)

Because algebraic topology is not a prerequisite for this course, today we will discuss the de Rham theorem.

Theorem 286. *The de Rham cohomology is equivalent to ordinary cohomology. Equivalently, it satisfies the Eilenberg-Steenrod axioms.*

Definition 287. A p-*simplex* is the convex hull $\langle v_1, \dots, v_p \rangle$, where the v_i are in general position. The *standard* p-*simplex* is given by the basis vectors in $\mathbb{R}^p$.

Definition 288. A *singular* p-*simplex* for a topological space M is a continuous map $\sigma : \Delta_o \to M$.

Definition 289. The p-*singular chains* on M of dimension p are $C_p(M) = \mathbb{R}\langle \text{p-singular chains on } M \rangle$.

For $0 \leqslant i \leqslant p$, define the ith *face map* $F_{i,p}$ by

$$(e_0, \dots, e_{p-1}) \mapsto (e_0, e_1 - e_0, \dots, \widehat{e_i}, \dots, e_{p-1} - e_0).$$

Now for a singular chain $\sigma : \Delta_p \to M$, we define the boundary

$$\partial \sigma = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}.$$

By linearity, we can extend to a map $\partial : C_p(M) \to C_{p-1}(M)$. This defines a chain complex $C_*(M)$, so we can define the *singular homology* $H_*(M)$. Then singular cohomology is defined by taking $\mathrm{Hom}(C_p, \mathbb{R})$ and then taking the cohomology of that.

Proposition 290. $H^p(M, \mathbb{R}) = H_p(M, \mathbb{R})^*$.

Proposition 291. *Singular homology and cohomology satisfy the same properties that de Rham cohomology satisfies.*

Theorem 292. *Singular homology and cohomology satisfy a Mayer-Vietoris sequence analogous to that of de Rham cohomology. The connecting homomorphisms satisfy* $\partial^* \gamma = \gamma \circ \partial_*$.

It turns out that we can take some triangulation or cell decomposition of our space to compute (co)homology. Now we let M be a smooth manifold. Then we can define the notion of a *smooth* p-*simplex*. Thus we can define the smooth chain complex $C_p^\infty(M)$. Because the face maps are smooth, we can define the smooth homology and cohomology H_*^∞, H_∞^*.

Note that we have a natural chain inclusion $C_p^\infty \to C_p(M)$.

Theorem 293. $i_* : H_p^\infty(M) \to H_p(M)$ *is an isomorphism.*

Proof of this uses the Whitney approximation theorem to construct a homotopy inverse to i.

The next part of relating analysis to topology is the de Rham theorem:

Theorem 294 (de Rham)**.** $H^*(M, \mathbb{R}) \simeq H^*_{dR}(M)$.

First, we define an integral over a smooth chain c by

$$\int_c \omega = \sum_{i=1}^{k} c_i \int_{\Delta_p} \sigma_i^* \omega.$$

Then we have

Theorem 295 (Stokes)**.** $\int_{\partial c} \omega = \int_c d\omega$.

Proof. We simply need to consider $c = \sigma$. We can check that $F_{i,p}$ is orientation preserving iff i is even. Then the result follows from the ordinary Stokes theorem. □

Now integration defines a homomorphism $\mathcal{J} : H^*_{dR}(M) \to H^p(M, \mathbb{R})$. To see that this is well-defined, we have for homologous chains c, c', we have

$$\int_c \omega - \int_{c'} \omega = \int_{\partial b} \omega = \int_b d\omega = 0.$$

Then if $\omega = d\eta$, we have

$$\int_c \omega = \int_c d\eta = \int_{\partial c} \eta = 0.$$

Lemma 296. *$\mathcal{J}$ is natural: it commutes with pullbacks and with the Mayer-Vietoris connecting homomorphisms.*

Proof. We will prove the Mayer-Vietoris part. We will show that for all $[\omega] \in H^{p-1}_{dR}(U \cap V)$ and $[e] \in H_p(M)$ that $\mathcal{J}(\delta([\omega]))[e] = (\delta^* \mathcal{J}[\omega])[e]$.

Choose $\sigma \in \Omega^p(M)$ and $c \in C_{p-1}(M)$ such that $[\sigma] = \delta^*[\omega]$ and $[c] = \partial_*[e]$. Then we have $\sigma = d\eta$ and $c = \partial d$, where $\omega = \eta - \eta'$, $[d + d'] = [e]$. Therefore

$$\begin{aligned}
\mathfrak{I}[\omega](\partial_*[e]) &= \int_c \omega = \int_{\partial d} \omega \\
&= \int_{\partial d} \eta - \int_{\partial d} \eta' \\
&= \int_d d\eta + \int_{d'} d\eta' \\
&= \int_d \sigma + \int_{d'} \sigma \\
&= \int_e \sigma \\
&= \mathcal{J}(\delta[\omega])[e].
\end{aligned}$$

□

Sketch of de Rham's Theorem. If M is a convex open set of $\mathbb{R}^n$, then $H^p_{dR}(M) = 0$ if $p \neq 0$ and $\mathbb{R}$ is $p = 0$. Because M is contractible, the same applies to $H^p(M, \mathbb{R})$. Then it is easy to check that $\mathcal{J}[1][\sigma] = 1$, so we have the base case.

Then suppose $M = U_1 \cup \cdots \cup U_k$, where each U_i is diffeomorphic to a contractible open set of $\mathbb{R}^n$. Assume J is an isomorphism for unions up to U_{k-1}. Then if $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$, we have the following diagram:

$$\begin{tikzcd}
H^{p-1}_{dR}(U) \oplus H^{p-1}_{dR}(V) \arrow[r] \arrow[d] & H^{p-1}_{dR}(U \cap V) \arrow[r] \arrow[d] & H^p_(dRM) \arrow[r] \arrow[d] & H^{p-1}_{dR}(U) \oplus H^{p-1}_{dR}(V) \arrow[r] \arrow[d] & H^p_{dR}(U \cap V) \arrow[d] \\
H^{p-1}(U) \oplus H^{p-1}(V) \arrow[r] & H^{p-1}(U \cap V) \arrow[r] & H^p M) \arrow[r] & H^{p-1}(U) \oplus H^{p-1}(V) \arrow[r] & H^p(U \cap V)
\end{tikzcd}$$

Now we use the inductive hypothesis and the five lemma. Finally, we go from a finite union to an arbitrary manifold using point-set topology and a second countable cover. □

24 Lecture 24 (Dec 4)

Today we will discuss integral curves and flows.

Definition 297. For $X \in H^0(M, TM)$, an *integral curve* of X is a map $\gamma : [a, b] \to M$ such that for all t, $\gamma'(t) = X(\gamma(t))$.

Definition 298. A *time-dependent vector field* X on M is a smooth map $X : I \times M \to TM$ that is a section for each T.

Definition 299. An *integral curve* of a time-dependent vector field X is a $\gamma : [a, b] \to M$ such that for all t, $\gamma'(t) = X(t, \gamma(t))$.

Example 300. Let $M = \mathbb{R}^4$ with the standard symplectic form. Then fix constants $m, g > 0$. Let $H : M \to \mathbb{R}$ (the total energy) be given by

$$(x_1, x_2, y_1, y_2) \mapsto mgx_2 + \frac{1}{2m}(y_1^2 + y_2^2).$$

Define $X = \omega(X, -) = dH$. Then the claim[36] is that an integral curve of X lies on $H^{-1}(c)$ for some $c \in \mathbb{R}$ (conservation of energy).

24.1 Local Representation Suppose $\gamma : (a, b) \to M$ is an integral curve of X and $\gamma(t) \in U$ for some chart (U, φ). Then locally, note that

$$\gamma'(t)(x^i) = \frac{d}{dt}(x^i \circ \gamma) = \frac{d}{dt}\gamma_i(t) = \gamma_i'(t).$$

[36] For proof, take Inanc's class.

In addition, we have

$$\begin{aligned} X(\gamma(t))(x^i) &= \sum_j X_j(\gamma(t))\frac{\partial}{\partial x^j}(x^i) \\ &= X_i(\gamma(t)) = (X_i \circ \varphi^{-1})(\varphi \circ \gamma(t)) \\ &= \widetilde{X_i}(\gamma_1(t), \dots, \gamma_n(t)). \end{aligned}$$

Therefore, the integral curve equation becomes

$$\begin{cases} \gamma_1'(t) = \widetilde{X_1}(\gamma_1(t), \dots, \gamma_n(t)) \\ \ddots \\ \gamma_n'(t) = \widetilde{X_1}(\gamma_1(t), \dots, \gamma_n(t)) \end{cases}. \tag{1}$$

Next we need a large existence and uniqueness theorem for ODEs:

Theorem 301. *Fix $U \subset \mathbb{R}^n$ with $X = (X_1, \dots, X_n) : U \to \mathbb{R}^n$ smooth. Then for all $t_0 \in \mathbb{R}$ and $x_0 = (x^1, \dots, x^n) \in \mathbb{R}^n$, consider the first order ODE (1) with initial consition $\gamma_i(t_0) = x^i$. Then:*

1. *There exists $t_0 \in (a, b) \subset \mathbb{R}$ and $x_0 \in U_0 \subset U$ such that for all $x \in U_0$, there exists a solution γ_x of (1) with the initial condition.*
2. *Any two differentiable solutions agree on their common domain.*
3. *Define $\Theta : (a, b) \times U_0 \to U$ given by $(t, x) \mapsto \gamma_x(t)$. Then Θ is smooth.*

Remark 302. Integral curves are *translation invariant*, where the translation is in the source interval.

Lemma 303. *For all $X \in H^0(M, TM)$, for all $p \in M$, there exists a unique maximal integral curve of X at p, $\gamma : J_p \to M$ where $0 \in J_p, \gamma(0) = p$.*

Proof. For uniqueness, we use uniqueness from Theorem 301. For existence, we know that at least one (γ, J) exists. Then we order all γ, J by inclusion. Then we use Zorn's lemma. □

Definition 304. X is complete if for all $p \in M$, $J_p = \mathbb{R}$.

Example 305. Let $M = (\mathbb{R}^2 \setminus \{(\pm 1, 0)\})/(5\mathbb{Z})^2$ and $X = \partial_x$. X is not complete because $J_{(0,0)} = (-1, 1)$.

Lemma 306 (Escape Lemma)**.** *If $X \in H^0(M, TM)$ and $\gamma : J \to M$ an integral curve and $J \neq \mathbb{R}$, then $\gamma(J)$ does not lie in a compact subset of M.*

Proposition 307. *Any compactly supported vector field X is complete.*

Proof. If $X(\gamma(t)) \neq 0$, then use the escape lemma. Otherwise, if there exists $t_0 \in J_p$ such that $X(\gamma(t_0)) = 0$. Then we define the constant curve at $\gamma(t_0)$. □

Definition 308. A *global flow* $\Theta : \mathbb{R} \times M \to M$ is a smooth map with the following properties:

1. For all $s \in \mathbb{R}$, the map $\Theta_s : M \to M$ is a diffeomorphism.

2. $\Theta_0 = \mathrm{Id}_M$.
3. For all $s, t \in \mathbb{R}$, $\Theta_s \circ \Theta_t = \Theta_{s+t}$.

Theorem 309. *There is a bijection between smooth complete time independent vector fields and global flows on* M.

Example 310. Let M be a smooth compact Riemannian manifold and let $f \in C^\infty(M)$. fix $a, b \in \mathbb{R}$ such that there no critical values of f in $[a, b]$. Let $M_r = f^{-1}(r)$ and $M_{[r_1,r_2]}$ similarly. Then $M_{[a,b]} \simeq M_a \times [0, b-1]$. In particular, $M_a \simeq M_b$. Informally, this means that the topology only changes when passing through critical values.

Proof. Let $\langle -, - \rangle$ be a metric. Define $\operatorname{grad} f$ by $\langle \operatorname{grad} f, y \rangle = df(Y)$. Now let $X = \frac{\operatorname{grad} f}{\langle \operatorname{grad} f, \operatorname{grad} f \rangle}$. Let M' be a neighborhood of $M_{[a,b]}$ such that $\operatorname{grad} f$ is nonzero on M'. Then fix $p \in M'$ and let $\gamma_p : J_p \to M$ be a maximal integral curve. Then we use the escape lemma to show that if $J_p \ni t \leqslant b - a$, then $b - a \in J_p$. Then we define a flow $\Theta : M_a \times [0, b-a] \to M_{a,b}$. This is well-defined. Then with work we show this is a diffeomorphism. □

25 Lecture 25 (Dec 09)

Today we will discuss Lie derivatives. First, we mention some properties of flows. Recall that the ODE theorem is a *local* existence, uniqueness, and smoothness theorem. Thus for any $X \in H^0(M, TM)$, there exists a unique local flow $\Theta : D \subset R \times M \to M$. We say that X *generates* Θ, or is the *infinitesimal generator* of Θ. Our goal will be to measure how a vector field Y and k-form ω varies under flow associated to a vector field X.

Let Θ_t be the local flow generated by X defined on $D = (-\varepsilon, \varepsilon) \times U$ for some open $U \ni p$. We will define

$$(L_X\omega)_p = \lim_{t \to 0} \frac{\Theta_t^*(\omega(\Theta_t(p))) - \omega(p)}{t}$$
$$(L_X Y)_p = \lim_{t \to 0} \frac{(\Theta_{-t})_*(Y(\Theta_t(p))) - Y(p)}{t}$$

Lemma 311. *The assignments* $p \mapsto (L_X\omega)_p$ *and* $p \mapsto (L_X Y)_p$ *define smooth forms and vector fields.*

Proposition 312. *Let* X, Y *be vector fields,* f *a function, and* ω, η *are forms of degree* k, ℓ.

1. $L_X f = X(f)$;
2. $L_X(\omega \otimes \eta) = (L_X\omega) \otimes \eta + \omega \otimes L_X\eta$;
3. $L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge L_X\eta$;
4. $L_X(d\omega) = d(L_X)\omega$;
5. $L_X(\alpha(Y)) = (L_X\alpha)(Y) + \alpha(L_X Y)$ *for a* 1*-form* α;
6. $L_X Y = [X, Y]$;
7. $L_X(i_Y\omega) = i_{L_X Y}\omega + i_Y(L_X\omega)$.

8. $L_X(\omega(Y_1, \ldots, Y_k)) = (L_X\omega)(Y_1, \ldots, Y_k) + \omega(L_XY, \ldots, Y_k) + \cdots + \omega(Y_1, \ldots, L_XY_k)$.

Theorem 313 (Cartan's Formula)**.** *With the same assumptions as in proposition 312,*

$$L_X\omega = d(i_X\omega) + i_X(d\omega).$$

Corollary 314. *With the same notation as Proposition 312,*

(a) $L_XY = -L_YX$;

(b) $L_X[Y, Z] = [L_XY, Z] + [Y, L_XZ]$;

(c) $L_{X,Y}Z = L_XL_YZ - L_YL_XZ$;

(d) $L_X(fY) = (L_Xf)Y + f(L_XY)$;

(e) If X', Y' *are* F*-related to* X, Y*, then* $F_*(L_XY) = L_{X'}Y'$.

Proof of Proposition 312 (1). Note that

$$\begin{aligned}(L_Xf)_p &= \lim_{t\to 0} \frac{(\Theta_t^* f)(\Theta_t(p)) - f(p)}{t} \\ &= \lim_{t\to 0} \frac{f(\Theta_t(p)) - f(p)}{t} \\ &= \gamma_p'(0)f = X_p(f) = (Xf)_p.\end{aligned}$$

□

Proof of Proposition 312 (4) for functions. Let $\partial_t|_0$ denote the limit of the difference quotient. Then

$$\begin{aligned}L_X(df) &= \partial_t|_0(\Theta_t^* df) \\ &= \partial_t|_0 d(\Theta_t^* f) \\ &= \partial_t|_0 d(f(\Theta(t, p))) \\ &= \partial_t|_0 \sum_i \partial_i f(\Theta(t, p))dx^i \\ &= \sum_i \partial_i \partial_t|_0 f(\Theta(t, p))dx^i \\ &= \sum_i \partial_i(L_Xf)dx^i \\ &= d(L_Xf).\end{aligned}$$

□

Proof of Proposition 312 (6). Fix a smooth function f. Then first we have $L_X(df(Y)) = L_X(Yf) = X(Y(f))$. In addition, we have $L_X(df)(Y) = d(L_Xf)(Y) = Y(L_Xf) = Y(X(f))$. Thus $L_X(df(Y)) = (L_X(df))(Y) + df(L_XY)$, so

$$df(L_XY) = L_X(df(Y)) - L_X(df)(Y) = X(Y(F)) - Y(X(F)) = d(df(X, Y)) + df([X, Y]) = df[X, Y].$$

□

Proof of Cartan's Formu4la. We induct. On functions, we have $i_X(df) + d(i_X f) = i_X(df) = df(X) = X(f) = L_X f$. For 1-forms $\alpha = u\,dv$, we have

$$\begin{aligned} i_X(d(udv)) + d(i_X(udv)) &= i_X(du \wedge dv) + d(u(Xv)) \\ &= (i_X du) \wedge dv - du \wedge i_X dv + ud(Xv) + (Xv)du \\ &= X(u)dv - Xvdu + ud(Xv) + (Xv)du \\ &= X(u)dv + ud(Xv). \end{aligned}$$

On the other hand,

$$L_X(udv) = L_X udv + u(L_X dv) = (Xu)dv + ud(Xv).$$

For the inductive step, we write $\omega = \sum_{I \in \binom{[n]}{k}} \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, so it suffices to check Cartan holds for $\alpha \wedge \beta$. The rest is left to Lee. □

Definition 315. Vector fields X, Y *commute* if $[X, Y] = 0$.

Definition 316. The vector field W is *invariant under flow* Θ if $(\Theta_t)_* W_p = W_{\Theta_t(p)}$.

Proposition 317. *Let X, Y be vector fields that generate flows Θ, Φ. Then the following are equivalent:*

1. *X, Y commute;*
2. *$L_X Y = 0$;*
3. *$L_Y X = 0$;*
4. *X is invariant under Φ and Y is invariant under Θ.*
5. *$\Phi_t \circ \Theta_s = \Theta_s \circ \Phi_t$.*

Example 318. If $M = \mathbb{R}^2$, we can check that ∂_x, ∂_y commute, but $x\partial_y + y\partial_x, x\partial_x - y\partial_y$ do not commute.

Theorem 319. *Let M be a smooth n-manifold and linearly independt vector fields $X_1, \ldots, X_k$ for all $p \in U \subset M$ open. Then TFAE:*

1. *$[X_i, X_j] = 0$ for all i, j;*
2. *There exist smooth coordinates such that $X_i = \partial_i$.*

Example 320. Let (M, ω) with vector field X. Let Θ be the flow generated by X. Then Θ_t is a symplectomorphism iff $i_X\omega$ is closed.[37]

Proof. $\Theta_t^*\omega = \omega$ for all t if and only if $0 = \frac{d}{dt}\Theta_t^*\omega = \Theta^*(L_X\omega)$ if and only if $L_X\omega = 0$. By Cartan, this is iff $di_X\omega + i_X d\omega = 0$, which happens iff $d(i_X\omega) = 0$ because ω is closed. □

[37] This was accompanied by a plug for Math 705.