ARTIN L-FUNCTIONS

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Recall the Dirichlet L-function

$$L(\chi, s) = \sum \frac{\chi(n)}{n^s}$$

attached to a character $\chi\colon \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = (\mathbb{Z}/m)^\times \to \mathbb{C}$. These admit a generalization both to any finite Galois extension L/K of number fields and any representation χ of $G = \mathrm{Gal}(L/K)$. Here, a representation is a map $\rho\colon G \to GL(V)$, and it has an associated character $\chi(\sigma) = \mathrm{Tr}(\rho(\sigma))$. These new L-functions are called *Artin L-functions* and they are central objects in the conjectural non-abelian class field theory.

Throughout this paper, L/K will be a finite Galois extension of number fields of degree n. We will assume that the reader has knowledge of class field theory and the representation theory of finite groups; see my notes[1] for the assumed number theory background.

1. Preliminary: Hecke L-functions

Let K be a number field, $\mathcal{O}_K$ be the ring of integers, and $\mathfrak{m}$ be a finite modulus (an ideal of $\mathcal{O}_K$). Write $I^\mathfrak{m}$ for the fractional ideals prime to $\mathfrak{m}$ and consider a character $\chi\colon I^\mathfrak{m} \to S^1$. Then we may define an *L-function*

$$L(\chi, x) = \sum_{(\mathfrak{a},\mathfrak{m})=1} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}}.$$

Write $X = \mathrm{Hom}(K, \mathbb{C})$ for the space of embeddings of K into $\mathbb{C}$ with size n, and consider the $\mathbb{C}$-algebra $\mathbf{C} = \prod_{\tau \in X} \mathbb{C}$. Now define the determinant map

$$N\colon \mathbf{C}^\times \to \mathbb{C}^\times \qquad D(z) = \prod_{\tau} z_\tau.$$

Also consider the involution $z \mapsto \overline{z}$ given by $(\overline{z})_\tau = \overline{z_{\overline{\tau}}}$. Also we have the involution $z \mapsto z^*$ given by $z^*_\tau = z_{\overline{\tau}}$. Now let $\mathbf{R} \subset \mathbf{C}$ be the fixed locus of $z \mapsto \overline{z}$ and $\mathbf{R}_\pm = \{x \in \mathbf{R} \mid x = x^*\}$. Of course we have an embedding $K \to \mathbf{C}$ given by taking the product of all of the embeddings $\tau \in \mathrm{Hom}(K, \mathbb{C})$ and the diagonal embedding of $\mathbb{C}$.

Proposition 1.1. *The L-function* $L(\chi, s)$ *converges absolutely and uniformly on* $\mathrm{Re}(s) \geqslant 1 + \delta$ *for all* $\delta > 0$.

Suppose that for all principal ideals $\mathfrak{a} = (a)$ prime to $\mathfrak{m}$, we have

$$\chi((a)) = \chi_f(a)\chi_\infty(a).$$

[1] https://math.columbia.edu/~plei/docs/NT.pdf

This gives us two characters $\chi_f\colon (\mathcal{O}/\mathfrak{m})^\times \to S^1, \chi_\infty\colon \mathbf{R}^\times \to S^1$. In this situation, we say that χ is a *Hecke character*. Now the L-function associated to a Hecke character is called a *Hecke L-function*.

Using the above notation, by [Neu99, Proposition VII.6.7], we have

$$\chi_\infty(x) = D(x^p|x|^{-p+iq}).$$

Here, $p \in \prod_\tau \mathbb{Z}$ is such that $p_\tau \in \{0,1\}$ if $\tau = \overline{\tau}$; $p_\tau p_{\overline{\tau}} = 0$ otherwise; and all $p_\tau \geqslant 0$, and $q \in \mathbf{R}_\pm$. So now write $\mathbf{s} = s(1,1,\ldots,1) + p - iq$ for some $s \in \mathbb{C}$, and now write

$$L_\infty(\chi, s) = D(\pi^{-\mathbf{s}/2}) \int_{\mathbf{R}_+^\times} D(e^{-y}y^{\mathbf{s}/2})\frac{dy}{y}.$$

Here, the integral is a higher-dimensional generalization of the gamma function.

Note that the L-function only has information about the finite places, so consider the *completed Hecke L-function*

$$\Lambda(\chi, s) = (|d_K|N\mathfrak{m})^{s/2}L_\infty(\chi, s)L(\chi, s).$$

Theorem 1.2. *The completed Hecke L-function satisfies the functional equation*

$$\Lambda(\chi, s) = W(\chi)\Lambda(\overline{\chi}, 1-s)$$

and is entire if $\mathfrak{m} \neq 1$ *or* $p \neq 0$ *and otherwise has at most two poles.*

2. Artin L-functions

Let $\mathfrak{p}$ be a prime in K, and $\mathfrak{P} \mid \mathfrak{p}$ be a prime of L above $\mathfrak{p}$. Write $G_\mathfrak{p}$ for the decomposition group and $I_\mathfrak{p}$ for the inertia group. We know $G_\mathfrak{p}/I_\mathfrak{p} = \mathrm{Gal}(k(\mathfrak{P})/k(\mathfrak{p}))$ is generated by $\mathrm{Frob}_\mathfrak{p}$, and that the determinant $\det(1 - \mathrm{Frob}_\mathfrak{p}\, t)$ on $V^{I_\mathfrak{p}}$ is independent of $\mathfrak{P} \mid \mathfrak{p}$ because all decomposition groups for $\mathfrak{P} \mid \mathfrak{p}$ are conjugate.

Definition 2.1. Let χ be any character associated to a representation $\rho\colon G \to GL(V)$ of V. Define the *Artin L-function* for χ as

$$\mathcal{L}(L/K, \chi, s) = \prod_\mathfrak{p} \frac{1}{\det(1 - \mathrm{Frob}_\mathfrak{p}\, N\mathfrak{p}^{-s})}.$$

This converges absolutely and uniformly in $\mathrm{Re}(s) \geqslant 1 + \delta$ for all s and is therefore analytic on $\mathrm{Re}(s) > 1$. To prove this, note that the eigenvalues of $\mathrm{Frob}_\mathfrak{p}$ are roots of unity.

Example 2.2. If χ corresponds to the trivial representation, then we recover the Dedekind zeta function $\zeta_K(s)$.

Remark 2.3. No additive expression for Artin L-functions exists in general.

Proposition 2.4.

1. *For two characters* χ, χ'*, we have* $\mathcal{L}(L/K, \chi + \chi', s) = \mathcal{L}(L/K, \chi, s)\mathcal{L}(L/K, \chi', s)$.
2. *Let* $L' \supset L \supset K$ *be a tower of fields and* χ *be a character of* $\mathrm{Gal}(L/K)$*. This induces a character of* $\mathrm{Gal}(L'/K)$ *(where we take the representation* $\mathrm{Gal}(L'/K) \twoheadrightarrow \mathrm{Gal}(L/K) \to GL(V)$*). Then* $\mathcal{L}(L'/K, \chi, s) = \mathcal{L}(L/K, \chi, s)$.
3. *Let* $L \supset M \supset K$ *be a tower of fields and* χ *be a character of* $\mathrm{Gal}(L/M)$*. Denote by* χ_* *the induced character of* $\mathrm{Gal}(L/K)$*. Then* $\mathcal{L}(L/M, \chi, s) = \mathcal{L}(L/K, \chi_*, s)$.

Note that the induced representation $\mathrm{Ind}_1^G \mathbb{C} = \mathrm{Hom}_{\mathsf{Set}}(G, \mathbb{C})$ is the regular representation of G. Because $\mathrm{Ind}_1^G \mathbb{C} = \bigoplus_{V \text{ irrep}} V \otimes V^*$, we obtain

Corollary 2.5. *For a finite Galois extension* L/K, *we have*

$$\zeta_L(s) = \zeta_K(s) \prod_{\chi \neq 1} \mathcal{L}(L/K, \chi, s)^{\chi(1)},$$

where χ *ranges over the characters of irreducible representations of* $\mathrm{Gal}(L/K)$.

The original question that Artin studied was whether or not the meromorphic function $\zeta_L(s)/\zeta_K(s)$ was entire. This follows from the the following conjecture:

Conjecture 2.6 (Artin)**.** *For all finite Galois* L/K *and irreducible nontrivial characters* χ *of* $\mathrm{Gal}(L/K)$, *the Artin L-function* $\mathcal{L}(L/K, \chi, s)$ *is entire.*

The Artin conjecture is true for abelian extensions by the theory of Hecke L-functions. If L/K is abelian and $\mathfrak{f}$ is the conductor of L/K, we have a surjection $I^{\mathfrak{f}}/P^{\mathfrak{f}} \twoheadrightarrow \mathrm{Gal}(L/K)$ from the ray class field of $\mathfrak{f}$. Note that all irreducible representations of abelian groups have dimension 1, so we obtain a Hecke character $\widetilde{\chi}\colon I^{\mathfrak{f}} \to \mathbb{C}^\times$ from a character $\chi\colon \mathrm{Gal}(L/K) \to \mathbb{C}^\times$.

Theorem 2.7. *Define* $S = \{\mathfrak{p} \mid f : \chi(I_{\mathfrak{p}}) = 1\}$. *Then the Artin L-function for* χ *and the Hecke L-function for* $\widetilde{\chi}$ *satisfy*

$$\mathcal{L}(L/K, \chi, s) = \prod_{\mathfrak{p} \in S} \frac{1}{1 - \chi(\mathrm{Frob}_{\mathfrak{p}}) N\mathfrak{p}^{-s}} L(\widetilde{\chi}, s).$$

Remark 2.8. If χ is injective, then $S = \emptyset$, so we have $\mathcal{L}(L/K, \chi, s) = L(\widetilde{\chi}, s)$. On the other hand, if χ is the trivial character, then

$$\zeta_K(s) = \prod_{\mathfrak{p} \mid \mathfrak{f}} \frac{1}{1 - N\mathfrak{p}^{-s}} L(\widetilde{\chi}, s).$$

Now we show that the Artin conjecture holds for all irreducible characters χ of abelian Galois groups. To see this, if $L_\chi = L^{\ker \chi}$ and we consider $\chi\colon \mathrm{Gal}(L_\chi/K) \hookrightarrow \mathbb{C}^\times$, then we obtain

$$\mathcal{L}(L/K, \chi, s) = \mathcal{L}(L_\chi/K, \chi, s) = L(\widetilde{\chi}, s),$$

and now by Theorem 1.2, $L(\widetilde{\chi}, s)$ is entire. The Artin conjecture also holds for any representation induced from a 1-dimensional representation by Theorem 2.7.[2]

3. Artin Conductor

Our goal is to prove a functional equation for the Artin L-functions. First, however, we need to construct certain ideals called *Artin conductors* which are related to the discriminant of L/K. For each character χ, the Artin conductor will be denoted $\mathfrak{f}(\chi)$, and we will see that

$$\mathfrak{d} \coloneqq \mathfrak{d}_{L/K} = \prod \mathfrak{f}(\chi)^{\chi(1)},$$

where χ ranges over the irreducible characters of $\mathrm{Gal}(L/K)$. We will construct the Artin conductors locally.

Let L/K be a Galois extension of local fields and $G = \mathrm{Gal}(L/K)$. Choose $x \in L$ such that $\mathcal{O}_L = \mathcal{O}_K[x]$ and $i_G(\sigma) = \nu_L(\sigma x - x)$, where ν_L is the normalized valuation of L. Write G_i for the i-th ramification group of L/K. Now clearly i_G is

[2]Neukirch claims that the result is true for all solvable extensions, but according to this Math.SE post, the Artin conjecture for solvable extensions is still open.

a class function, and if $H \subseteq G$ is a subgroup, we have $i_H(\sigma) = i_G(\sigma)$. Now i_G is a class function on G, and if L/K is unramified, then $i_G \equiv 0$. Now write

$$a_G(\sigma) = \begin{cases} -f i_G(\sigma) & \sigma \neq 1 \\ f \sum_{\tau \neq 1} i_G(\tau) & \sigma = 1. \end{cases}$$

Again a_G is a class function on G so we may write

$$a_G = \sum_{\chi} f(\chi)\chi,$$

where χ ranges over the irreducible characters of G. Here, $f(\chi) \in \mathbb{C}$, but we need to show that $f(\chi) \in \mathbb{Z}_{\geqslant 0}$, so we can form the ideal $\mathfrak{p}^{f(\chi)}$, which will be the local Artin conductor.

Proposition 3.1.

(1) Let H be a normal subgroup of G. Then $a_{G/H} = (a_G)_*$ is the induced character of a_G along the quotient map $G \twoheadrightarrow G/H$.

(2) If $H \subseteq G$ is any subgroup and $K' = L^H$ has discriminant $\mathfrak{d}_{K'/K} = \mathfrak{p}^{\nu}$, then $a_G\big|_H = \nu r_H + f_{K'/K} a_H$, where r_H is the regular representation of H.

(3) Let u_i be a character of G_i and $(u_i)_*$ be the induced character of G. Then

$$a_G = \sum_i \frac{1}{[G_0 : G_i]}(u_i)_*.$$

Note that $f(\chi) = (a_G, \chi)$ by Schur orthogonality, so now we can write $f(\varphi)$ for all class functions φ on G. As a corollary of the previous proposition, we have

Proposition 3.2.

(1) Let φ be a class function on G/H and φ' be the corresponding class function on G. Then $f(\varphi) = f(\varphi')$.

(2) If φ is a class function on a subgroup $H \subseteq G$ and φ_* is the induced class function on G, then

$$f(\varphi_*) = \nu_K(\mathfrak{d}_{K'/K})\varphi(1) + f_{K'/K} f(\varphi).$$

(3) For all class functions φ on G, we have

$$f(\varphi) = \sum_{i \geqslant 0} \frac{g_i}{g_0}(\varphi(1) - \varphi(G_i))$$

where $g_i \coloneqq |G_i|$ and $\varphi(G_i) \coloneqq \frac{1}{g_i}\sum_{\sigma \in G_i} \varphi(\sigma)$.

Now recall that if χ is the character of some representation V of G, then $\chi(1) = \dim V$ and $\chi(G_i) = \dim V^{G_i}$, so we obtain

$$f(\chi) = \sum_{i \geqslant 0} \frac{g_i}{g_0} \operatorname{codim} V^{G_i}.$$

Now consider the function[3] $\eta_{L/K}$ defined by $\eta(0) = 0, \eta(-1) = -1$, and for $m \geqslant 1$,

$$\eta_{L/K}(m) = \sum_{i=1}^{m} \frac{g_i}{g_0}.$$

[3]The original definition in [Neu99, Ch. II.10] is an integral and works for all $x \in \mathbb{R}$, but this definition suffices for our purposes.

Proposition 3.3. *Let χ be the character of a 1-dimensional irreducible representation. Then let j be the largest integer such that $\chi\big|_{G_j} \neq \mathbb{1}_{G_j}$. When χ is the trivial character, set $j = -1$. Then $f(\chi) = \eta_{L/K}(j) + 1$.*

In particular, this means that $f(\chi)$ is a nonnegative integer for any 1-dimensional irreducible representation. By Brauer's theorem, we know $\chi = \sum n_i(\chi_i)_*$, where χ_i is the character of a 1-dimensional representation of some subgroup $H_i \subseteq G$ and $n_i \in \mathbb{Z}$, so by Proposition 3.2, we obtain

$$f(\chi) = \sum n_i(\nu_K(\mathfrak{d}_{K_i/K})\chi_i(1) + f_{K_i/K} f(\chi_i)),$$

where $K_i = L^{H_i}$. This establishes integrality of $f(\chi)$ for arbitrary characters. Next, we note that $g_0 a_G$ is the character of some representation of G, so $f(\chi) \geqslant 0$, and therefore we have

Theorem 3.4. *Let χ be a character of* $\mathrm{Gal}(L/K)$. *Then $f(\chi) \in \mathbb{Z}_{\geqslant 0}$.*

Definition 3.5. If χ is a character of $\mathrm{Gal}(L/K)$, then the *local Artin conductor* of χ is $\mathfrak{f}_\mathfrak{p}(\chi) = \mathfrak{p}^{f(\chi)}$.

The following result links the local Artin conductor to abelian extensions of local fields.

Proposition 3.6. *Let L/K be a Galois extension of local fields and χ be a character of a 1-dimensional representation of* $\mathrm{Gal}(L/K)$. *Let $L_\chi = L^{\ker\chi}$ and $\mathfrak{f}$ be the conductor of L_χ/K. Then $\mathfrak{f} = \mathfrak{f}_\mathfrak{p}(\chi)$.*

We are now ready to consider the global situation. Let L/K be a Galois extension of number fields and $\mathfrak{p}$ be a prime in K. Then note that $\mathfrak{f}_\mathfrak{p}(\chi) = 1$ if $\mathfrak{p}$ is unramified in L, so we can define the *global Artin conductor* of χ to be

$$\mathfrak{f}(\chi) = \prod_{\mathfrak{p}\nmid\infty} \mathfrak{f}_\mathfrak{p}(\chi).$$

There are analogous results for global Artin conductors to the ones that we stated for Artin L-functions.

Proposition 3.7.

(1) If χ, χ' are characters of $G = \mathrm{Gal}(L/K)$, then $\mathfrak{f}(\chi + \chi') = \mathfrak{f}(\chi), \mathfrak{f}(\chi')$.
(2) If L'/K is a Galois subextension of L/K and χ is a character of $\mathrm{Gal}(L'/K)$, *then $\mathfrak{f}(L/K, \chi) = \mathfrak{f}(L'/K, \chi)$.*
(3) If $H \subseteq G$, $K' = L^H$, and χ is a character of H, then

$$\mathfrak{f}(L/K, \chi_*) = \mathfrak{d}_{K'/K}^{\chi(1)} \mathrm{Nm}_{K'/K}(\mathfrak{f}(L/K', \chi)).$$

Corollary 3.8. *Let $\chi = \mathbb{1}_H$ and $s_{G/H} \coloneqq \chi_*$. Then $\mathfrak{d}_{K'/K} = \mathfrak{f}(L/K, s_{G/H})$.*

Theorem 3.9. *Let L/K be finite Galois. Then $\mathfrak{d}_{L/K} = \prod_\chi \mathfrak{f}(\chi)^{\chi(1)}$, where χ ranges over all characters of irreducible representations of* $\mathrm{Gal}(L/K)$.

Proposition 3.10. *Let L/K be finite Galois and χ be a 1-dimensional character of* $\mathrm{Gal}(L/K)$. *Let $L_\chi = L^{\ker\chi}$ and $\mathfrak{f}$ be the conductor of L_χ/K. Then $\mathfrak{f} = \mathfrak{f}(\chi)$.*

Now we will form some integer invariants. First consider the ideal

$$c(L/K,\chi) = \mathfrak{d}_{K/\mathbb{Q}}^{\chi(1)} \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{f}(L/K,\chi)).$$

This is generated by the positive integer

$$c(L/K,\chi) = |d_K|^{\chi(1)} N(\mathfrak{f}(L/K,\chi)).$$

As a simple application of Proposition 3.7, we have

Proposition 3.11. *Use the same notation as in Proposition 3.7*

(1) $c(L/K,\chi+\chi') = c(L/K,\chi)\cdot c(L/K,\chi')$.
(2) $c(L/K,\chi) = c(L'/K,\chi)$.
(3) $c(L/K,\chi_*) = c(L/K',\chi)$.

4. Functional Equation

Before we prove a functional equation for Artin L-functions, we need to complete the L-function with factors coming from infinite places. For an infinite place $v \mid \infty$, write

$$\mathcal{L}_v(L/K,\chi,s) = \begin{cases} L_{\mathbb{C}}(s)^{\chi(1)} & K_v = \mathbb{C} \\ L_{\mathbb{R}}(s)^{n^+} L_{\mathbb{R}}(s+1)^{n^-} & K_v = \mathbb{R}, \end{cases}$$

where $n^{\pm} = \frac{\chi(1)\pm\chi(\varphi_v)}{2}$ and

$$L_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) \qquad L_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Note that $n^{\pm}$ are the dimensions of the (± 1)-eigenspaces of the generator φ_v of $\mathrm{Gal}(L^v/K_v)$. These infinite factors $\mathcal{L}_v(L/K,\chi,s)$ have the same behavior under change of field and change of character as the Artin L-functions and Artin conductor.

Proposition 4.1.

(1) *For two characters* χ,χ', $\mathcal{L}_v(L/K,\chi+\chi',s) = \mathcal{L}_v(L/K,\chi,s)\mathcal{L}_v(L/K,\chi',s)$.
(2) *If* L'/K *is a Galois subextension of* L/K *and* χ *is a character of* $\mathrm{Gal}(L'/K)$*, then* $\mathcal{L}_v(L/K,\chi,s) = \mathcal{L}_v(L'/K,\chi,s)$.
(3) *If* $K \subset K' \subset L$ *is a tower of extensions and* χ *is a character of* $\mathrm{Gal}(L/K')$*, then* $\mathcal{L}_v(L/K,\chi_*,s) = \prod_{w\mid v}\mathcal{L}_w(L/K',\chi,s)$.

Now we may combine all of the infinite places and write

$$\mathcal{L}_\infty(L/K,\chi,s) = \prod_{v\mid\infty}\mathcal{L}_v(L/K,\chi,s).$$

The same results from the previous proposition hold for $\mathcal{L}_\infty$.

Definition 4.2. Define the *completed Artin L-function* for a finite Galois L/K and character χ of $\mathrm{Gal}(L/K)$ by

$$\Lambda(L/K,\chi,s) = \left(|d_K|^{\chi(1)} N(\mathfrak{f}(L/K,\chi))\right)^{s/2}\mathcal{L}_\infty(L/K,\chi,s)\mathcal{L}(L/K,\chi,s).$$

From the properties of change of character and change of field that we have seen before, we obtain

Proposition 4.3.

(1) *For two characters* χ,χ', $\Lambda_v(L/K,\chi+\chi',s) = \Lambda_v(L/K,\chi,s)\Lambda_v(L/K,\chi',s)$.

(2) *If* L'/K *is a Galois subextension of* L/K *and* χ *is a character of* $\mathrm{Gal}(L'/K)$*, then* $\Lambda_v(L/K, \chi, s) = \Lambda_v(L'/K, \chi, s)$.
(3) *If* $K \subset K' \subset L$ *is a tower of extensions and* χ *is a character of* $\mathrm{Gal}(L/K')$*, then* $\Lambda_v(L/K, \chi_*, s) = \prod_{w|v} \Lambda_w(L/K', \chi, s)$.

If $\chi(1) = 1$, then we claim that $\Lambda(L/K, \chi, s)$ is a completed Hecke L-function. Let $L_\chi = L^{\ker \chi}$ and $\mathfrak{f}$ be the conductor of L_χ. Recall that $\mathfrak{f} = \mathfrak{f}(\chi)$. Via the isomorphism $I^{\mathfrak{f}}/P^{\mathfrak{f}} \to \mathrm{Gal}(L_\chi/K)$ from the ray class group of $\mathfrak{f}$, we obtain a Hecke character $\widetilde{\chi}$.

Proposition 4.4. $\Lambda(L/K, \chi, s) = \Lambda(\widetilde{\chi}, s)$.

Now using this, Brauer's theorem on induced characters, and the functional equation for Hecke L-functions, we obtain

Theorem 4.5. *The completed Artin L-function* $\Lambda(L/K, \chi, s)$ *admits a meromorphic continuation to* $\mathbb{C}$ *and*

$$\Lambda(L/K, \chi, s) = W(\chi)\Lambda(L/K, \overline{\chi}, 1 - s)$$

for some constant $W(\chi)$ *of absolute value* 1.

References

[Neu99] Jürgen Neukirch. *Algebraic Number Theory*. Berlin: Springer-Verlag, 1999.