ALGEBRAIC CURVES AND INTEGRABLE HIERARCHIES

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ABSTRACT. In the 1990s, a remarkable correspondence was discovered between the geometry of algebraic curves and infinite-dimensional systems of differential equations. The correspondence has its origin in the study of two-dimensional quantum field theories and is related to many different areas of mathematics. After introducing the relevant objects, I will then state the first result in this story, which was conjectured by Witten and proved by Kontsevich.

1. Algebraic curves and their moduli

For the purposes of this lecture, we will only define smooth curves. Also, space will really mean algebraic variety/scheme/stack, whichever is needed to make the statements of the results true.

Definition 1.1. A *smooth curve* is a compact 1-dimensional complex manifold; i.e. a Riemann surface.

Note that curves will be 1-dimensional over $\mathbb{C}$ and 2-dimensional over $\mathbb{R}$. We will also need our curves to develop singularities, but fortunately we only need one kind of singularity:

Definition 1.2. A *nodal singularity* is one that locally looks like $\{xy = 0\} \subseteq \mathbb{C}^2$.

1.1. Moduli spaces.

As the simplest algebraic varieties, much has been written on the geometry of curves. For this lecture, we will focus on the study of familes of curves – a very productive technique in algebraic geometry is to study how properties vary in families. We will begin with a very natural question.

Question 1.3. *Can we classify all algebraic curves?*

If we first restrict to smooth curves, we know that these are topologically surfaces, and the topological classification of surfaces is very simple: they are controlled by a discrete parameter, the genus. However, for a surface of genus $g \geqslant 1$, there will be many ways to give it the structure of a complex manifold or algebraic variety.

Example 1.4. For any $\lambda \neq 0, 1$, there is a genus 1 smooth curve given by the equation

$$y^2 = x(x-1)(x-\lambda),$$

and different λ give non-isomorphic curves.

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It is natural to wonder if for a given genus g, there is a space M_g paramterizing curves of genus g, known as a moduli space. More precisely, for any space X, we want the set $\mathsf{Maps}(X, \mathcal{M}_g)$ of maps from X to $\mathcal{M}_g$ to correspond to familes of smooth curves of genus g over X. In fact, such a space exists, and the following is true:

Proposition 1.5 (Riemann, Deligne–Mumford)**.** *For any g, n such that $2g-2+n > 0$, there is a space $\mathcal{M}_{g,n}$ of dimension $3g-3+n$ parameterizing smooth curves of genus g with n distinct marked points.*

Unfortunately, $\mathcal{M}_{g,n}$ is not compact, which geometrically means that there are families of smooth curves over $\mathbb{C} \setminus 0$ which cannot be filled in (meaning adding a fiber over 0) with a smooth curve. In order to compactify $\mathcal{M}_{g,n}$, we will add nodal curves.

Proposition 1.6 (Deligne–Mumford)**.** *For any g, n such that $2g-2+n > 0$, there exists a compact space $\overline{\mathcal{M}}_{g,n}$ of dimension $3g-3+n$ parameterizing families of possibly nodal curves C of genus g with n distinct marked points satisfying the following conditions:*

(1) The marked points are away from the nodes; i.e. they lie on smooth points of C.

(2) For any irreducible component C' of C with genus g', if n' denotes the number of nodes and marked points on C', then $2g'-2+n' > 0$.

Now, the identity map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ corresponds to some family of curves over $\overline{\mathcal{M}}_{g,n}$, and this is the *universal family*, which we will denote $\mathcal{C}_{g,n}$. This means that for any map $f\colon X \to \overline{\mathcal{M}}_{g,n}$, the corresponding family is given by $f^*\mathcal{C}_{g,n}$.

1.2. **Integrals on the moduli space of curves.** We will define line bundles L_i for $i = 1, \ldots, n$ on $\overline{\mathcal{M}}_{g,n}$ as follows. For a point $[C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$, we will define

$$(L_i)_{[C,x_1,\ldots,x_n]} = T_{x_i}C.$$

More precisely, if $x_i\colon \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}$ is the map corresponding to the i-th marked point, then $L_i = x_i^*\Omega^1_{\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}}$, where $\Omega^1_{\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ is the relative cotangent bundle. Now, we will define $\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n})$ to be the first Chern class of L_i. These ψ classes are known as *gravitational descendents* in the physics literature, where the integrals below are related to 2D topological gravity.

Definition 1.7. For any integers $a_1, \ldots, a_n \geqslant 0$, define

$$\left\langle \psi_1^{a_1} \cdots \psi_n^{a_n} \right\rangle_{g,n} \coloneqq \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\psi_2^{a_2} \cdots \psi_n^{a_n}.$$

2. The KdV hierarchy

The behavior of waves in shallow water is described by the *Korteweg-de Vries equation:*

$$\frac{\partial u}{\partial t_1} = u\frac{\partial u}{\partial t_0} + \frac{1}{12}\frac{\partial^3 u}{\partial t_0^3}.$$

The coefficients can be scaled to any nonzero real numbers by scaling the variables. The KdV equation has a solution given by

$$u_1(t_0, t_1) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(t_0 - ct_1 + \delta)\right),$$

where c and δ are constants and we impose $u(x,t) = f(t_0 - ct_1)$ to be a travelling wave. In fact, there are infinitely many exact solutions $u_n(t_0, t_1)$ to the KdV equation, which behave like independent waves for large time (for both $t_1 \ll 0$ and $t_1 \gg 0$) but can collide in small time. Waves which exhibit this type of behavior are called *solitons*. The infinitely many exact solutions make the KdV equation an *integrable system* with infinitely many degrees of freedom.

The KdV equation generates an infinite-dimensional system of differential equations using the following procedure. Define the *Schrodinger operator* by

$$L \coloneqq (\partial_0^2 + u).$$

Its square root can be written as

$$(\partial_0^2 + u)^{\frac{1}{2}} = \partial_0 + \frac{1}{2}u\partial_0^{-1} - \frac{1}{4}u_0\partial_0^{-2} + \left(\frac{u_{00}}{8} - \frac{u^2}{8}\right)\partial_0^{-3} + \cdots$$

Now for an expression of the form $M = \sum_{\ell=0}^{\infty} g_\ell \partial^{n-\ell}$, define

$$M_+ \coloneqq \sum_{\ell=0}^{n} g_\ell \partial^{n-\ell}, \qquad M_- \coloneqq M - M_+.$$

We can now define the operators

$$K_a(u) \coloneqq -[L, L_+^{\frac{2a+1}{2}}]$$

for nonnegative integers a and obtain the *KdV hierarchy*

$$\frac{\partial u}{\partial t_a} = K_a(u).$$

The first two equations are in fact

$$\frac{\partial u}{\partial t_0} = \frac{\partial u}{\partial t_0}, \qquad \frac{\partial u}{\partial t_1} = \frac{3}{2}u\frac{\partial u}{\partial t_0} + \frac{1}{4}\frac{\partial^3 u}{\partial t_0^3},$$

and we see the first is a tautology and the second is the KdV equation up to scaling. What justifies calling this an integrable hierarchy is the commutation relation

$$\frac{\partial}{\partial x_b}K_a(u) = \frac{\partial}{\partial x_a}K_b(u)$$

for all a, b. This states that the evolution of the system in the t_b direction and the evolution of the system in the t_a direction commute.

3. The Kontsevich-Witten theorem

Define the partition function (precisely, this is the all-genus Gromov-Witten potential of a point)

$$\mathcal{Z} = \exp\left(\sum_{\substack{g=0 \\ n=1 \\ 2g-2+n>1}}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{a_1,\ldots,a_n} \left\langle \psi_1^{a_1} \cdots \psi_n^{a_n} \right\rangle_{g,n} t_{a_1} \cdots t_{a_n} \right).$$

Theorem 3.1 (Kontsevich 1992)**.** *The function*

$$u(t_0, t_1, \ldots) = \frac{\partial^2}{\partial t_0^2} \log \mathcal{Z}$$

is the unique solution to the KdV hierarchy with initial condition $u(t_0, 0, 0, \ldots) = t_0$.

This was proven by Kontsevich using a combinatorial description of $\overline{\mathcal{M}}_{g,n}$ in terms of thickened graphs and computing various matrix integrals. This theorem has several other proofs using various techniques. We will briefly outline three of them.

- Okounkov and Pandharipande (2001) gave a proof which proceeds first by relating the integrals of ψ-classes on $\overline{\mathcal{M}}_{g,n}$ to counts of permutations known as *Hurwitz numbers*, continues by constructing a different matrix model than the one considered by Kontsevich, and concludes by relating their matrix model to that of Kontsevich using techniques from probability theory.
- Mirzakhani (2007) gave a proof which proceeds first by defining a metric (the so-called *Weil-Petersson metric*) on $\overline{\mathcal{M}}_{g,n}$ and computing a recursive formula for the volume of $\overline{\mathcal{M}}_{g,n}$ and concludes by relating the volumes to the integrals we defined in the first section.
- Kazarian and Lando (2007) gave a proof which relates the Hurwitz numbers considered by Okounkov and Pandharipande to a different integrable hierarchy known as the KP hierarchy, and then concludes by reducing the KP hierarchy to the KdV hierarchy.