WHAT'S THE DEAL WITH NAKAJIMA QUIVER VARIETIES?

PATRICK LEI

ABSTRACT. I will introduce Nakajima quiver varieties and attempt to explain why so many people care about them.

1. INTRODUCTION AND HISTORY

Nakajima quiver varieties are related to many areas of mathematics and originally appeared as moduli spaces of Yang-Mills instantons, for example in

- Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients* (1989);
- Kronheimer-Nakajima, *Yang-Mills instantons on ALE gravitational instantons* (1990).

In the representation theoretic context, they were introduced and originally studied by Nakajima in two papers in the 1990s to study representations of Kac-Moody algebras:

- *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras* (1994);
- *Quiver varieties and Kac-Moody algebras* (1998).

There are several expositions of the basic ideas of Nakajima, for example:

- Schiffmann, *Varietés carquois de Nakajima* (Séminaire Bourbaki, 2006-2007);
- Ginzburg, *Lectures on Nakajima's quiver varieties* (2012);
- Nakajima, *Introduction to quiver varieties – for ring and representation theorists* (2016).

There is also an entire book by Kirillov that starts from the basics of quivers and finishes with the work of Nakajima. Nakajima varieties are also used to study geometric representations of other objects, for example in

- Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras* (2001);
- Maulik-Okounkov, *Quantum groups and quantum cohomology* (2019);
- Okounkov, *Lectures on K-theoretic computations in enumerative geometry* (2017);
- Aganagic-Okounkov, *Elliptic stable envelopes* (2021);

2. WHAT IS A NAKAJIMA VARIETY?

Let Q be a quiver with vertex set I. Recall that a representation of Q is the same thing as a functor $Q \to \mathrm{Vect}_{\mathbb{C}}$. Nakajima quiver varieties will be moduli spaces of framed representations of Q in the following sense.

Date: May 11, 2022.

First, define a new quiver Q' by adding a vertex i' for all $i \in I$ and an arrow $i' \to i$. Also let $\mathbf{v} = (v_i)_{i \in I}$ and $\mathbf{w} = (w_i)_{i \in I}$ be dimension vectors and $V = \bigoplus_{i \in I}$ and $W = \bigoplus_{i \in I} W_i$ be $\mathbf{v}, \mathbf{w}$-dimensional vector spaces, respectively. Consider the space

$$\mathbf{M} := T^* \mathrm{Rep}(Q', \mathbf{v}, \mathbf{w}) = \bigoplus_{\substack{i \to j \\ \text{edge of } Q}} \mathrm{Hom}(V_i, V_j) \oplus \mathrm{Hom}(V_j, V_i)$$

$$\oplus \bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i)$$

$$= \mathrm{Rep}(Q, \mathbf{v}) \oplus \mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{v}) \oplus \mathrm{Hom}_I(W, V) \oplus \mathrm{Hom}_I(V, W)$$

of representations of the union of Q' and $(Q')^{\mathrm{op}}$ with the vertices identified. This has a natural action of $\mathrm{GL}_{\mathbf{v}} = \prod_{i \in I} \mathrm{GL}_{v_i}$.

The Nakajima variety will be a hyperkähler reduction of this vector space, which is given the natural symplectic form of a cotangent bundle. The moment map is defined by

$$\mu \colon \mathbf{M} \to \mathfrak{gl}_{\mathbf{v}} \qquad (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto \sum_e [x_e, y_{\overline{e}}] + \mathbf{i} \circ \mathbf{j},$$

where $\mathfrak{gl}_n$ is identified with its dual by $x \mapsto \mathrm{Tr}(x \cdot -)$.

It remains to define stability conditions. Let $\theta \in \mathbb{Z}^I$. (Often, we will choose $\theta^+ = (1, \dots, 1)$ or $\theta^- = (-1, \dots, -1)$.) The following definition of stability is not the standard definition of stability as defined for quiver representations, but is equivalent and most useful in practice. It also coincides with GIT stability by work of King.

Definition 2.1. A quadruple $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$ is θ-*semistable* if the following condition holds: If $S = \bigoplus_{i \in I} S_i \subseteq V$ is stable under $\mathbf{x}$ and $\mathbf{y}$, we have

$$S_i \subset \ker j_i \ \forall i \in I \implies \theta \cdot (\dim_I S) \leqslant 0;$$
$$S_i \supset \mathrm{Im}\, i_i \ \forall i \in I \implies \theta \cdot (\dim_I S) \leqslant \theta \cdot \mathbf{v}.$$

Finally, we define the *Nakajima quiver variety*

$$\mathcal{M}_{\lambda, \theta}(\mathbf{v}, \mathbf{w}) := \mu^{-1}(\lambda) \mathbin{/\!\!/}_\theta \mathrm{GL}_{\mathbf{v}}.$$

If we omit λ, this means that we take $\lambda = 0$. For a generic λ, θ, there are no strictly semistable points, and there are no nontrivial finite stabilizers, so $\mathcal{M}_{\lambda, \theta}(\mathbf{v}, \mathbf{w})$ is a holomorphic symplectic variety.

Example 2.2. Let Q be the quiver with one vertex and no edges and let $\theta > 0$. Set $\mathbf{v} = r$ and $\mathbf{w} = n$. Then we see that

$$\mu(\mathbf{i}, \mathbf{j}) = \mathbf{i} \circ \mathbf{j}.$$

In addition, a point $(\mathbf{i}, \mathbf{j})$ is semistable if and only if $\ker \mathbf{j} = 0$, so we see that

$$\mu^{-1}(0)^{\mathrm{ss}} = \{(\mathbf{j} \colon \mathbb{C}^r \hookrightarrow \mathbb{C}^n, \mathbf{i} \colon \mathbb{C}^n / \mathrm{Im}(\mathbf{j}) \to \mathbb{C}^r)\},$$

and so quotienting out by GL_r, we see that $\mathcal{M}_{\theta^+}(r, n) = T^* \mathrm{Gr}(r, n)$. If we choose the other stability condition, then $(\mathbf{i}, \mathbf{j})$ is semistable if and only if $\mathbf{i}$ is surjective, so

$$\mu^{-1}(0)^{\mathrm{ss}} = \{(\mathbf{i} \colon \mathbb{C}^n \twoheadrightarrow \mathbb{C}^r, \mathbf{j} \colon \mathbb{C}^r \to \ker \mathbf{i})\},$$

and again we have $\mathcal{M}_{\theta-}(r,n) = T^*Gr(r,n)$.

Example 2.3. Let Q be the quiver with one vertex and one edge. Let $\mathbf{v} = n$ and $\mathbf{w} = 1$ and choose $\theta = -1$. Then $\mathbf{i} \circ \mathbf{j}$ is a rank 1 operator, so we see that in $\mu^{-1}(0)$,

$$\langle \mathbf{i}, \mathbf{j} \rangle = \text{Tr}(\mathbf{i} \circ \mathbf{j}) = -\text{Tr}([x,y]) = 0$$

and thus $[x,y]$ is a nilpotent rank 1 operator. Then by some linear algebra magic, we can choose a basis of $\mathbb{C}^n$ such that x,y are both upper-triangular. Now in this upper triangular form, we have a morphism

$$\mathcal{M}_0(n,1) \to \mathbb{C}^{2n}/S_n \qquad (x,y,\mathbf{i},\mathbf{j}) \mapsto (\text{eigenvalues of } x,y),$$

and in fact this is an isomorphism.

Now adding in the stability condition, we see that $\mathbf{i}$ must be a cyclic vector for x and y. But then we know $\langle \mathbf{j}, a\mathbf{i} \rangle = \text{Tr}(a \circ \mathbf{i} \circ \mathbf{j}) = -\text{Tr}(a \circ [x,y]) = 0$, so $\mathbf{j} = 0$ and thus $[x,y] = 0$. Therefore,

$$\mu^{-1}(0)^{ss} = \{(x,y,\mathbf{i},\mathbf{j}) \mid [x,y] = 0, \mathbf{j} = 0, \mathbf{i} \text{ is cyclic for } x,y\}.$$

Taking the quotient by GL_n, we see that $\mathcal{M}_{\theta-}(n,1) = (\mathbb{C}^2)^{[n]}$ is the Hilbert scheme of n points on $\mathbb{C}^2$ and the natural map $\mathcal{M}_{\theta-}(n,1) \to \mathcal{M}_0(n,1)$ is the Hilbert-Chow morphism.

3. Kac-Moody Lie algebras

Let $C = (a_{ij})$ be a matrix such that $a_{ii} = 2$, $a_{ij} \leqslant 0$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Then define the Lie algebra $\widetilde{\mathfrak{g}}$ by

$$\widetilde{\mathfrak{g}} = \langle e_i, f_i, h_i \rangle \Big/ \left(\begin{array}{l} [h_i, e_i] = a_{ij} e_j \\ [h_j, f_j] = -a_{ij} f_j \\ [e_i, f_j] = \delta_{ij} h_i \end{array} \right).$$

Then the *Kac-Moody algebra* $\mathfrak{g}$ is defined to be the quotient of $\widetilde{\mathfrak{g}}$ by the maximal submodules in $\widetilde{\mathfrak{n}_+}$ and $\widetilde{\mathfrak{n}_-}$. In the case where C is symmetrizable (for example for semisimple Lie algebras), then the maximal submodules in $\widetilde{\mathfrak{n}}_\pm$ are generated by the *Chevalley-Serre relations*

$$\text{ad}(e_i)^{1-a_{ij}} e_j = \text{ad}(f_i)^{1-a_{ij}} f_j = 0 \qquad \text{for all } i,j.$$

Note that for every i, we have a copy of $\mathfrak{sl}_2$ given by $\mathfrak{sl}_2^i = \langle e_i, h_i, f_i \rangle$.

Definition 3.1. A $\mathfrak{g}$-module M is *integrable* if for every i, M decomposes as a sum of finite-dimensional $\mathfrak{sl}_2^i$-modules.

For our purposes (where $\mathfrak{g}$ is associated to a quiver with no loops), we will modify $\mathfrak{g}$ by replacing the root lattice in the Cartan decomposition with the weight lattice (still having $[h, e_i] = \langle h, \alpha_i \rangle e_i$ and $[h, f_j] = -\langle h, \alpha_j \rangle f_j$). However, this point is not essential for our discussion.

4. BOREL-MOORE HOMOLOGY

We will let $H_*(X)$ denote the Borel-Moore homology of a space X. This is defined by $H_*(X) := H^*(M, M \setminus X)$, where $X \hookrightarrow M$ is any embedding into a smooth manifold, and is independent of the choice of M. This has an advantage over ordinary homology in that even noncompact manifolds have fundamental classes. For any proper map $p \colon X \to Y$, there is a pushforward p_*. There is also a cap product $\cap \colon H_i(X) \times H_j(Y) \to H_{i+j-\dim M}(X)$, which does depend on the ambient smooth manifold M.

If M_1, M_2, M_3 are smooth manifolds, $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$, and $p_{13} \colon (p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23}) \to M_1 \times M_3$ is proper, define

$$Z_{12} \circ Z_{13} := p_{13}((p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23})).$$

Then the convolution product is defined by

$$* := p_{13*}((- \boxtimes [M_3]) \cap ([M_1] \boxtimes -)) \colon H_i(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-\dim M_2}(Z_{12} \circ Z_{23}).$$

5. CONSTRUCTING REPRESENTATIONS OF KAC-MOODY LIE ALGEBRAS

From now on, let Q be a quiver with no loops (including loops containing more than one edge) and $\mathfrak{g} = \mathfrak{g}_Q$ be the Kac-Moody Lie algebra produced from the Cartan matrix $C_Q = 2\mathrm{Id} - A_Q$, where A_Q is the adjacency matrix of the underlying undirected graph of Q. We will denote the generators e_i, h_i, f_i for $i \in I$, the simple roots by α_i, and the corresponding fundamental weights by ϖ_i. We will also make the universal choice of $\theta = \theta^+$ and write $\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{\theta^+}(\mathbf{v}, \mathbf{w})$ and $M(\mathbf{v}, \mathbf{w}) = \mathcal{M}_0(\mathbf{v}, \mathbf{w})$ for the affine quotient.

Consider the natural morphism $\pi \colon \mathcal{M}(\mathbf{v}, \mathbf{w}) \to M(\mathbf{v}, \mathbf{w})$, which is projective. Define the subscheme

$$\Lambda(\mathbf{v}, \mathbf{w}) := \pi^{-1}(M(\nu, w)^{\mathbb{C}^\times})_{\mathrm{red}},$$

where the action of $\mathbb{C}^\times$ on $\mathbf{M} = T^* \mathrm{Rep}(Q', \mathbf{v}, \mathbf{w})$ by scaling the cotangent directions. This action preserves $\mu^{-1}(0)$ and in fact descends to $\mathcal{M}(\mathbf{v}, \mathbf{w})$. In fact, in our case where Q has no loops, $M(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times} = \{0\}$, so $\Lambda(\mathbf{v}, \mathbf{w}) = \pi^{-1}(0)$.

Theorem 5.1. *Every irreducible component of $\Lambda(\mathbf{v}, \mathbf{w})$ is a Lagrangian subvariety of Q.*

Recall that a Lagrangian subvariety L of a symplectic variety X if for any smooth point $x \in L$, $T_x L \subset T_x X$ is a maximal isotropic subspace with respect to the symplectic form ω_X. In addition, the irreducible components of $\Lambda(\mathbf{v}, \mathbf{w})$ are the closures of attracting loci of components F_s of $\mathcal{M}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times}$, which are defined by

$$\mathrm{Attr}(F_s) = \left\{ z \in \mathcal{M}(\mathbf{v}, \mathbf{w}) \mid \lim_{t \to \infty} t(z) \in F_s \right\}.$$

These attracting loci play an important role also in the work of Okounkov and collaborators.

Now, for $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^I$, we will define a *Steinberg variety* $Z(\mathbf{v}, \mathbf{v}', \mathbf{w}) \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}', \mathbf{w})$ as the fiber product in the following diagram:

$$
\begin{array}{ccccc}
Z(\mathbf{v}, \mathbf{v}', \mathbf{w}) & \longrightarrow & \mathcal{M}(\mathbf{v}', \mathbf{w}) & \xrightarrow{\ \pi\ } & M(\mathbf{v}', \mathbf{w}) \\
\downarrow & & & & \uparrow \\
\mathcal{M}(\mathbf{v}, \mathbf{w}) & \xrightarrow{\ \pi\ } & M(\mathbf{v}, \mathbf{w}) & \longhookrightarrow & M(\mathbf{v} + \mathbf{v}', \mathbf{w}).
\end{array}
$$

All irreducible components of $Z(\mathbf{v}, \mathbf{v}', \mathbf{w})$ are half-dimensional in our case, and good components are Lagrangian subvarieties. Also, if $d = \dim_{\mathbb{C}}(\mathcal{M}(\mathbf{v}, \mathbf{w})) + \dim_{\mathbb{C}}(\mathcal{M}(\mathbf{v}', \mathbf{w}))$, we will define a new grading on Borel-Moore homology by

$$H_{[i]}(Z(\mathbf{v}, \mathbf{v}', \mathbf{w})) := H_{d-i}(Z(\mathbf{v}, \mathbf{v}', \mathbf{w})).$$

Now we may introduce the following schemes with infinitely many components:

$$\mathcal{M}(\mathbf{w}) := \bigsqcup_{\mathbf{v} \in \mathbb{Z}^I} \mathcal{M}(\mathbf{v}, \mathbf{w}), \qquad M(\mathbf{w}) := \bigsqcup_{\mathbf{v} \in \mathbb{Z}^I} M(\mathbf{v}, \mathbf{w}), \qquad Z(\mathbf{w}) := \bigsqcup_{(\mathbf{v}, \mathbf{v}')} Z(\mathbf{v}, \mathbf{v}', \mathbf{w}).$$

Also, define the algebra

$$H_{\mathbf{w}} = \bigoplus_{m \geqslant 0} \left(\prod_{\substack{\mathbf{v}, \mathbf{v}' \geqslant 0 \\ |\mathbf{v} - \mathbf{v}'| \leqslant m}} H_{[0]}(Z(\mathbf{v}, \mathbf{v}', \mathbf{w})) \right)$$

and the vector space

$$L_{\mathbf{w}} := \bigoplus_{\mathbf{v} \in \mathbb{Z}^I} H_{\mathrm{top}}(\Lambda(\mathbf{v}, \mathbf{w})).$$

Theorem 5.2 (Nakajima)**.** *There is a natural algebra homomorphism $\Psi \colon \widetilde{\mathcal{U}}\mathfrak{g} \to H_{\mathbf{w}}$ such that the induced action on $L_{\mathbf{w}}$ makes $L_{\mathbf{w}}$ a simple integrable $\mathfrak{g}$-module with highest weight $\sum_{i \in I} w_i \cdot \varpi_i$.*

This action is defined in the following way. Let $i \in I$. Then define

$$Z^i(\mathbf{v}, \mathbf{w}) := \left\{ (\rho, \rho') \mid \rho \overset{\mathrm{framed}}{\subset} \rho' \right\} \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v} + \mathbf{e}^i, \mathbf{w}).$$

This is an irreducible component of $Z(\mathbf{v}, \mathbf{v}', \mathbf{w})$. Thus we have operators

$$F_i(\mathbf{v}, \mathbf{w}) := [Z^i(\mathbf{v}, \mathbf{w})] * - \colon H_{\mathrm{top}}(\Lambda(\mathbf{v}, \mathbf{w})) \to H_{\mathrm{top}}(\Lambda(\mathbf{v} + \mathbf{e}^i, \mathbf{w}))$$

$$E_i(\mathbf{v}, \mathbf{w}) := [Z^i(\mathbf{v}, \mathbf{w})] * - \colon H_{\mathrm{top}}(\Lambda(\mathbf{v} + \mathbf{e}^i, \mathbf{w})) \to H_{\mathrm{top}}(\Lambda(\mathbf{v}, \mathbf{w}))$$

Finally, we define the following operators:

$$f_i = \sum_{\mathbf{v}} F_i(\mathbf{v}, \mathbf{w});$$

$$e_i = \sum_{\mathbf{v}} (-1)^{\dim_{\mathbb{C}}(\mathcal{M}(\mathbf{v} + \mathbf{e}^i, \mathbf{w}) - \dim_{\mathbb{C}}(\mathcal{M}(\mathbf{v}, \mathbf{w})))} E_i(\mathbf{v}, \mathbf{w});$$

$$h_i = \sum_{\mathbf{v}} \left(w_i - \sum_j a_{ij} v_j \right) \mathrm{id}_{H_{\mathrm{top}}(\Lambda(\mathbf{v}, \mathbf{w}))}.$$