Student Learning Seminar on Galois Deformations.

Rafah Hajjar, September 20, 2023

These notes are based on Mazur's paper *An Introduction to the Deformation Theory of Galois Representations* and Tobe Gee's paper *Modularity Lifting Theorems.*

1 Representations of profinite groups

Let Π be a profinite group, A a coefficient ring (complete, noetherian, local, finite residue field) with residue field k. Let

$$\rho : \Pi \to \mathrm{GL}_n(A)$$

Remark 1. An equivalent definition of a representation is a free A-module M of rank n with a continuous action of Π and a choice of basis (changing the basis gives an equivalent representation), with the obvious correspondence. We will use both notions interchangeably.

Why coefficient rings

We will mostly be interested in the case of representations $\rho : \Pi \to \mathrm{GL}_n(A)$ where A is an extension of $\mathbb{Q}_p$. These extensions are not coefficient rings (their residue field is not finite). However, their rings of integers are, and this is enough because of the following

Fact. If $L/\mathbb{Q}_p$ is an algebraic extension, and $\rho : \Pi \to \mathrm{GL}_n(L)$ is a continuous representation, then ρ is equivalent to a representation in $\mathrm{GL}_n(\mathcal{O}_L)$.

Definition 1.1. The *underlying residual representation* to ρ,

$$\overline{\rho} : \Pi \to \mathrm{GL}_n(k)$$

is the composition of ρ with the natural projection map $\mathrm{GL}_n(A) \to \mathrm{GL}_n(k)$.

Definition 1.2. We say $\overline{\rho} : \Pi \to \mathrm{GL}_n(k)$ is absolutely irreducible if it is irreducible over $\overline{k}$ (equivalently, over any field extension of k).

Let $A[[\Pi]]$ denote the completed group-ring of Π with coefficients in A, i.e.

$$A[[\Pi]] = \varprojlim A[[\Pi/\Pi_0]]$$

where Π_0 runs through the normal open subgroups of Π of finite index. Then

Proposition 1.1. There is a correspondence between continuous representations $\rho : \Pi \to \mathrm{GL}_n(A)$ and continuous A-algebra homomorphisms

$$r : A[[\Pi]] \to M_N(A)$$

given by restriction (viewing $\Pi \subset A[[\Pi]]^\times$).

Proposition 1.2. The residual representation associated to ρ is absolutely irreducible if and only if the homomorphism r is surjective.

Corollary 1.1 (Schur's lemma)**.** If the residual representation $\overline{\rho}$ is absolutely irreducible, any matrix in $M_n(A)$ which commutes with all the elements in the image of ρ is a scalar.

Proof. The elements commuting with $\mathrm{Im}(\rho)$ must commute with the completion $\widehat{\mathrm{Im}(\rho)} = \mathrm{Im}(r) = M_n(A)$, where the last equality follows from the previous proposition. Therefore such an element must lie in the center $Z(M_n(A)) = \{\lambda \cdot \mathrm{Id}_n \mid \lambda \in A\}$ □

Definition 1.3. Given a representation $\rho : \Pi \to \mathrm{GL}_n(A)$, the *character* associated to ρ is the map $\chi_\rho : \Pi \to A$ given by $\chi_\rho(g) = \mathrm{tr}_A(\rho(g))$.

Proposition 1.3. Let $\rho, \rho' : \Pi \to \mathrm{GL}_n(A)$ be a representation with the same character, i.e. $\mathrm{tr}_A(\rho(g)) = \mathrm{tr}_A(\rho'(g))$ for all $g \in \Pi$. Suppose that one of these representations is residually absolutely irreducible. Then ρ and ρ' are equivalent representations.

Remark 2. Without the irreducibility assumption, we have the weaker consequence $\rho^{\mathrm{ss}} \sim \rho'^{\mathrm{ss}}$ (Recall that the semisimplification of ρ is the direct sum of all the composition factors in a composition series)

A direct application of Chebotarev Density Theorem gives

Corollary 1.2. Let $\rho, \rho' : G_{K,S} \to \mathrm{GL}_n(A)$ be continuous representations. Suppose that one of these representations is residually absolutely irreducible. Suppose further that

$$\mathrm{tr}_A(\rho(\mathrm{Frob}_\ell)) = \mathrm{tr}_A(\rho'(\mathrm{Frob}_\ell))$$

for ℓ running through a set of prime numbers (outside S) which is of Dirichlet density 1. Then ρ is equivalent to ρ'.

Assume $\rho : \Pi \to \mathrm{GL}_n(A)$ is a representation whose character $\chi_\rho = \mathrm{tr}_A(\rho)$ has values in a subring $A_0 \subset A$. A *descent* from ρ to A_0 is a representation $\rho_0 : \Pi \to \mathrm{GL}_n(A_0)$ which, after extension of scalars, becomes equivalent to ρ.

Theorem 1.1. For A a coefficient ring, there always exists a descent of ρ in the situation above.

Remark 3. The proof uses a result of Carayol and Serre about Azumaya algebras.

2 Local Galois Representations

Let ℓ be a prime, and let $K/\mathbb{Q}_\ell$. As usual, ϖ_K denotes a uniformizer, $\mathcal{O}_K$ is the ring of integers, and k is the residue field. We want to study representations $G_K \to \mathrm{GL}_n(A)$, for A a coefficient ring with residual characteristic p. In this section, we will focus on the case of a finite extension $L/\mathbb{Q}_p$ and will consider representations $G_K \to \mathrm{GL}(V)$, for V a finite-dimensional L-vector space. (recall this is equivalent to a representation over $\mathrm{GL}_n(\mathcal{O}_L)$ for $\dim(V) = n$)

2.1 Case $l \neq p$

Recall the short exact sequence

$$0 \to I_K \to G_K \to G_k \to 0$$

Let $\mathrm{Frob}_k \in G_k$ be a Frobenius element (i.e. a topological generator for $G_k \cong \widehat{\mathbb{Z}}$). We define the Weil group via the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_K & \longrightarrow & G_K & \longrightarrow & G_k & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathrm{Frob}_k^{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

so that W_K is the subgroup of elements of G_K that map to an integral power of the Frobenius.

Local Class Field Theory is summarized in the following

Theorem 2.1. Let W^{ab} denote the group $W_K/\overline{[W_K, W_K]}$. Then there are unique isomorphisms $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ such that

1. if K'/K is a finite extension, then $\mathrm{Art}_{K'} = \mathrm{Art}_K \circ N_{K'/K}$, and
2. (2) we have a commutative square

$$\begin{array}{ccc} K^\times & \xrightarrow{\mathrm{Art}_K} & W_K^{\mathrm{ab}} \\ {\scriptstyle \mathrm{val}_K}\downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathrm{Frob}_K^{\mathbb{Z}} \end{array}$$

   where the bottom arrow is the isomorphism sending $a \mapsto \mathrm{Frob}_K^a$.

Recall that we have a character $t : I_K/P_K \xrightarrow{\sim} \prod_{p \neq l} \mathbb{Z}_p$. Any other character is of the form $t' = ut$ for some $u \in \prod_{p \neq l} \mathbb{Z}_p^\times$. Fix t for the rest of the section, and let t_p be the composite of t with the projection to $\mathbb{Z}_p$.

Remark 4. The continuous irreducible representations of the group W^{ab} are just the continuous characters of W_K, and local class field theory gives a simple description of them, as representations of $K^\times$

Definition 2.1. A *Weil–Deligne representation* of W_K on a finite-dimensional L-vector space V is a pair (r, N) consisting of a representation $r : W_K \to \mathrm{GL}(V)$ with open kernel, and a nilpotent endomorphism $N \in \mathrm{End}(V)$ such that for all $\sigma \in W_K$,

$$r(\sigma) N r(\sigma)^{-1} = (\#k)^{-v_K(\sigma)} N$$

where $v_K : W_K \to \mathbb{Z}$ is determined by $\sigma \mid_{K^{\mathrm{ur}}} = \mathrm{Frob}_K$.

Theorem 2.2 (Grothendieck's Monodromy theorem). Fix $\varphi \in W_K$ a lift of Frob_K. If $\rho : G_K \to \mathrm{GL}(V)$ is a continuous representation then there is a finite extension K'/K and a uniquely determined nilpotent $N \in \mathrm{End}(V)$ s.t.

$$\rho(\sigma) = \exp(N t_p(\sigma)), \quad \forall\, \sigma \in I_{K'}$$
$$\rho(\sigma) N \rho(\sigma)^{-1} = (\#k)^{-v_K(\sigma)} N, \quad \forall\, \sigma \in W_{K'}$$

We have an equivalence of categories from the category of continuous representations of G_K on f.d. L-v.s. to the category of bounded ($r(\sigma)$ stabilizes an $\mathcal{O}_L$-lattice) Weil-Deligne representations on f.d. L-v.s., taking

$$\rho \mapsto (V, r, N), \quad r(\sigma) = \rho(\sigma) \exp(-t_p(\varphi^{-v_K(\sigma)} \sigma) N)$$

Remark 5. One significant advantage of Weil-Deligne representations over Galois representations is that there are no subtle topological issues: the topology on the Weil-Deligne representation is the discrete topology.

It turns out that the Frobenius Weil-Deligne representations are in bijection with irreducible admissible representations of the $\mathrm{GL}_n(F)$, thus linking two kinds of representations

2.2 Case $l = p$

This case is far more complicated than the case $l \neq p$, largely because wild inertia can act in a highly nontrivial fashion. The study of representations $G_K \to \mathrm{GL}_n(\mathbb{Q}_p)$ with $K/\mathbb{Q}_p$ finite is part of what is called p-adic Hodge theory. We will give a very brief account of the big picture.

There is a hierarchy of classes of representations

$$\{\text{crystalline}\} \subset \{\text{semistable}\} \subset \{\text{de Rham}\} \subset \{\text{Hodge–Tate}\}$$

For any of these classes, we say that ρ is potentially X if there is a finite extension K'/K such that $\rho\mid_{G_{K'}}$ is X. A representation is potentially de Rham if and only if it is de Rham, and potentially HT if and only if it is HT.

The notion of a de Rham representation is designed to capture the representations arising in geometry. Similarly, the definitions of crystalline and semistable are designed to capture the notions of good and semistable reduction, respectively. (Given X/K smooth projective variety with one of these properties, $H^1_t(X \times_K \overline{K}, \overline{\mathbb{Q}_p})$ is a representation of the corresponding kind)

A useful heuristic when comparing to the $l \neq p$ case is that crystalline representations correspond to unramified representations, semistable representations correspond to representations for which inertia acts unipotently, and de Rham representations correspond to all representations. The analogous notion of inertia acting quasi-unipotently as in Grothendieck's monodromy theorem is potential semistability. Therefore the following result (conjectured by Fontaine) can be viewed as an analog of Grothendieck's monodromy theorem.

Theorem 2.3 (The p-adic monodromy theorem)**.** A representation is de Rham if and only if it is potentially semistable.

3 Global Galois Representations

The global Galois representations that we will care about are those that Fontaine and Mazur call geometric. Let $L/\mathbb{Q}_p$ be an algebraic extension.

Definition 3.1. If K is a number field, then a continuous representation $\rho : G_K \to \mathrm{GL}_n(L)$ is geometric if it is unramified outside of a finite set of places of K, and if for each place $v \mid p$, $\rho\mid_{G_{K_v}}$ is de Rham.

In practice (and conjecturally always), geometric Galois representations arise as part of a compatible system of Galois representations.

Suppose that K and F are number fields, that S is a finite set of places of K and that n is a positive integer.

Definition 3.2. By a weakly compatible system of n-dimensional p-adic representations (for varying p) of G_K defined over F and unramified outside S we mean a family of continuous semisimple representations

$$r_\lambda : G_K \to \mathrm{GL}(\overline{F_\lambda})$$

where λ runs over the finite places of M, with the following properties.

- For each $v \notin S$ and λ not dividing the same prime p as v, r_λ is unramified at v and the characteristic polynomial of $r_\lambda(\mathrm{Frob}_v)$ lies in $F[X]$ and is independent of λ.
- Each representation r_λ is de Rham at all places above the residue characteristic of λ, and crystalline at any place $v \notin S$ which divides the residue characteristic of λ.
- For each $\tau : K \hookrightarrow \overline{F}$, the τ-HT weight of r_λ are independent of λ.

Definition 3.3. We say that a weakly compatible system is *strictly compatible* if for each finite place v of K there is a Weil–Deligne representation WD_v of W_{K_v} over F such that for each finite place λ of F and every F-linear embedding $\iota : \overline{F} \hookrightarrow \overline{F_\lambda}$, we have $\iota\mathrm{WD}_v \cong \mathrm{WD}(r_\lambda\, |_{G_{K_v}})^{F-\mathrm{ss}}$.

Conjecturally, every weakly compatible system is strictly compatible. We also have the following consequence of the Fontaine–Mazur conjecture:

Conjecture. Any semisimple geometric representation $G_K \to \mathrm{GL}_n(L)$ is part of a strictly compatible system of Galois representations.

In practice, most progress on understanding these conjectures has been made by using automorphy lifting theorems to prove special cases of the following conjecture.

Conjecture. Any weakly compatible system of Galois representations is strictly compatible, and is in addition automorphic, in the sense that there is an algebraic automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ with the property that $\mathrm{WD}_v \cong \mathrm{rec}_{K_v}(\pi|\det|^{(1-n)/2})$ for each finite place v of K, where rec_{K_v} is the local Langlands correspondence.

The main source (and conjecturally the only source) of compatible systems of Galois representations is the étale cohomology of algebraic varieties.

Theorem 3.1. Let K be a number field, and let X/K be a smooth projective variety. Then for any i, j, the $H^i_{\text{ét}}(X \times_K \overline{K}, \mathbb{Q}_p)^{\mathrm{ss}}(j)$ (the (j) denoting a Tate twist) form a weakly compatible system (defined over $\mathbb{Q}$) as p varies.

Conjecturally, it is a strictly compatible system, and there is no need to semisimplify the representations.

Conjecture (Fontaine-Mazur)**.** Any irreducible geometric representation $\rho : G_K \to \mathrm{GL}_n(\mathbb{Q}_p)$ is (the extension of scalars to $\mathbb{Q}_p$ of) a subquotient of a representation arising from étale cohomology as in the previous theorem