Representability of Deformation Functors

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October 4, 2023

1 Setup and Definition of Deformation Functors

Let Π be a profinite group. We define a finiteness condition for Π:

Condition Φ_p: For every open subgroup of finite index $\Pi_0 \subset \Pi$ there exists only finitely many continuous homomorphisms $\Pi_0 \to \mathbb{F}_p$.

In fact, both $G_{\mathbb{Q}_\ell}$ and $G_{\mathbb{Q},S}$ satisfy condition Φ_p.

Choose and fix a finite field k of characteristic p. Let $\mathcal{C}$ denote the category (of coefficient rings) whose objects are complete Noetherian local rings with residue field k and whose morphisms are local homomorphisms which induce identity on k. Let $\mathcal{C}^0$ be the full subcategory whose objects are artinian. Let Λ be an object of $\mathcal{C}$. We define $\mathcal{C}_\Lambda$ to be the category whose objects are coefficient Λ-algebras and the homomorphism respect the Λ-algebra structure. We let $\mathcal{C}_\Lambda^0$ be the artinian full subcategory. Note that we have $\mathcal{C} = \mathcal{C}_{W(k)}$.

Define Deformation Functors $\mathbf{D}$, $\mathbf{D}_\Lambda$.

2 Schlessinger's Criteria

Recall that any object of $\mathcal{C}$ is an inverse limit of objects of $\mathcal{C}^0$. We say a functor $\mathbf{F}$ on $\mathcal{C}$ is *continuous* if

$$\mathbf{F}(R) = \varprojlim_n \mathbf{F}(R/\mathfrak{m}^n),$$

where R is a complete noetherian local ring with maximal ideal $\mathfrak{m}$. Then $\mathbf{F}$ is completely determined by its values on $\mathcal{C}^0$. It may happen that $\mathbf{F}$ is not representable as a functor on $\mathcal{C}^0$, but that there exists an object $\mathcal{R}$ of the larger category $\mathcal{C}$ such that we have

$$\mathbf{F}(A) = \mathrm{Hom}(\mathcal{R}, A)$$

for every artinian coefficient ring A. In this case, we say that the functor $\mathbf{F}$ on the category $\mathcal{C}^0$ is *pro-representable*. In fact,

Lemma 2.1. If $\mathbf{F}$ is continuous, then it is pro-representable as a functor on $\mathcal{C}^0$ if and only if it is representable as a functor on $\mathcal{C}$.

Proof. A consequence of $\mathrm{Hom}(R, -)$ commuting with inverse limit in the second factor. □

Since we have shown last time that the deformation functor is continuous, instead of showing the deformation functor is representable, we show that it is pro-representable under suitable hypothesis.

A Theorem of Grothendieck gives a sufficient condition for a functor on $\mathcal{C}^0$ to be pro-representable involving the MV property. However, the hypothesis of the theorem is hard to check. Schlessinger obtained a set of criteria for pro-representability of functors on the categories of artinian rings which are much easier to apply.

Let $\mathbf{F}$ be a covariant functor

$$\mathbf{F} : \mathcal{C}^0_\Lambda \rightsquigarrow \mathbf{Sets},$$

and assume that $\mathbf{F}(k)$ consists of one element. Before stating the criteria for $\mathbf{F}$ to be pro-representable by a ring $\mathcal{R}$ in $\mathcal{C}_\Lambda$, we need a definition.

Definition 2.1. If R and S are two coefficient Λ-algebras, we say a homomorphism

$$\phi : R \to S$$

is *small* if it is surjective and if $\ker(\phi)$ is principal and is annihilated by $\mathfrak{m}_R$.

Lemma 2.2. Any surjective homomorphism in $\mathcal{C}^0_\Lambda$ factors as the composition of small homomorphisms.

Remark 2.0.1. One prototypical example of a small homomorphism which will be of great importance is the homomorphism

$$k[\varepsilon] \to k$$

which sends $\varepsilon \mapsto 0$.

Now we are ready to set up the Schlessinger's criteria. Consider rings R_0, R_1, and R_2 in $\mathcal{C}^0_\Lambda$. Suppose we have morphisms $\phi_1 : R_1 \to R_0$ and $\phi_2 : R_2 \to R_0$. We define the fibre product $R_3 = R_1 \times_{R_0} R_2$. Since for $i = 1, 2$ we have projections $\pi_i : R_3 \to R_i$ and $\mathbf{F}$ is a functor, we get a map (*)

$$\mathbf{F}(R_3) \to \mathbf{F}(R_1) \times_{\mathbf{F}(R_0)} \mathbf{F}(R_2)$$

by universal property of the fibre product.

H1: If the map $\phi_2 : R_2 \to R_0$ is small, then (*) is surjective.

H2: If $R_0 = k$ and $R_2 = k[\varepsilon]$, then (*) is bijective.

If **H2** holds, applying it to the case $R_1 = k[\varepsilon]$ shows that the tangent space hypothesis is satisfied, and hence we can think of $t_{\mathbf{F}} = \mathbf{F}(k[\varepsilon])$ as a k-vector space (can define addition).

H3: The vector space $t_{\mathbf{F}} = \mathbf{F}(k[\varepsilon])$ is finite-dimensional.

H4: If $R_1 = R_2$, $\phi_1 = \phi_2$, and the ϕ_i's are small, then (*) is bijective.

Remark 2.0.2. The first two conditions say that the map (*) should be nice when ϕ_2 is simple.

Theorem 2.1 (Schlessinger)**.** *Let $\mathbf{F}$ be a set-valued covariant functor on $\mathcal{C}^0_\Lambda$ such that $\mathbf{F}(k)$ has exactly one element. If $\mathbf{F}$ satisfies conditions **H1** to **H4**, then $\mathbf{F}$ is pro-representable.*

Let $d = \dim_k t_{\mathbf{F}}$. The proof structures $\mathcal{R}$ as an inverse limit of quotients of $\Lambda[[X_1, X_2, \cdots, X_d]]$.

3 Existence of Universal Deformation

We now apply Schlessinger's theorem to the deformation functor. As we will see, the first three conditions will always hold, but the fourth will depend on the residual represenation $\overline{\rho}$.

We first make a definition.

Definition 3.1. Let $\overline{\rho}$ be a residual representation. We let

$$C(\overline{\rho}) = \mathrm{Hom}_{\Pi}(k^n, k^n) = \{P \in M_n(k) \mid P\overline{\rho}(g) = \overline{\rho}(g)P \text{ for all } g \in \Pi\}$$

Let ρ be a deformation of $\overline{\rho}$ to a coefficient Λ-algebra A. We define

$$C_A(\rho) = \mathrm{Hom}_{\Pi}(A^n, A^n) = \{P \in M_n(A) \mid P\rho(g) = \rho(g)P \text{ for all } g \in \Pi\}$$

In particular, $C(\overline{\rho}) = C_k(\overline{\rho})$.

Theorem 3.1. *Suppose Π is a profinite group that satisfies property Φ_p, $\overline{\rho} : \Pi \to GL_n(k)$ is a continuous representation, and Λ is a complete Noetherian ring with residue field k. Then the deformation functor $\mathbf{D}_\Lambda$ always satisfies property **H1**, **H2**, and **H3**. Furthermore, if $C(\overline{\rho}) = k$, then $\mathbf{D}_\Lambda$ also satisfies property **H4**.*

We will now start the proof of the theorem. We fix the following notations throughout.

Let R_0, R_1, R_2 be artinian coefficient Λ-algebras and suppose we are given $\phi_i : R_i \to R_0$ as above. Let

$$E_i := \mathrm{Hom}_{\overline{\rho}}(\Pi, GL_n(R_i))$$

be the set of homomorphisms from Π to $GL_n(R_i)$ which reduce to $\overline{\rho}$ modulo the maximal ideal. Let

$$\Gamma_n(R_i) := \ker(GL_n(R_i) \to GL_n(k)) = 1 + M_n(\mathfrak{m}_{R_i}).$$

Then

$$\mathbf{D}_\Lambda(R_i) = E_i/\Gamma_n(R_i).$$

Therefore, the map of concern in Schlessinger's (*) translates to

$$E_3/\Gamma_n(R_3) \to E_1/\Gamma_n(R_1) \times_{E_0/\Gamma_n(R_0)} E_2/\Gamma_n(R_2).$$

By the property of local ring map and the explicit description of $\Gamma_n(R_i)$, it is not hard to see that if $R_i \to R_0$ is surjective, then so is $\Gamma_n(R_i) \to \Gamma_n(R_0)$.

Lemma 3.1. Property **H1** is true.

Proof. Suppose ϕ_2 is small (in fact we only need that it is surjective). Given a pair of deformation (ρ_1, ρ_2) of deformations to R_1 and R_2 which induce the same deformation to R_0, we want to show that we can paste them together to get a deformation to R_3. The pasting is clear if we are only considering the E_i's. For the actual map (*), the key is to pick the representatives properly so they match when projected down to R_0.

Pick any pair of representative (π_1, π_2) for the equivalence class of (ρ_1, ρ_2) and by assumption we know that $\phi_1(\pi_1) \sim \phi_2(\pi_2)$ when projected down to R_0, i.e. there exists an element $\overline{M} \in \Gamma_n(R_0)$ such that,

$$\overline{M}^{-1}(\phi_1(\pi_1))\overline{M} = \phi_2(\pi_2).$$

Since ϕ_2 is surjective, we know that $\Gamma_n(R_2) \twoheadrightarrow \Gamma_n(R_0)$. Thus, we can lift $\overline{M}$ to $M \in \Gamma_n(R_2)$. As a result, π_1 and $M^{-1}\pi_2 M$ has the same image in $GL_n(R_0)$, which implies the pair $(\pi_1, M^{-1}\pi_2 M)$ defines an element $\pi_3 \in E_3$. The equivalence class of ϕ_3 is mapped to the equivalence class of (ρ_1, ρ_2). ☐

Now to prove **H2**, we first establish a criterion for (*) to be injective. Let $\pi_2 \in E_2$ and let $\pi_0 \in E_0$ denote its image when projected down to R_0. Set

$$G_i(\pi_i) = \{g \in \Gamma_n(R_i) \mid g \text{ commutes with the image of } \pi_i\}.$$

Note that $G_i(\phi_i) \subset C_{R_i}(\phi_i)$ but they are not identical.

Lemma 3.2. If for all $\pi_2 \in E_2$ the map
$$G_2(\pi_2) \to G_0(\pi_0)$$
is surjective, then the map (*) is injective.

Proof. Suppose ρ and π are elements of E_3 that induce elements ρ_i, π_i in E_i for $i = 0, 1, 2$. Suppose that ρ, π have the same image under (*), then there exists $M_i \in \Gamma_n(R_i)$ such that $\rho_i = M_i^{-1}\pi_i M_i$. Mapping down to E_0 we see that
$$\rho_0 = \overline{M_1}^{-1}\pi_0\overline{M_1} = \overline{M_2}^{-1}\pi_0\overline{M_2},$$
so that $\overline{M_2M_1}^{-1} \in G_0(\pi_0)$. By the surjectivity in the assumption, we can find $N \in G_2(\pi_2)$ that is mapped to $\overline{M_2M_1}^{-1}$.

Define $N_2 = N^{-1}M_2$, then we have
$$N_2^{-1}\pi_2 N_2 = M_2^{-1}N\pi_2N^{-1}M_2 = M_2^{-1}\pi_2M_2 = \rho_2.$$
On the other hand, N_2 is mapped to $\overline{M_1} \in \Gamma_0(R_0)$. Thus, the pair (M_1, N_2) defines an element $M \in \Gamma_n(R_3)$ and have $\rho = M^{-1}\pi M$. Thus, ρ and π are equivalent and we are done. □

Now we are ready to prove **H2**.

Lemma 3.3. Property **H2** is true.

Proof. If $R_0 = k$ and $R_2 = k[\varepsilon]$, we already know (*) is surjective by **H1**. We want to check that
$$G_2(\pi_2) \to G_0(\pi_0)$$
is surjective for these rings. When $R_0 = k$, $G_0 = \Gamma_n(k) = \{1\}$, thus surjectivity holds trivially. □

Lemma 3.4. Property **H3** is true.

Proof. Let $\Pi_0 = \ker(\overline{\rho})$ and let ρ be a lift of $\overline{\rho}$ to $k[\varepsilon]$. If $x \in \Pi_0$, we know that $\overline{\rho}(x) = 1$, which implies $\rho(x) \in \Gamma_n(k[\varepsilon])$. Thus, ρ defines a map
$$\rho : \Pi/\Pi_0 \to GL_n(k[\varepsilon])/\Gamma_n(k[\varepsilon]) = GL_n(k),$$
which should agree with $\overline{\rho}$. Thus, if ρ, ρ' are two lifts that determine the same map on Π_0, then $\rho \sim \rho'$. Since Π_0 is an open subgroup of Π and $\Gamma_n(k[\varepsilon])$ is a finite p-elementary abelian group. By property Φ_p[1], there are only finitely many maps from Π_0 to $\Gamma_n(k[\varepsilon])$. Thus, by the argument we just established, there are finitely many elements in $\mathbf{D}_\Lambda(k[\varepsilon])$. □

Now we move on to prove **H3**. We first establish a Lemma.

Lemma 3.5. If $C(\overline{\rho}) = k$, then for any deformation ρ of $\overline{\rho}$ to an artinian ring A we have $C_A(\rho) = A$.

Proof. Since $A \twoheadrightarrow k$ is surjective, it factors as a sequence of small extensions. We know that $C(\overline{\rho}) = k$ by assumption. It is then suffices to show the alternative claim:

If $A \to B$ is small and $C_B(\rho_B) = B$, then $C_A(\rho_A) = A$. Here ρ_B is induced from ρ_A by $A \to B$.

[1] every p-elementary abelian group is a vector space over $\mathbb{F}_p$

To prove this, take $c \in C_A(\rho_A)$. By our assumption, the image of c in $M_n(B)$ is a scalar matrix. Suppose $c \mapsto \bar{r}$, where $\bar{r} \in B$ is the image of some $r \in A$. Then we can write

$$c = r + tM,$$

where t is a generator of the kernal $A \to B$ (this is possible from the small assumption) and $M \in M_n(A)$. Since c commutes with the image of ρ_A, we have that for every $g \in \Pi$,

$$(r + tM)\rho_A(g) = \rho_A(g)(r + tM),$$

which implies

$$M\rho_A(g) = \rho_A(g)M$$

Reducing modulo $\mathfrak{m}_A$ and using the fact that $C(\overline{\rho}) = k$, we know that $M = s + M_1$ with $s \in A$ and $M_1 \in M_n(\mathfrak{m}_A)$. Since $A \to B$ is small, we know that $tM_n(\mathfrak{m}_A) = 0$. Thus, we have proven that

$$c = r + ts,$$

which is a scalar. □

Lemma 3.6. Suppose $C(\overline{\rho}) = k$, Then property **H4** is true.

Proof. For $i = 0, 1, 2$, we know from the Lemma that $G_i(\pi_i) \subset C_{R_i}(\pi_i) = R_i$. In fact, $G_i(\pi_i) \cong 1 + \mathfrak{m}_{R_i}$. For $i = 1, 2$, we know that ϕ_i is surjective. Thus, $G_i(\pi_i) \twoheadrightarrow G_0(R_0)$. By Lemma 3.2 about injectivity of (*) and **H1** about surjectivity of (*), we have shown that (*) is bijective. □

Consequently, we have proven the following theorem.

Theorem 3.2 (Mazur, Ramakrishna)**.** *Suppose Π is a profinite group that satisfies property Φ_p, $\overline{\rho} : \Pi \to GL_n(k)$ is a continuous representation such that $C(\overline{\rho}) = k$, and Λ is a complete Noetherian ring with residue field k. Then there exists a ring $\mathcal{R} = \mathcal{R}(\Pi, k, \overline{\rho})$ in $\mathcal{C}_\Lambda$, and a deformation $\boldsymbol{\rho}$ of $\overline{\rho}$ to $\mathcal{R}$,*

$$\boldsymbol{\rho} : \Pi \to GL_n(\mathcal{R}),$$

such that any deformation of $\overline{\rho}$ to a coefficient Λ-algebra A is obtained from $\boldsymbol{\rho}$ via a unique morphism $\mathcal{R} \to A$.

We call $\mathcal{R}$ the *universal deformation ring* and $\boldsymbol{\rho}$ the *universal deformation* of $\overline{\rho}$.

Note that the condition $C(\overline{\rho}) = k$ is weaker than saying that $\overline{\rho}$ is absolutely irreducible. In fact, there exists reducible representations satisfying $C(\overline{\rho}) = k$.

4 Example of Universal Deformation Ring

4.1 Π is finite

If Π is finite of order not divisible by p, and suppose that $\overline{\rho}$ is an inclusion. Then there exists a lift of $\overline{\rho}$ to $GL_n(W(k))$, and hence to $GL_n(\Lambda)$ via the canonical map. In fact, the universal deformation ring is Λ.

4.2 $n = 1$

The universal deformation ring for a character $\overline{\chi} : \Pi \to k^\times$ is $\Lambda[[\Gamma]]$, with Γ the abelianization of the pro-p-completion of Π.