Student Learning Seminar on Galois Deformations.

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These notes are based on Taylor-Wiles paper *Ring-theoretic properties of certain Hecke Algebras* and Ray's lecture *Overview of the Taylor-Wiles method.*

1 An overview of the Taylor-wiles Method

The goal of the lecture is to sketch the proof of the following theorem

Theorem 1.1 (Wiles)**.** Every semistable elliptic curve over $\mathbb{Q}$ is modular

Let $E/\mathbb{Q}$ be a semistable elliptic curve and let $\rho_{E,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p)$ be its associated p-adic Galois representation. The strategy will be to find a global deformation problem satisfied by $\rho_{E,p}$ where we can lift modularity. To do that, we first need that $\overline{\rho}_{E,p}$ is modular. It is conjectured by Serre that every Galois representation over a finite field is modular. For our purposes, it is enough with the following result.

Lemma. At least one of the representations $\overline{\rho}_{E,3}$ and $\overline{\rho}_{E,5}$ is both modular and irreducible.

Sketch of the proof. We know that at least one of $\overline{\rho}_{E,3}$ and $\overline{\rho}_{E,5}$ is absolutely irrducible. If $\overline{\rho}_{E,3}$ is absolutely irreducible, then it is modular by a result of Langlands and Tunnell. If $\overline{\rho}_{E,5}$ is absolutely irreducible, by a $3-5$ switch we get an elliptic curve E' such that $\overline{\rho}_{E',3}$ is absolutely irreducible (hence modular), and $\overline{\rho}_{E,5} \cong \overline{\rho}_{E',5}$. The result follows from the fact that modularity of $\overline{\rho}_{E,p}$ for some p implies modularity for all p. □

Fix p such that $\overline{\rho}_{E,p}$ is modular and irreducible.

Let N_0 be the minimal level of a modular form f such that $\overline{\rho}_{f,p} \cong \overline{\rho}_{E,p}$ (it is the prime to p part of the Artin conductor of $\overline{\rho}$). Let

$$\mathbb{T}(N_0) = \mathbb{Z}[T_\ell, \langle d \rangle] \subseteq \mathrm{End}(\mathcal{S}_2(\Gamma_1(N_0)))$$

be the Hecke algebra of endomorphisms of weight 2 cusforms of level N_0, and denote $\mathbb{T}_0 = \mathbb{T}(N_0)_{\mathfrak{m}}$ its localization at an appropriate maximal ideal $\mathfrak{m}$.

We have a Galois representation associated to $\mathbb{T}_0$,

$$\rho_{\mathbb{T}_0} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}_0),$$

coming from the action of $G_{\mathbb{Q}}$ on the Jacobian $J_1(N)$ of the modular curve $X_1(N)$, given the relation $\mathrm{Ta}_{\mathfrak{m}}\left(J_1(N_0)(\overline{\mathbb{Q}})\right) \cong \mathbb{T}_0^2$ (which in turn comes from the iso $J_1(N_0)(\overline{\mathbb{Q}})[\mathfrak{m}] \cong (\mathbb{T}(N_0)/\mathfrak{m})^2$).

1.1 $R = \mathbb{T}$ theorems

We define a minimal deformation type $\mathcal{D}_0 = (S, \{C_\ell\}_{\ell \in S})$ via the deformation conditions

- ρ has determinant χ_p,
- ρ is unramified outside $S \cup \{p\}$,

- ρ is semistable at S,
- ρ is flat at p (if $\overline{\rho}$ is flat at p).

Here, S is the set of primes dividing N_0p. Let R_0 be the universal deformation ring $R_{\mathcal{D}_0}$.

The representation $\rho_{\mathbb{T}_0}$ defined above satisfies all deformation conditions C_ℓ prescribed by $\mathcal{D}_0$. By the universal property, we obtain a map $\varphi_0 : R_0 \to \mathbb{T}_0$.

The goal is to show that φ_0 is an isomorphism. Such a result needs to be proven at non-minimal levels as well, but this requires a slightly more involved argument.

A result establishing an isomorphism between a deformation ring R and a localized Hecke algebra $\mathbb{T}$ is known as an "$R = \mathbb{T}$´´ theorem. Modularity follows from this isomorphism in the following way:

The representation $\rho_{E,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p)$ corresponds to a map $R \to \mathbb{Z}_p$ by the universal property. Since $R \cong \mathbb{T}$, it follows that this map is the same as a map $\mathbb{T} \to \mathbb{Z}_p$. Finally, it is not hard to show that any homomorphism $\phi : \mathbb{T} \to \mathbb{Z}_p$ corresponds to a normalized Hecke eigenform f of weight 2 with Fourier coefficients $a_n(f) = \varphi(T_n)$, and one can check $\rho_{f,p} \simeq \rho_{E,p}$. This is a consequence of the perfect pairing $\mathbb{T} \times \mathcal{S}_2(\Gamma_1(N))(\mathcal{O}) \to \mathcal{O}$ given by $\langle T, f\rangle = a_1(Tf)$.

1.2 Taylor-Wiles primes

Definition 1.1. We say that a primer q is a Taylor-Wiles prime if

- $q \notin$ S (i.e. $\overline{\rho}$ is unramified at q)
- $q \cong 1 \pmod p$
- $\overline{\rho}(\mathrm{Frob}_q)$ is semisimple with distinct eigenvalues

Let $Q = \{q_1, \dots, q_r\}$ be a finite set of Taylor-Wiles primes. Define a new deformation condition $\mathcal{D}_Q = (S \cup Q, \{C_\ell\}_{\ell \in S \cup Q})$ by allowing ramification at the primes $q \in Q$. Denote by R_Q the associated deformation ring.

Since we have only loosened the conditions at the primes in Q, the universal deformation of type $\mathcal{D}_0$ is also of type $\mathcal{D}_Q$, so there is a natural homomorphism

$$R_Q \to R_0$$

Let Δ_q be the p-primary part of $(\mathbb{Z}/q\mathbb{Z})^\times$, and set Δ_Q to be the product

$$\Delta_Q = \prod_{q \in Q} \Delta_q$$

The deformation ring R_Q is an $\mathcal{O}[\Delta_Q]$-algebra. Letting $\mathfrak{a}_Q$ be the augmentation ideal in $\mathcal{O}[\Delta_Q]$, there is an isomorphism

$$R_Q/\mathfrak{a}_Q R_Q \cong R_0.$$

Likewise, there is a localized Hecke algebra T_Q of level $N_Q := N_0 \cdot \prod_{q \in Q} q$ which is of type $\mathcal{D}_Q$, and hence a map $\phi_Q : R_Q \to \mathbb{T}_Q$ which makes the square commute. $\mathbb{T}_Q$ is also naturally an $\mathcal{O}[\Delta_Q]$-algebra via the map $\mathcal{O}[\Delta_Q] \to \mathbb{T}_Q$ which sends $x \in \Delta_Q$ to the diamond operator $\langle d\rangle$ with $d \equiv x \pmod p$ and $d \equiv 1 \pmod{N_0}$. It can be shown that

$$\mathbb{T}_Q/\mathfrak{a}_Q\mathbb{T}_Q \cong \mathbb{T}_0.$$

1.3 Patching

There exists $r \geq 1$ such that for every $n \geq 1$, there is a set Q_n of r Taylor-Wiles primes such that $q \equiv 1 \pmod{p^n}$. Set $R_n := R_{Q_n}$ and $\mathbb{T}_n := \mathbb{T}_{Qn}$.

Given Q_n, the set of primes Q_{n+1} can be constructed in a way so that the following diagram commutes

$$\begin{array}{ccc} R_{n+1} & \xrightarrow{\varphi_{n+1}} & \mathbb{T}_{n+1} \\ \downarrow & & \downarrow \\ R_n & \xrightarrow[\varphi_n]{} & \mathbb{T}_n \end{array}$$

Set $\Delta_n := \Delta_{Q_n}$. Note that R_n and $\mathbb{T}_n$ are algebras over

$$\mathcal{O}[\Delta_n] \cong \frac{\mathcal{O}[T_1, \ldots, T_r]}{((1+T_1)^{p^n} - 1) \cdots ((1+T_r)^{p^n} - 1)}$$

Taking the inverse limit $\mathcal{O}_\infty := \varprojlim \mathcal{O}[\Delta_n]$we get a formal power series ring over $\mathcal{O}$ in r-variables

$$\mathcal{O}_\infty \cong \mathcal{O}[[T_1, \ldots, T_r]]$$

Set $R\infty := \varprojlim R_n$ and $\mathbb{T}_\infty := \varprojlim \mathbb{T}_n$, and let $\varphi_\infty : R_\infty \to \mathbb{T}_\infty$ be the inverse limit of the maps $\varphi_n : R_n \to \mathbb{T}_n$.

Note that $R_0 = R_\infty/(T_1, \ldots, T_r)$ and $\mathbb{T}_0 = \mathbb{T}_\infty/(T_1, \ldots, T_r)$, so if it is shown that $\varphi_\infty : R_\infty \to \mathbb{T}_\infty$ is an isomorphism, then it shall follow that $\varphi_0 : R_0 \to \mathbb{T}_0$ is an isomorphism as well.

Each Hecke-algebra $\mathbb{T}_n$ acts faithfully on a space of modular forms M_n which is finitely generated and free as an $\mathcal{O}[\Delta_n]$-module. Letting $M_\infty := \varprojlim M_n$, we find that M_∞ is a finitely generated free $\mathcal{O}_\infty$-module. It follows from this that $\mathbb{T}_\infty$ is also a finitely generated and faithful $\mathcal{O}_\infty$-module.

On the other hand, it follows from Galois theoretic arguments that R_∞ is a quotient of $\mathcal{O}[[T_1, \ldots, T_r]]$. By dimension considerations, $R_\infty = \mathcal{O}[[T_1, \ldots, T_r]]$.

Since $R_\infty \to \mathbb{T}_\infty$ is surjective and $\mathbb{T}_\infty$ is faithful over $\mathcal{O}_\infty$ this implies that φ_∞ must be an isomorphism.