Student Learning Seminar on Galois Deformations.

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These notes are based on Taylor-Wiles paper *Ring-theoretic properties of certain Hecke Algebras* and Ray’s lecture *Overview of the Taylor-Wiles method.*

1 An overview of the Taylor-wiles Method

The goal of the lecture is to sketch the proof of the following theorem

**Theorem 1.1** (Wiles). Every semistable elliptic curve over $\mathbb{Q}$ is modular

Let $E/\mathbb{Q}$ be a semistable elliptic curve and let $\rho_{E,p} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Z}_p)$ be its associated $p$-adic Galois representation. The strategy will be to find a global deformation problem satisfied by $\rho_{E,p}$ where we can lift modularity. To do that, we first need that $\rho_{E,p}$ is modular. It is conjectured by Serre that every Galois representation over a finite field is modular. For our purposes, it is enough with the following result.

**Lemma.** At least one of the representations $\rho_{E,3}$ and $\rho_{E,5}$ is both modular and irreducible.

**Sketch of the proof.** We know that at least one of $\rho_{E,3}$ and $\rho_{E,5}$ is absolutely irreducible. If $\rho_{E,3}$ is absolutely irreducible, then it is modular by a result of Langlands and Tunnell. If $\rho_{E,5}$ is absolutely irreducible, by a $3-5$ switch we get an elliptic curve $E'$ such that $\rho_{E',3}$ is absolutely irreducible (hence modular), and $\rho_{E,5} \cong \rho_{E',5}$. The result follows from the fact that modularity of $\rho_{E,p}$ for some $p$ implies modularity for all $p$. □

Fix $p$ such that $\rho_{E,p}$ is modular and irreducible.

Let $N_0$ be the minimal level of a modular form $f$ such that $\overline{\rho}_{f,p} \cong \rho_{E,p}$ (it is the prime to $p$ part of the Artin conductor of $\overline{\rho}$). Let

\[ T(N_0) = \mathbb{Z}[T_\ell, \langle d \rangle] \subseteq \text{End}(S_2(\Gamma_1(N_0))) \]

be the Hecke algebra of endomorphisms of weight 2 cusforms of level $N_0$, and denote $T_0 = T(N_0)_m$ its localization at an appropriate maximal ideal $m$.

We have a Galois representation associated to $T_0$,

\[ \rho_{T_0} : G_{\mathbb{Q}} \to \text{GL}_2(T_0), \]

coming from the action of $G_{\mathbb{Q}}$ on the Jacobian $J_1(N)$ of the modular curve $X_1(N)$, given the relation $T_{\alpha m} (J_1(N_0)(\overline{\mathbb{Q}})) \cong T_0$ (which in turn comes from the iso $J_1(N_0)(\overline{\mathbb{Q}})[m] \cong (T(N_0)/m)^2$).

1.1 $R = T$ theorems

We define a minimal deformation type $\mathcal{D}_0 = (S, \{ C_\ell \}_{\ell \in S})$ via the deformation conditions

- $\rho$ has determinant $\chi_p$,
- $\rho$ is unramified outside $S \cup \{ p \}$,
• \( \rho \) is semistable at \( S \),
• \( \rho \) is flat at \( p \) (if \( \mathfrak{p} \) is flat at \( p \)).

Here, \( S \) is the set of primes dividing \( N_0p \). Let \( R_0 \) be the universal deformation ring \( R_{D_0} \).

The representation \( \rho_{T_0} \) defined above satisfies all deformation conditions \( C_\ell \) prescribed by \( D_0 \). By the universal property, we obtain a map \( \varphi_0 : R_0 \to T_0 \).

The goal is to show that \( \varphi_0 \) is an isomorphism. Such a result needs to be proven at non-minimal levels as well, but this requires a slightly more involved argument.

A result establishing an isomorphism between a deformation ring \( R \) and a localized Hecke algebra \( T \) is known as a "\( R = \mathcal{T} \) theorem. Modularity follows from this isomorphism in the following way:

The representation \( \rho_{E,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p) \) corresponds to a map \( R \to \mathbb{Z}_p \) by the universal property. Since \( R \cong = T \), it follows that this map is the same as a map \( T \to \mathbb{Z}_p \). Finally, it is not hard to show that any homomorphism \( \phi : T \to \mathbb{Z}_p \) corresponding to a normalized Hecke eigenform \( f \) of weight 2 with Fourier coefficients \( a_n(f) = \phi(T_n) \), and one can check \( \rho_{f,p} \cong \rho_{E,p} \). This is a consequence of the perfect pairing \( T \times S_2(\Gamma_1(N))(\mathcal{O}) \to \mathcal{O} \) given by \( \langle T, f \rangle = a_1(Tf) \).

### 1.2 Taylor-Wiles primes

**Definition 1.1.** We say that a primer \( q \) is a Taylor-Wiles prime if

- \( q \notin S \) (i.e. \( \mathfrak{p} \) is unramified at \( q \))
- \( q \equiv 1 \pmod{p} \)
- \( \mathfrak{p}(\text{Frob}_q) \) is semisimple with distinct eigenvalues

Let \( Q = \{q_1, \ldots, q_r\} \) be a finite set of Taylor-Wiles primes. Define a new deformation condition \( D_Q = (S \cup Q, \{C_\ell\}_{\ell \in S \cup Q}) \) by allowing ramification at the primes \( q \in Q \). Denote by \( R_Q \) the associated deformation ring.

Since we have only loosened the conditions at the primes in \( Q \), the universal deformation of type \( D_0 \) is also of type \( D_Q \), so there is a natural homomorphism

\[ R_Q \to R_0 \]

Let \( \Delta_q \) be the \( p \)-primary part of \((\mathbb{Z}/q\mathbb{Z})^\times\), and set \( \Delta_Q \) to be the product

\[ \Delta_Q = \prod_{q \in Q} \Delta_q \]

The deformation ring \( R_Q \) is an \( \mathcal{O}[\Delta_Q] \)-algebra. Letting \( a_Q \) be the augmentation ideal in \( \mathcal{O}[\Delta_Q] \), there is an isomorphism

\[ R_Q/a_QR_Q \cong R_0. \]

Likewise, there is a localized Hecke algebra \( T_Q \) of level \( N_Q := N_0 \cdot \prod_{q \in Q} q \) which is of type \( D_Q \), and hence a map \( \phi_Q : R_Q \to T_Q \) which makes the square commute. \( T_Q \) is also naturally an \( \mathcal{O}[\Delta_Q] \)-algebra via the map \( \mathcal{O}[\Delta_Q] \to T_Q \) which sends \( x \in \Delta_Q \) to the diamond operator \( \langle d \rangle \) with \( d \equiv x \pmod{p} \) and \( d \equiv 1 \pmod{N_0} \). It can be shown that

\[ T_Q/a_QT_Q \cong T_0. \]
1.3 Patching

There exists $r \geq 1$ such that for every $n \geq 1$, there is a set $Q_n$ of $r$ Taylor-Wiles primes such that $q \equiv 1 \pmod{p^n}$. Set $R_n := R_{Q_n}$ and $T_n := T_{Q_n}$.

Given $Q_n$, the set of primes $Q_{n+1}$ can be constructed in a way so that the following diagram commutes

$$
\begin{array}{ccc}
R_{n+1} & \xrightarrow{\varphi_{n+1}} & T_{n+1} \\
\downarrow & & \downarrow \\
R_n & \xrightarrow{\varphi_n} & T_n
\end{array}
$$

Set $\Delta_n := \Delta_{Q_n}$. Note that $R_n$ and $T_n$ are algebras over

$$
O[\Delta_n] \cong O[T_1, \ldots, T_r] / ((1 + T_1)^{p^n} - 1) \cdots ((1 + T_r)^{p^n} - 1)
$$

Taking the inverse limit $O_{\infty} := \varprojlim O[\Delta_n]$ we get a formal power series ring over $O$ in $r$-variables

$$
O_{\infty} \cong O[[T_1, \ldots, T_r]]
$$

Set $R_{\infty} := \varprojlim R_n$ and $T_{\infty} := \varprojlim T_n$, and let $\varphi_{\infty} : R_{\infty} \to T_{\infty}$ be the inverse limit of the maps $\varphi_n : R_n \to T_n$.

Note that $R_0 = R_{\infty} / (T_1, \ldots, T_r)$ and $T_0 = T_{\infty} / (T_1, \ldots, T_r)$, so if it is shown that $\varphi_{\infty} : R_{\infty} \to T_{\infty}$ is an isomorphism, then it shall follow that $\varphi_0 : R_0 \to T_0$ is an isomorphism as well.

Each Hecke-algebra $T_n$ acts faithfully on a space of modular forms $M_n$ which is finitely generated and free as an $O[\Delta_n]$-module. Letting $M_{\infty} := \varprojlim M_n$, we find that $M_{\infty}$ is a finitely generated free $O_{\infty}$-module. It follows from this that $T_{\infty}$ is also a finitely generated and faithful $O_{\infty}$-module.

On the other hand, it follows from Galois theoretic arguments that $R_{\infty}$ is a quotient of $O[[T_1, \ldots, T_r]]$. By dimension considerations, $R_{\infty} = O[[T_1, \ldots, T_r]]$.

Since $R_{\infty} \to T_{\infty}$ is surjective and $T_{\infty}$ is faithful over $O_{\infty}$ this implies that $\varphi_{\infty}$ must be an isomorphism.