Examples of changing the order in triple integrals

Example 1: A tetrahedron T is defined by the inequalities $x, y, z \geq 0$ and $2x + 3y + z \leq 6$. The tetrahedron has three faces which are triangles in the coordinate planes. For example, the face of T in the xy-plane is given by $x, y \geq 0$ and $2x + 3y \leq 6$. The remaining face of T is the triangle with vertices $(3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 6)$. It's straightforward to draw a picture of T, as we did in class. If we want to describe T as a standard region, corresponding to the order of integration given by $dx\,dy\,dz$, then T is defined by the inequalities

$$\begin{aligned} &0 \leq x \leq 3; \\ &0 \leq y \leq \tfrac{1}{3}(6 - 2x); \\ &0 \leq z \leq 6 - 2x - 3y. \end{aligned}$$

If we wanted to use a triple integral to compute the volume of T, we would get

$$\begin{aligned} \text{volume}(T) &= \iiint_T 1\,dV = \int_0^3 \int_0^{\frac{1}{3}(6-2x)} \int_0^{6-2x-3y} 1\,dz\,dy\,dx \\ &= \int_0^3 \int_0^{\frac{1}{3}(6-2x)} (6 - 2x - 3y)\,dy\,dx = \int_0^3 \left[(6-2x)y - \frac{3}{2}y^2\right]_0^{\frac{1}{3}(6-2x)} dx \\ &= \int_0^3 \left[\frac{(6-2x)^2}{3} - \frac{3}{2}\frac{(6-2x)^2}{9}\right] dx \\ &= \frac{1}{6}\int_0^3 (6-2x)^2\,dx = \left(\frac{1}{6}\right)\cdot\left(-\frac{1}{6}\right)\left[(6-2x)^3\right]_0^6 \\ &= \frac{6^3}{6^2} = 6. \end{aligned}$$

(Here, we have skipped a few steps in the computation. Also, note that we have saved ourselves a lot of calculation by not expanding out the squared terms but rather by grouping them carefully.) The area of the base of T is one half the base times the height $= \frac{1}{2}(2)(3) = 3$, and the volume of T is one third the area of the base times the height $= \frac{1}{3}(3)(6) = 6$, which agrees with the computation above.

If we wanted to change the order of integration above to, say, $dx\,dy\,dz$, then T would lie to the front of the triangle in the yz-plane given by $y, z \geq 0$

and $3y + z \leq 6$. The inequalities defining T would then take the form

$$\begin{aligned} 0 &\leq z \leq 6; \\ 0 &\leq y \leq \tfrac{1}{3}(6 - z); \\ 0 &\leq x \leq \tfrac{1}{2}(6 - 3y - z). \end{aligned}$$

The corresponding integral expressing the volume of T would then be

$$\iiint_T 1\, dV = \int_0^6 \int_0^{\frac{1}{3}(6-z)} \int_0^{\frac{1}{2}(6-3y-z)} 1\, dx\, dy\, dz.$$

A similar calculation shows that this triple integral is equal to 6 (as it must).

Example 2: Consider the triple integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx,$$

where $f(x, y, z)$ is a generic function of three variables. This integral is an integral over the region in $\mathbb{R}^3$ defined by the inequalities

$$\begin{aligned} 0 &\leq x \leq 1; \\ \sqrt{x} &\leq y \leq 1; \\ 0 &\leq z \leq 1 - y. \end{aligned} \qquad (*)$$

It's not that difficult to draw a picture of the region defined by these inequalities, as we did in class, but it's not too helpful for what follows. If we just want to change the order to $dz\, dx\, dy$, we just want to change the Type I region in the plane defined by

$$\begin{aligned} 0 &\leq x \leq 1; \\ \sqrt{x} &\leq y \leq 1 \end{aligned}$$

to a Type II region, and we have seen how to do this: the Type II region is

$$\begin{aligned} 0 &\leq y \leq 1; \\ 0 &\leq x \leq y^2. \end{aligned}$$

(You can and should draw a picture of this!) So the integral becomes

$$\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, dz\, dx\, dy.$$

It's more complicated to change the order to $dy\,dx\,dz$. To do so, we look for the strongest inequalities we can get on z in terms of constants alone, on x in terms of z, and on y in terms of x and z. For example, we see that $0 \le z$. Also, $z \le 1 - y$ is the only inequality we have on z in the other direction. But we need an inequality only involving constants. However, since $y \ge 0$, $1 - y \le 1$, so we get a combined inequality $0 \le z \le 1 - y \le 1$. Similarly, we see that $0 \le x$ and $\sqrt{x} \le y$, so $x \le y^2$. However, we can't use an inequality for x which involves y, so we also use $z \le 1 - y$, hence $y \le 1 - z$, and thus $0 \le x \le y^2 \le (1-z)^2$. Finally, for y, we have $\sqrt{x} \le y \le 1$ and also $y \le 1-z$, which is a better inequality since $z \ge 0$ and hence $1 - z \le 1$. So in all we get three inequalities

$$\begin{aligned} 0 &\le z \le 1; \\ 0 &\le x \le (1-z)^2; \\ \sqrt{x} &\le y \le 1 - z. \end{aligned} \tag{$**$}$$

How do we know that we have done this correctly? What this means is, how do we know that the system of inequalities $(*)$ and $(**)$ are equivalent, i.e. that they define the same set of points (x, y, z) in $\mathbb{R}^3$? We have showed that, if (x, y, z) satisfy the system $(*)$ of inequalities, then they satisfy the system $(**)$. If we show conversely that every (x, y, z) satisfying the system $(**)$ also satisfy the system $(*)$, then we see that the regions defined by $(*)$ and $(**)$ are the same. **While we won't normally ask for this on HW or exams**, let us just check it in this case. Starting with (x, y, z) satisfying $(**)$, we see that $0 \le x \le (1 - z)^2 \le 1$, since $0 \le z \le 1$. Hence $0 \le x \le 1$. Next, $\sqrt{x} \le y \le 1 - z \le 1$, since $z \ge 0$. Hence $\sqrt{x} \le y \le 1$. Finally, $0 \le z$ and, from $y \le 1 - z$, we see that $z \le 1 - y$. (Of course, we also have $z \le 1$, but $z \le 1 - y$ is stronger, and $\sqrt{x} \le 1 - z$, hence $z \le 1 - \sqrt{x}$, but since $\sqrt{x} \le y$, $z \le 1 - y$ is a stronger inequality because $1 - y \le 1 - \sqrt{x}$.) This gives $0 \le z \le 1 - y$. So the system $(**)$ of inequalities implies the system $(*)$, and hence they are equivalent in the sense of defining the same points in $\mathbb{R}^3$.

In terms of triple integrals,

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz.$$

Example 3: Consider the triple integral

$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} f(x, y, z)\, dy\, dz\, dx.$$

Thus corresponds to the system of inequalities

$$\begin{aligned} 0 &\le x \le 1; \\ 0 &\le z \le 1 - x; \\ 0 &\le y \le 2 - 2z \end{aligned} \tag{†}$$

The first two inequalities define a triangle in the xz-plane bounded by the lines $x = 0$, $z = 0$, and $x + z = 1$. The region R defined by these inequalities is all of the y lying to the right of this region in the xz-plane and to the left of the plane $y + 2z = 2$. If you draw the picture, you can see that R is also described as a pyramid, whose base is the rectangle in the xy-plane defined by $0 \le x \le 1$ and $0 \le y \le 2$, and whose vertex is the point $(0, 0, 1)$.

Now let's try the order $dz\, dy\, dx$. First, we see that $0 \le x \le 1$, and the inequality $z \le 1 - x$, which is the same as $x \le 1 - z$, doesn't give anything better since z has to disappear from the final answer. Also, $0 \le y \le 2 - 2z$ and the inequality $x \le 1 - z$, i.e. $2x \le 2 - 2z$, is not comparable with this. As we have to make z disappear from the final inequality, we use $z \ge 0$ to conclude that $0 \le y \le 2 - 2z \le 2$.

What about inequalities for z? We see that $0 \le z \le 1 - x$. and also, from $y \le 2 - 2z$, that $2z \le 2 - y$, or equivalently that $z \le 1 - \frac{1}{2}y$. Neither of the inequalities $z \le 1 - x$ or $z \le 1 - \frac{1}{2}y$ is necessarily better that the other, so for the moment we just record the inequalities we have learned:

$$\begin{aligned} 0 &\le x \le 1; \\ 0 &\le y \le 2; \\ 0 &\le z \le 1 - x; \\ 0 &\le z \le 1 - \tfrac{1}{2}y. \end{aligned} \tag{††}$$

The way to understand this system of inequalities is as follows. The first two define a rectangle in the xy-plane. When $1 - x \ge 1 - \frac{1}{2}y$, the **second** inequality for z is stronger than the first (i.e. imposes a stronger condition on z). If $1 - x \le 1 - \frac{1}{2}y$, the **first** inequality for z is stronger than the second (i.e. imposes a stronger condition on z). This says that we should divide up the rectangle $0 \le x \le 1$, $0 \le y \le 2$, according to whether $1 - x \ge 1 - \frac{1}{2}y$ or $1 - x \le 1 - \frac{1}{2}y$. The first condition is that $y \ge 2x$, the second that

$y \le 2x$. So we define two regions R_1 and R_2 as follows: R_1 is the region where $0 \le x \le 1$, $0 \le y \le 2$, and $y \ge 2x$, and then we use the inequality $0 \le z \le 1 - \frac{1}{2}y$. The second region R_2 is the region where $0 \le x \le 1$, $0 \le y \le 2$, and $y \le 2x$, and then we use the inequality $0 \le z \le 1 - x$. Thus, the region R_1 is defined by

$$\begin{aligned} 0 &\le x \le 1; \\ 2x &\le y \le 2; \\ 0 &\le z \le 1 - \tfrac{1}{2}y. \end{aligned} \tag{$\dagger\dagger_1$}$$

The region R_2 is defined by

$$\begin{aligned} 0 &\le x \le 1; \\ 0 &\le y \le 2x; \\ 0 &\le z \le 1 - x. \end{aligned} \tag{$\dagger\dagger_2$}$$

You can also see this from a good picture of the pyramid: over the triangle in the xy-plane below the diagonal line $y = 2x$ in the rectangle $0 \le x \le 1$, $0 \le y \le 2$, the upper boundary of the pyramid is the plane $z = 1 - x$. Over the triangle in the rectangle lying above the diagonal line $y = 2x$, the upper boundary of the pyramid is the plane $z = 1 - \frac{1}{2}y$.

The upshot is that, while the original region R is a standard region for the order $dy\,dz\,dx$, it is **not** a standard region for the order $dz\,dy\,dx$. In fact, it is a union of two standard regions R_1 and R_2 for the order $dz\,dy\,dx$. Thus we can write (for a generic function $f(x, y, z)$)

$$\begin{aligned} \iiint_R f(x,y,x)\,dV &= \int_0^1 \int_0^{1-x} \int_0^{2-2z} f(x,y,z)\,dy\,dz\,dx \\ &= \iiint_{R_1} f(x,y,x)\,dV + \iiint_{R_2} f(x,y,x)\,dV \\ &= \int_0^1 \int_{2x}^2 \int_0^{1-\frac{1}{2}y} f(x,y,z)\,dz\,dy\,dx + \int_0^1 \int_0^{2x} \int_0^{1-x} f(x,y,z)\,dz\,dy\,dx. \end{aligned}$$

Other examples of this type are in the HW for Monday (Stewart 15.6, exercises 34, 36).