Some distance, area and volume formulas

The distance from a point to a line. We begin with the case of a point $\mathbf{v}$ and a line L through the origin. Then L is the set of all scalar multiples $t\mathbf{w}$ of a nonzero vector $\mathbf{w}$. In the special case where $\mathbf{w} = \mathbf{u}$ is a unit vector, we define the *component of* $\mathbf{v}$ *along* $\mathbf{u}$ or the *scalar projection of* $\mathbf{v}$ *to* L to be $\mathbf{v} \cdot \mathbf{u}$. The *vector projection* or simply *projection* of $\mathbf{v}$ to L is then

$$p_{\mathbf{u}}(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{u})\mathbf{u}.$$

Note that $p_{\mathbf{u}}(\mathbf{v})$ is a scalar multiple of $\mathbf{u}$, so that $p_{\mathbf{u}}(\mathbf{v}) \in L$. The vector $\mathbf{v} - p_{\mathbf{u}}(\mathbf{v})$ is then perpendicular to L, because

$$\begin{aligned}(\mathbf{v} - p_{\mathbf{u}}(\mathbf{v})) \cdot \mathbf{u} &= (\mathbf{v} \cdot \mathbf{u}) - ((\mathbf{v} \cdot \mathbf{u})\mathbf{u}) \cdot \mathbf{u} \\ &= (\mathbf{v} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{u})(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{v} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{u})\|\mathbf{u}\|^2 = 0,\end{aligned}$$

since $\|\mathbf{u}\|^2 = 1$ under the assumption that $\mathbf{u}$ is a unit vector. The distance from $\mathbf{v}$ to L is then

$$\boxed{\text{distance } (\mathbf{v}, L) = \|\mathbf{v} - p_{\mathbf{u}}(\mathbf{v})\|} \tag{1}$$

Next consider the case where L is still a line through the origin, hence is the set of all scalar multiples $t\mathbf{w}$ of a nonzero vector $\mathbf{w}$, but $\mathbf{w}$ is not necessarily a unit vector. Then $\mathbf{u} = \mathbf{w}/\|\mathbf{w}\|$ is a unit vector and it defines the same line through the origin as $\mathbf{w}$. We then define the *scalar projection of* $\mathbf{v}$ *to* L to be $(\mathbf{v} \cdot \mathbf{w})/\|\mathbf{w}\|$, and the *vector projection* or simply *projection* of $\mathbf{v}$ to L to be

$$p_{\mathbf{w}}(\mathbf{v}) = \frac{(\mathbf{v} \cdot \mathbf{w})}{\|\mathbf{w}\|^2}\mathbf{w}.$$

(These are the formulas we would get by substituting in $\mathbf{u} = \mathbf{w}/\|\mathbf{w}\|$.) Then we have the same formula

$$\boxed{\text{distance } (\mathbf{v}, L) = \|\mathbf{v} - p_{\mathbf{w}}(\mathbf{v})\|} \tag{2}$$

(Compare the discussion in Stewart p. 811 as well as Figures 4 and 5 on that page.) Note that the scalar projection satisfies

$$\frac{(\mathbf{v} \cdot \mathbf{w})}{\|\mathbf{w}\|} = \pm\|p_{\mathbf{w}}(\mathbf{v})\| = \frac{\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta}{\|\mathbf{w}\|} = \|\mathbf{v}\| \cos\theta,$$

and this is the length of the leg of the right triangle whose hypotenuse is $\mathbf{v}$ and which has one side parallel to $\mathbf{w}$, at least if $\cos\theta$ is positive (or equivalently, θ is an acute angle). What happens if $\cos\theta$ is negative?

Finally consider the case where L does not necessarily pass through the origin. Then L is the set of all vectors of the form $\mathbf{p}_0 + t\mathbf{w}$, for some fixed nonzero vector $\mathbf{w}$, as t runs through all real numbers. Let $\mathbf{p}$ be a point. Subtracting $\mathbf{p}_0$ replaces L by the set of all scalar multiples $t\mathbf{w}$ of $\mathbf{w}$, and replaces $\mathbf{p}$ by $\mathbf{p} - \mathbf{p}_0 = \mathbf{v}$, and doesn't change distances. Thus

$$\text{(3)} \qquad \boxed{\text{distance } (\mathbf{p}, L) = \|(\mathbf{p} - \mathbf{p}_0) - p_{\mathbf{w}}(\mathbf{p} - \mathbf{p}_0)\|}$$

In practice, it is usually simpler to start off by subtracting $\mathbf{p}_0$ to be in the situation described by Equation (2).

Example: Find the distance from (i) the point $(1, 2, 4)$ to the line L through $(2, 3, 2)$ which is parallel to $(-1, -1, 5)$; (ii) the point $(1, 1, -2)$ to the line L through $(3, -3, 2)$ which is parallel to $(1, -2, 2)$.

Solution: (i) Here L is the set of all points of the form $(2, 3, 2) + t(-1, -1, 5)$. Subtracting $(2, 3, 2)$, we see that we are in the situation of (2) with $\mathbf{v} = (1, 2, 4) - (2, 3, 2) = (-1, -1, 2)$ and $\mathbf{w} = (-1, -1, 5)$. Then

$$p_{\mathbf{w}}(\mathbf{v}) = \frac{12}{27}(-1, -1, 5) = \frac{4}{9}(-1, -1, 5),$$

and

$$\mathbf{v} - p_{\mathbf{w}}(\mathbf{v}) = (-1, -1, 2) - \frac{4}{9}(-1, -1, 5) = (-\frac{5}{9}, -\frac{5}{9}, -\frac{2}{9}) = \frac{1}{9}(-5, -5, -2).$$

So finally the distance is

$$\|\mathbf{v} - p_{\mathbf{w}}(\mathbf{v})\| = \frac{1}{9}\|(-5, -5, -2)\| = \frac{\sqrt{54}}{9} = \frac{3\sqrt{6}}{9} = \frac{\sqrt{6}}{3}.$$

(ii) Here L is the set of all points of the form $(3, -3, 2) + t(1, -2, 2)$. Subtracting $(3, -3, 2)$, we see that we are in the situation of (2) with $\mathbf{v} = (1, 1, -2) - (3, -3, 2) = (-2, 4, -4)$ and $\mathbf{w} = (1, -2, 2)$. Then

$$p_{\mathbf{w}}(\mathbf{v}) = \frac{-18}{9}(1, -2, 2) = (-2)(1, -2, 2) = (-2, 4, -4) = \mathbf{v},$$

and

$$\mathbf{v} - p_{\mathbf{w}}(\mathbf{v}) = (-2, 4, -4) - (-2, 4, -4) = \mathbf{0}.$$

The distance from $(1, 1, -2)$ to L is then $\|\mathbf{0}\| = 0$. This just means that $(1, 1, -2)$ lies on L; in fact, $(1, 1, -2) = (3, -3, 2) + (-2)(1, -2, 2)$.

There is another formula that works for point and lines in $\mathbb{R}^3$ (and, by extension, for $\mathbb{R}^2$, but not in higher dimensions). As before, let L be the set of all vectors of the form $\mathbf{p}_0 + t\mathbf{w}$, for some fixed nonzero vector $\mathbf{w}$. For a point $\mathbf{p}$, we set $\mathbf{v} = \mathbf{p} - \mathbf{p}_0$. Then

$$\text{(4)} \qquad \boxed{\text{distance }(\mathbf{p}, L) = \frac{\|\mathbf{v} \times \mathbf{w}\|}{\|\mathbf{w}\|}}$$

To see this, we can subtract off $\mathbf{p}_0$ as in the discussion before Equation (3), so we may as well assume that $\mathbf{p}_0 = \mathbf{0}$ and that $\mathbf{p} = \mathbf{v}$. Then $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, and hence

$$\frac{\|\mathbf{v} \times \mathbf{w}\|}{\|\mathbf{w}\|} = \|\mathbf{v}\| \sin\theta.$$

In class, we have identified $\|\mathbf{v}\| \sin\theta$ with the distance from $\mathbf{v}$ to L (see also Stewart, p. 817, Figure 2.)

Let's redo the example above using Formula (4):

(i) Here as noted $\mathbf{v} = (-1, -1, 2)$ and $\mathbf{w} = (-1, -1, 5)$. Also, a computation shows that $\mathbf{v} \times \mathbf{w} = (-1, -1, 2) \times (-1, -1, 5) = (-3, 3, 0)$. Thus $\|\mathbf{v} \times \mathbf{w}\| = \|(-3, 3, 0)\| = \sqrt{18}$ and $\|\mathbf{w}\| = \sqrt{27}$, so that the distance is

$$\frac{\sqrt{18}}{\sqrt{27}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3},$$

agreeing with our previous computation.

(ii) Here as noted $\mathbf{v} = (-2, 4, -4)$ and $\mathbf{w} = (1, -2, 2)$. Also, a computation shows that $\mathbf{v} \times \mathbf{w} = (-2, 4, -4) \times (1, -2, 2) = (0, 0, 0)$, so the distance is 0. This again agrees with the previous computation.

The distance from a point to a plane in $\mathbb{R}^3$. As before, we begin with a plane P through the origin, given by the equation $Ax + By + Cz = 0$. Here $\mathbf{n} = (A, B, C)$ is the normal vector. Let $\mathbf{v}$ be a point. Then the distance from $\mathbf{v}$ to P is given by the length of the vector projection of $\mathbf{v}$ onto $\mathbf{n}$ (see for example Stewart p. 829 Figure 12). This length is

$$\frac{|\mathbf{v} \cdot \mathbf{n}|}{\|\mathbf{n}\|^2} \|\mathbf{n}\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}.$$

Explicitly, if $\mathbf{v} = (x_1, y_1, z_1)$, then the distance is given by

$$\text{(5)} \qquad \boxed{\text{distance }(\mathbf{v}, P) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|Ax_1 + By_1 + Cz_1|}{\sqrt{A^2 + B^2 + C^2}}}$$

Now assume that the plane P does not necessarily pass through the origin, and fix a point $\mathbf{p}_0 = (x_0, y_0, z_0)$ on P. We have seen that the plane P is given by the equation $Ax + By + Cz = D$, where $\mathbf{n} = (A, B, C)$ is the normal vector and $D = \mathbf{n} \cdot \mathbf{p}_0 = Ax_0 + By_0 + Cz_0$. Let $\mathbf{p}_1 = (x_1, y_1, z_1)$ be a point in $\mathbb{R}^3$. We wish to find the distance from $\mathbf{p}_1$ to P. Subtracting $\mathbf{p}_0$ replaces P by the plane through the origin defined by $Ax + By + Cz = 0$, replaces $\mathbf{p}$ by $\mathbf{p}_1 - \mathbf{p}_0 = \mathbf{v}$, and doesn't change distances. So by applying Equation (5), we see that

$$\text{(6)} \qquad \boxed{\text{distance } (\mathbf{p}_1, P) = \frac{|\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_0)|}{\|\mathbf{n}\|} = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}}$$

Example: the distance from $(0, 0, 1)$ to the plane defined by $2x - 3y + z = 5$: here $(A, B, C) = (2, -3, 1)$, $D = 5$, and $(x_1, y_1, z_1) = (0, 0, 1)$. The distance is then $|1 - 5|/\sqrt{14} = 4/\sqrt{14}$.

The distance between two parallel planes in $\mathbb{R}^3$. Suppose that P_1 and P_2 are two parallel planes in $\mathbb{R}^3$. Then they have the same normal vectors (up to a nonzero scalar multiple) and so we can assume that P_1 is given by the equation $Ax + By + Cz = D_1$ and that P_2 is given by the equation $Ax + By + Cz = D_2$. The distance from P_2 to P_1 is the same as the distance from any point of P_2 to P_1. Applying Equation (6), we see:

$$\text{(7)} \qquad \boxed{\text{distance } (P_1, P_2) = \frac{|Ax_2 + By_2 + Cz_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}}$$

The distance between two skew lines in $\mathbb{R}^3$. Let L_1 and L_2 be two skew lines in $\mathbb{R}^3$. Then L_1 is the set of all vectors of the form $\mathbf{p}_1 + t\mathbf{w}_1$, $t \in \mathbb{R}$, and likewise L_2 is the set of all vectors of the form $\mathbf{p}_2 + t\mathbf{w}_2$, $t \in \mathbb{R}$. The condition that L_1 and L_2 are skew is that $\mathbf{w}_1$ and $\mathbf{w}_2$ are not parallel. There is a unique vector perpendicular to both $\mathbf{w}_1$ and $\mathbf{w}_2$ up to multiplying by a nonzero scalar, namely $\mathbf{n} = \mathbf{w}_1 \times \mathbf{w}_2 = (A, B, C)$, say. Let $D_1 = \mathbf{n} \cdot \mathbf{p}_1$ and let $D_2 = \mathbf{n} \cdot \mathbf{p}_2$. Then L_1 is contained in the plane P_1 with equation $Ax + By + Cz = D_1$, and Then L_2 is contained in the parallel plane P_2 with equation $Ax + By + Cz = D_2$. A basic fact (which we shall not prove) is that the distance between L_1 and L_2 is the same as the distance between P_1 and P_2; equivalently, there is a line segment (in fact a unique one) joining L_1 and L_2 which is perpendicular to L_1 and L_2. Assuming this, we can use

Equation (7) to find the distance from L_1 to L_2:

$$\text{(8)} \qquad \boxed{\text{distance } (L_1, L_2) = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}}$$

Alternatively, we can write this as

$$\text{(9)} \qquad \boxed{\text{distance } (L_1, L_2) = \frac{|(\mathbf{w}_1 \times \mathbf{w}_2) \cdot \mathbf{v}|}{\|\mathbf{w}_1 \times \mathbf{w}_2\|}}$$

where as usual $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$. (We can recognize this quantity as the length of the projection of $\mathbf{v}$ to the line perpendicular to L_1 and L_2.)

Example: let L_1 be the line whose points are of the form $(-1, 2, -1) + t(2, 0, 3)$ and let L_2 be the line whose points are of the form $(0, 2, 0) + t(1, 1, 2)$. First, the normal to L_1 and L_2 is given by $(2, 0, 3) \times (1, 1, 2) = (-3, -1, 2)$. Then $D_1 = (-3, -1, 2) \cdot (-1, 2, -1) = -1$ and $D_2 = (-3, -1, 2) \cdot (0, 2, 0) = -2$. Thus, the distance between L_1 and L_2 is

$$\frac{|-2 - (-1)|}{\sqrt{9 + 1 + 4}} = \frac{1}{\sqrt{14}}.$$

Determinants, area, and volume. A *matrix* is a rectangular array of numbers. We shall only be concerned with 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and 3×3 matrices

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We define the determinant $\det M$ of a 2×2 matrix M by the formula

$$\det M = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bd.$$

Note that, in spite of the vertical lines in the notation above, $\det M$ **can be negative**. For example,

$$\det \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} = \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13.$$

The determinant $\det M$ of a 3×3 matrix M can be defined in various different ways: either inductively by the formula

$$\det M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

or directly by the formula

$$\det M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

Finally, if the rows of M are the vectors $\mathbf{v} = (a_1, a_2, a_3)$,$\mathbf{w} = (b_1, b_2, b_3)$, and $\mathbf{u} = (c_1, c_2, c_3)$, then direct computation shows that

$$\det M = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Example: $\det \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} = 1(1) - 3(4) + 2(5) = -1$. If we wanted to compute via cross products instead, we have $(4, 1, 0) \times (-1, 1, 1) = (1, -4, 5)$, and thus

$$\det M = (1, 3, 2) \cdot (1, -4, 5) = 1 - 12 + 10 = -1.$$

Determinants are use to compute are and volume as follows: First, if $\mathbf{v}_1 = (a, c)$ and $\mathbf{v}_2 = (b, d)$ are the two rows of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let P be the parallelogram in $\mathbb{R}^2$ with vertices $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_2$. Then

$$\boxed{\text{area } (P) = |\det M| = |ad - bd|} \tag{10}$$

Note that we have to take the **absolute value** of the determinant. For example, the parallelogram in $\mathbb{R}^2$ with vertices $\mathbf{0}$, $(1, 3)$, $(5, 2)$, and $(6, 5)$ is $|2 - 15| = |-13| = 13$.

A similar result holds for the parallelepiped P defined by by $\mathbf{0}$, $\mathbf{v}_1 = (a_1, a_2, a_3)$, $\mathbf{v}_2 = (b_1, b_2, b_3)$, and $\mathbf{v}_3 = (c_1, c_2, c_3)$ (so that its vertices are $\mathbf{0}$,

$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$). The volume of P is given by:

$$\text{(11)} \qquad \boxed{\text{volume } (P) = |\det M|}$$

As for the area of a parallelogram in the plane, the quantity $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ can be **negative**, so the volume is always given by the **absolute value** of the determinant.

For example, let P be the parallelepiped defined by $\mathbf{0}$, $\mathbf{v}_1 = (1, 3, 2)$, $\mathbf{v}_2 = (4, 1, 0)$, and $\mathbf{v}_3 = (-1, 1, 1)$ (so that its vertices are $\mathbf{0}$, $\mathbf{v}_1 = (1, 3, 2)$, $\mathbf{v}_2 = (4, 1, 0)$, $\mathbf{v}_3 = (-1, 1, 1)$, $\mathbf{v}_1 + \mathbf{v}_2 = (5, 4, 2)$, $\mathbf{v}_1 + \mathbf{v}_3 = (0, 4, 3)$, $\mathbf{v}_2 + \mathbf{v}_3 = (3, 2, 1)$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = (4, 5, 3)$), then the volume of P is $|\det M| = |-1| = 1$, where M is the 3×3 matrix whose determinant we computed on the last page.

Exercises

Exercise 1: Let $\mathbf{v} = (1, 0, -2)$ and let $\mathbf{w} = (-2, -2, 1)$.

(i) Find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{w}$.

(ii) Find the component of $\mathbf{v}$ along $\mathbf{u}$ and the (vector) projection $p_{\mathbf{u}}(\mathbf{v})$.

(iii) Find $\mathbf{v} - p_{\mathbf{u}}(\mathbf{v})$ and verify that it is perpendicular to $\mathbf{u}$.

(iv) Compute the distance from $\mathbf{v}$ to the line through the origin in $\mathbb{R}^3$ and $\mathbf{w}$.

Exercise 2: Find the following determinants:

$$\text{(a)} \det \begin{pmatrix} 2 & -4 \\ 3 & 7 \end{pmatrix}; \qquad \text{(b)} \det \begin{pmatrix} 3 & 5 \\ 3 & 2 \end{pmatrix};$$

$$\text{(c)} \det \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}; \qquad \text{(d)} \det \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 2 \\ 0 & -1 & -4 \end{pmatrix}; \qquad \text{(e)} \det \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

For (a), (b), what is the area of the parallelogram with vertices $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the two rows of the given matrix?

For (c), (d), (e), what is the volume of the parallelepiped defined by the vectors $\mathbf{0} = (0, 0, 0)$ and the three rows $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of the given matrix? (The 8 vertices of the parallelepiped are $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Compare Figure 3 on Stewart p. 819.) What is the meaning of your answer for (d)?

Exercise 3: Find the distance from the point $\mathbf{p}$ to the line L given as the set of all vectors of the form $\mathbf{p}_0 + t\mathbf{w}$, where

(i) $\mathbf{p} = (-1, 1, 0)$, $\mathbf{p}_0 = (2, -1, 4)$, and $\mathbf{w} = (1, -1, 1)$.

(ii) $\mathbf{p} = (4, -1, 3)$, $\mathbf{p}_0 = (1, 2, 3)$, and $\mathbf{w} = (1, -1, 0)$.

Exercise 4: Find the distance from the point $\mathbf{p} = (-1, -1, 5)$ to the plane defined by the equation $3x + 2y - 5z = 10$.

Exercise 5: Find the distance from the point $(2, 1, 4)$ to the plane containing the three points $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

Exercise 6: Let L_1 be the line given as the set of all vectors of the form $(1, 2, 0) + t(-1, -1, 4)$ and and let L_2 be the line given as the set of all vectors of the form $(0, 2, 1) + t(2, -3, 2)$. Find the distance from L_1 to L_2.