Integer random matrices, fluctuations of random groups, and an interacting particle system

Roger Van Peski (Columbia)

Columbia Probability Seminar
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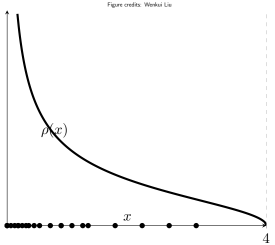
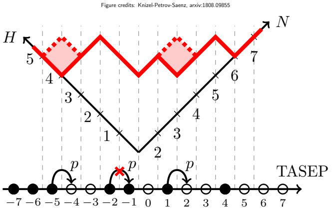
Based on joint work with Hoi Nguyen (https://arxiv.org/abs/2409.03099), with background from https://arxiv.org/abs/2310.12275, https://arxiv.org/abs/2312.11702

Random matrices and interacting particle systems



Random matrix theory over $\mathbb{R}, \mathbb{C}$



Discrete-space interacting particle systems (TASEP, ASEP, ...)

Many limit objects (e.g. Tracy-Widom) appear in both.

There is also discrete random matrix theory over $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$, etc.

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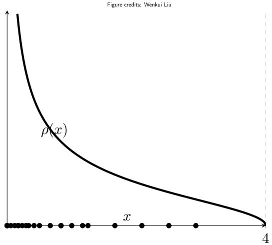

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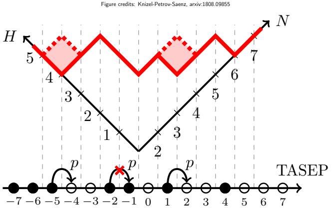

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There is also discrete random matrix theory over $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$, etc.

Today: Random matrix products over $\mathbb{Z} \rightsquigarrow$ discrete interacting particle system ('reflecting Poisson sea')

1. Products of random matrices over $\mathbb{Z}/p\mathbb{Z}$
2. A single random matrix over $\mathbb{Z}$
3. Matrix products over $\mathbb{Z}$ and an interacting particle system
4. The rescaled moment method

Products of random matrices over $\mathbb{Z}/p\mathbb{Z}$

Let $A_1, A_2, \ldots$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of $\mathrm{rank}(A_\tau \cdots A_2 A_1)$ look like for large N and τ?

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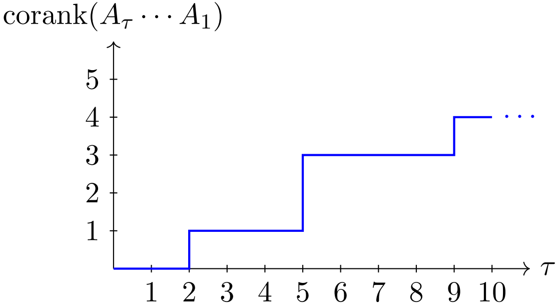

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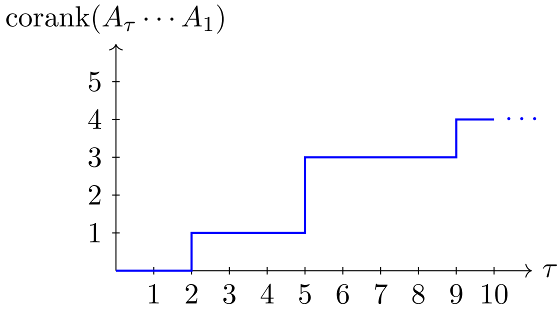

Fact: $\mathrm{corank}(A_\tau \cdots A_1) \approx \log_p \tau$, finite limit fluctuations.

Question 1: What is the limit of

$$\mathrm{corank}(A_\tau \cdots A_1) - \log_p \tau$$

as $N, \tau \to \infty$?.

Definition

For any $\chi \in \mathbb{R}_{>0}, p \in \mathbb{R}_{>1}$, $\mathcal{L}^{(1)}_{p^{-1},\chi}$ *is the* $\mathbb{Z}$*-valued random variable given by*

$$\Pr(\mathcal{L}^{(1)}_{p^{-1},\chi} = x) = \frac{1}{\prod_{i \geq 1}(1-p^{-i})} \sum_{j \geq 0} e^{-\chi p^{j-x}} \frac{(-1)^j p^{-\binom{j}{2}}}{\prod_{i=1}^{j}(1-p^{-i})}$$

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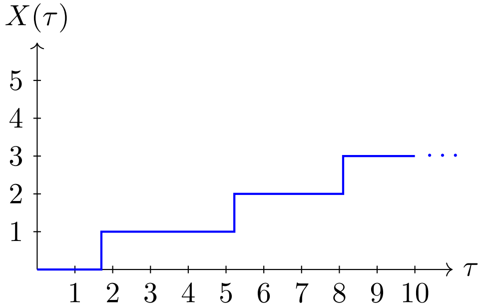
Theorem (VP '23, special case)

For each $N \geq 1$ *take* $A_1^{(N)}, A_2^{(N)}, \ldots$ *iid uniform in* $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$, *and* $(\tau_N)_{N\geq 1}$ *such that* $1 \ll \tau_N \ll p^N$ *and the fractional part* $\{-\log_p \tau_N\}$ *converges to some* $\zeta \in [0,1]$.
Then as $N \to \infty$,

$$\operatorname{corank}(A_{\tau_N}^{(N)} \cdots A_1^{(N)}) - \mathrm{Int}(\log_p \tau_N + \zeta) \to \mathcal{L}^{(1)}_{p^{-1},p^{-\zeta}/(p-1)}.$$

Continuous-time model process

Let $p > 1$ and $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be the $\mathbb{Z}_{\geq 0}$-valued process which jumps by 1, and waits at $x \in \mathbb{Z}_{\geq 0}$ for an $\mathrm{Exp}(p^{-x})$-distributed time.



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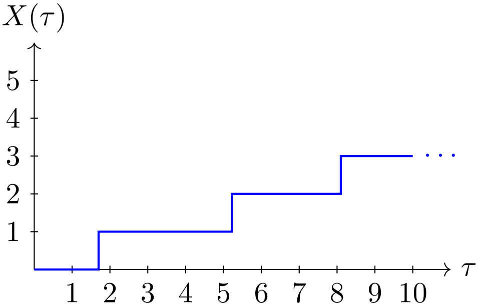

Theorem (VP '23, special case)

$$X(\tau) - (\log_p \tau + \zeta) \to \mathcal{L}^{(1)}_{p^{-1}, p^{-\zeta}/(p-1)}$$

in distribution as $\tau \to \infty$ along the sequence $\tau \in p^{\mathbb{N}-\zeta}$.

A single random matrix over $\mathbb{Z}$

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In many contexts, want asymptotic distribution of (pseudo-)random finite abelian groups G:

- Arithmetic statistics—distributions of class groups of number fields, Tate-Shafarevich groups—(Bhargava, Cohen, Ellenberg, Kane, Lenstra Jr., Nguyen, Poonen, Rains, Sawin, Venkatesh, Westerland, Wood... '83-present),
- Sandpile groups of random graphs—(Clancy, Fulman, Kaplan, Koplewitz, Leake, Nguyen, Payne, Wood... '14-present),
- (co)homology groups of random chain complexes—(Kahle, Lutz, Meszaros, Newman, Parsons,...)

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$A \in \mathrm{Mat}_N(\mathbb{Z})$ gives linear map $A : \mathbb{Z}^N \to \mathbb{Z}^N$, has **cokernel**

$$\mathrm{cok}(A) := \mathbb{Z}^N / A\mathbb{Z}^N,$$

an abelian group.

Singular numbers and random groups

For $A \in \mathrm{Mat}_N(\mathbb{Z})$,

$$A = U \operatorname{diag}(a_1, \ldots, a_N) V$$

for some $U, V \in \mathrm{GL}_N(\mathbb{Z})$ and 'discrete singular values' $a_1 \geq \ldots \geq a_N \in \mathbb{Z}_{\geq 0}$.

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$$\operatorname{cok}(A) := \mathbb{Z}^N / A\mathbb{Z}^N \cong \bigoplus_{i=1}^{N} \mathbb{Z}/a_i\mathbb{Z},$$

parametrized by the same $a_1, \ldots, a_N$.

Universality and decoupling of primes

Easier to look at p-Sylow subgroups:

$$\operatorname{cok}(A) \cong \bigoplus_{p \text{ prime}} \operatorname{cok}(A)_p.$$

Theorem (Wood 2015)

Fix p and integer random variable ξ with $\xi \pmod p$ nonconstant. For each $N \geq 1$ let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ have iid ξ entries. Then as $N \to \infty$, $\operatorname{cok}(A^{(N)})_p$ converges in law to the Cohen-Lenstra distribution on abelian p-groups,

$$\Pr(G) = \frac{\prod_{i=1}^{\infty}(1-p^{-i})}{\#\operatorname{Aut}(G)}.$$

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Furthermore,* $\operatorname{cok}(A^{(N)})_p, p = 2, 3, 5, \ldots$ *are asymptotically independent (as* $N \to \infty$*) for different* p*.

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Note: $\operatorname{cok}(A)_p \cong \bigoplus_i \mathbb{Z}/p^{\lambda_i^{(p)}}\mathbb{Z}$ for some p-**singular numbers** $(\lambda_1^{(p)}, \ldots, \lambda_N^{(p)})$ (random partition); also $a_i = 2^{\lambda_i^{(2)}} \cdot 3^{\lambda_i^{(3)}} \cdots$

Backstory

- 1983: Cohen and Lenstra observe in statistics of class groups
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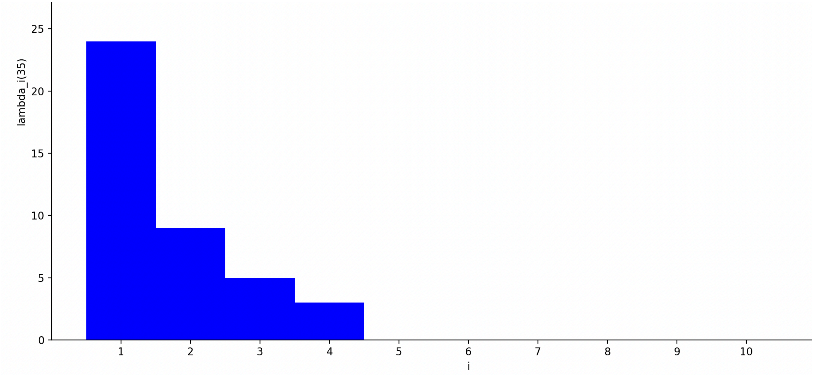
Question

How do p-singular numbers/cokernels of integer matrix products behave?

Remark

For singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random complex matrices, local limits are deformations of sine/Airy kernel ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018], ...).

Take $(\lambda_1^{(p)}(\tau), \ldots, \lambda_N^{(p)}(\tau))$ p-singular numbers of $A_\tau \cdots A_2 \cdot A_1$.



ith column $= \lambda_i^{(p)}(\tau)$.

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(Recall p-rank$(G) := \dim_{\mathbb{F}_p}(G/pG)$, an $\mathbb{F}_p$-vector space).

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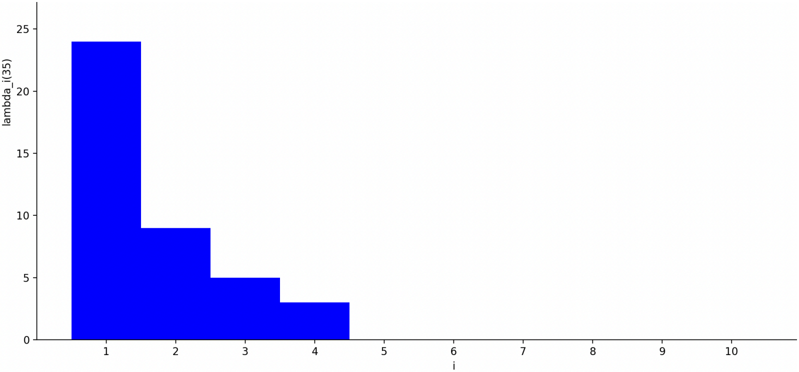

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First row $= \operatorname{corank}(A_\tau \cdots A_1 \pmod p)$.

Matrix products over $\mathbb{Z}$ and an interacting particle system

Theorem (Nguyen-VP '24)

Fix p prime, let ξ be a $\mathbb{Z}$-valued random variable such that $\xi \pmod p$ is nonconstant. For each $N \geq 1$ let $A_1^{(N)}, A_2^{(N)}, \ldots$ be iid $N \times N$ matrices over $\mathbb{Z}$ with iid ξ entries. Let

$$G_N := \operatorname{cok}(A_{\tau(N)}^{(N)} \cdots A_1^{(N)})_p,$$

where the number of matrices $\tau(N)$ satisfies

- $\tau(N) \to \infty$ *as* $N \to \infty$,
- $\tau(N) = O(e^{(\log N)^{1-\epsilon}})$ *for some* $0 < \epsilon < 1$, *and*
- *the fractional part* $\{-\log_p \tau(N)\}$ *converges to some* $\zeta \in [0,1]$.

Then

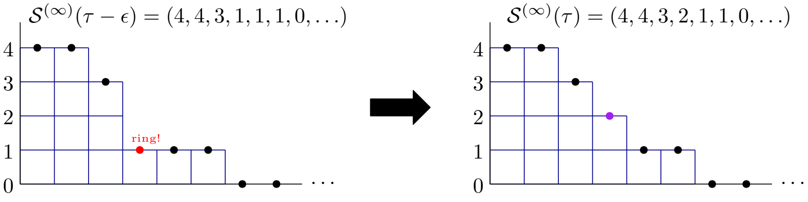
$$\underbrace{p\text{-}\operatorname{rank}(p^{i-1} G_N)}_{(i^{th} \text{ row of blue wave})} - \operatorname{Int}(\log_p(\tau(N)) + \zeta) \to \mathcal{L}^{(i)}_{p^{-1}, p^{-\zeta}/(p-1)}$$

in joint distribution for $i = 1, 2, \ldots$, where the $\mathcal{L}^{(i)}_{p^{-1}, p^{-\zeta}/(p-1)}$ are explicit (correlated) integer random variables.

Particle positions $\mathcal{S}^{(\infty)}(\tau) = (\mathcal{S}^{(\infty)}(\tau)_1, \mathcal{S}^{(\infty)}(\tau)_2, \ldots), \tau \in \mathbb{R}_{\geq 0}$.

Starts at $\mathcal{S}^{(\infty)}(0) = (0, 0, \ldots)$.

Indep. exp. clocks at $1, 2, \ldots$ of rates $p^{-1}, p^{-2}, \ldots$ control jumps.



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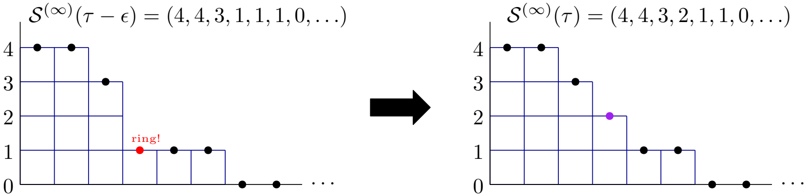
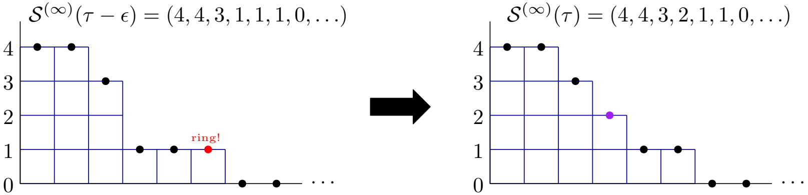


When particle is blocked, donates jump:



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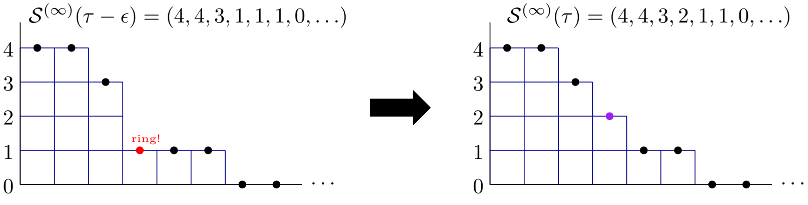

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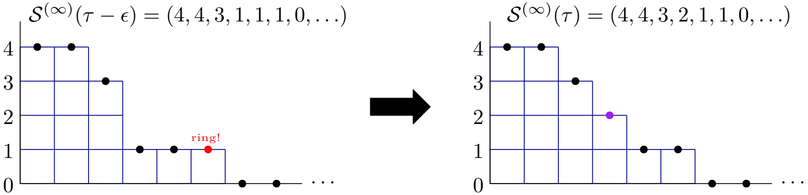

First row is $X(\frac{1}{1-p^{-1}}\tau)$ (jump rate $p^{-x} + p^{-x-1} + \dots = \frac{p^{-x}}{1-p^{-1}}$).

Theorem (VP '23)

Let $\tau_N = p^{N-\zeta}$. Then we have (joint) convergence in distribution

$$\underbrace{\mathcal{S}^{(\infty)}(\tau_N)'_i}_{i^{th}\ \textit{row}} - \log_p(\tau_N) - \zeta \xrightarrow{N\to\infty} \mathcal{L}^{(i)}_{p^{-1},p^{-\zeta}/(p-1)}.$$

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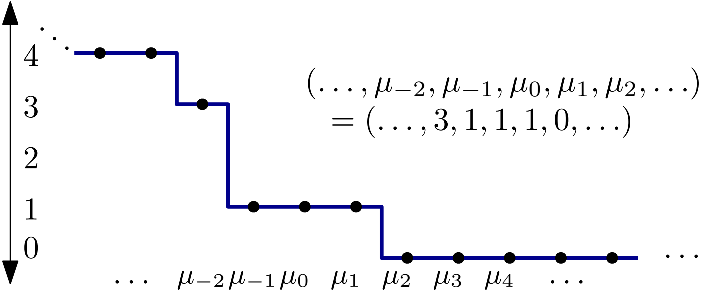
$\mathcal{L}^{(1)}_{p^{-1},p^{-\zeta}/(p-1)}, \mathcal{L}^{(2)}_{p^{-1},p^{-\zeta}/(p-1)}, \ldots$ also give fixed-time row marginals of reflecting Poisson sea, bulk limit of $\mathcal{S}^{(\infty)}$.

The reflecting Poisson sea



$$(\dots, \mu_{-2}, \mu_{-1}, \mu_0, \mu_1, \mu_2, \dots) = (\dots, 3, 1, 1, 1, 0, \dots)$$

Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\dots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \dots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*p^{-i} *exponential clock (independent of each other), donating move if blocked.*

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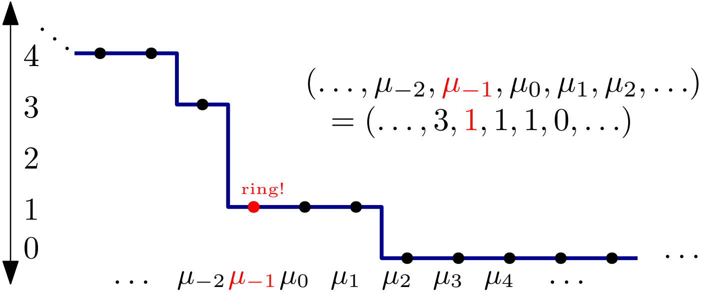

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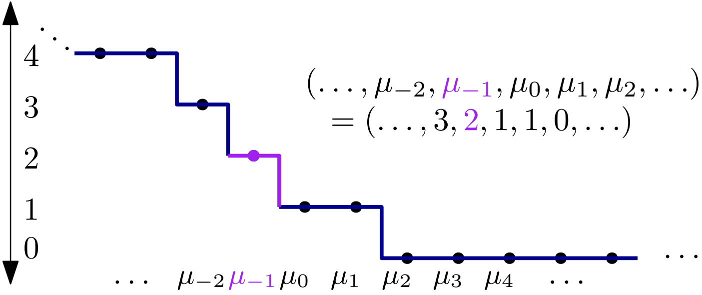

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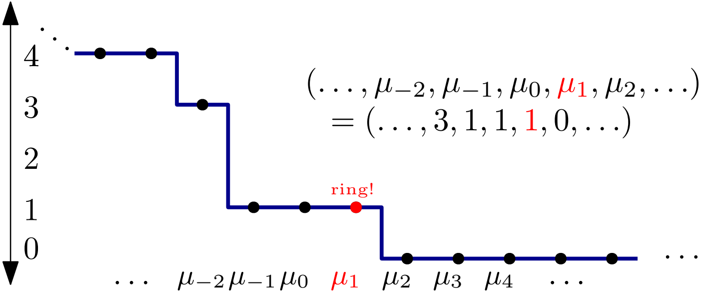

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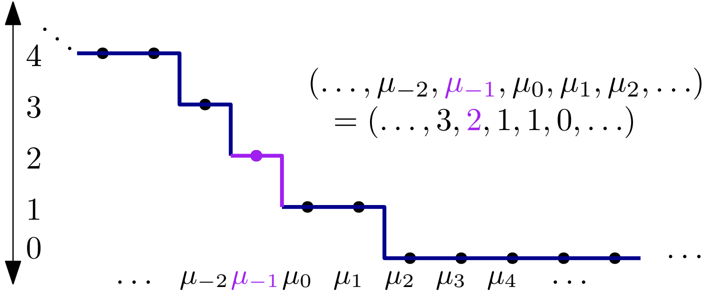

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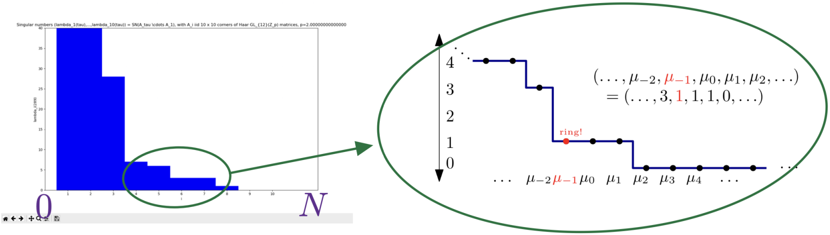
Moral

[Nguyen-VP '24]: Universal convergence of discrete 'time' stochastic process $\mathrm{cok}(A_\tau^{(N)} \cdots A_1)_p, \tau = 0, 1, 2, \ldots$ to $\mu(T)$ in 1-pt (single-time) distribution.



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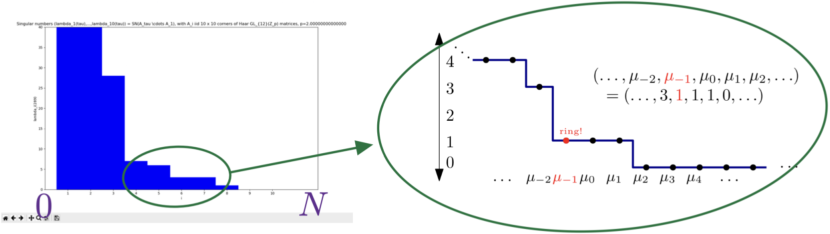

Multi-time convergence? ([VP '23a] does for 'uniform' matrices via Macdonald techniques)

The rescaled moment method

'Traditional' moment method [Wood 2014]: show $G_N \to G$ in law by showing $\mathbb{E}[\#\operatorname{Sur}(G_N, H)] \to \mathbb{E}[\#\operatorname{Sur}(G, H)]$ for all abelian p-groups H.

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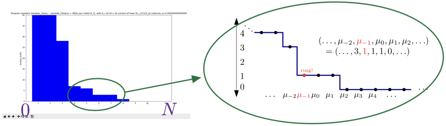
‘Rescaled moment method’ [Nguyen-VP ‘24]: If rescaled moments have nice limits

$$\frac{\mathbb{E}[\#\operatorname{Sur}(G_N, H)]}{C_N} \to C_H$$

for all H, then p-rank fluctuations (rows) converge to discrete random variables. *No integrable input*.

Outlook

p-singular numbers of products
$\rightsquigarrow$ reflecting Poisson sea,
universally (single-time)



	Single-time	Multi-time
Matrices w/ iid entries	[Nguyen-VP ‘24], [VP ‘23a]	?
GL_N-invariant matrices	?	[VP ‘23b]

Outlook

p-singular numbers of products
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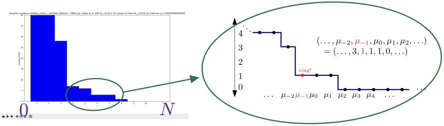

	Single-time	Multi-time
Matrices w/ iid entries	[Nguyen-VP '24], [VP '23a]	?
GL_N-invariant matrices	?	[VP '23b]

Thanks!