# Introduction to Khovanov Homology

Monday, September 13, 2021 3:32 PM

The Kauffman bracket of a link diagram $D$ has axioms

$$\langle \emptyset \rangle = 1 \ ; \ \langle O \sqcup D \rangle = (q+q^{-1}) \langle D \rangle ; \ \langle \times \rangle = \langle \asymp \rangle - q \langle )( \rangle$$

If $D$ has $n_+$ positive crossings, $n_-$ negative crossings, then the Jones polynomial of the underlying link $L$ is

$$J(L) = (-1)^{-n_-} q^{n_+ - 2n_-} \langle D \rangle$$

Knot / 3-manifold Invts, Pre-Khovanov

| Quantum | Gauge Theoretic |
| --- | --- |
| Jones, HOMFLY, WRT... | Instanton Floer Homology Casson Invariant... |

The Casson invt. is twice the Euler char. of Instanton Floer.

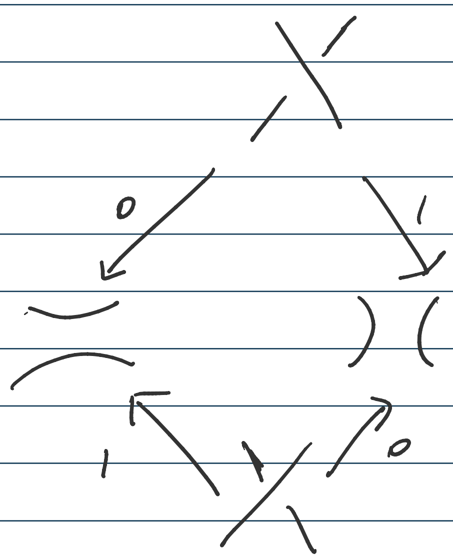
Khovanov homology:

link (diagram) $L \rightsquigarrow$ bigraded $\mathbb{Z}$-module $Kh^{*,*}(L)$

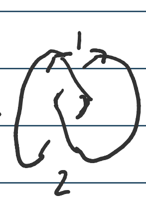
s.t. $J(L) = \sum_{i,j} (-1)^i q^j \operatorname{rk}_{\mathbb{Z}} Kh^{i,j}(L)$.

Jones polynomial from cube of resolutions

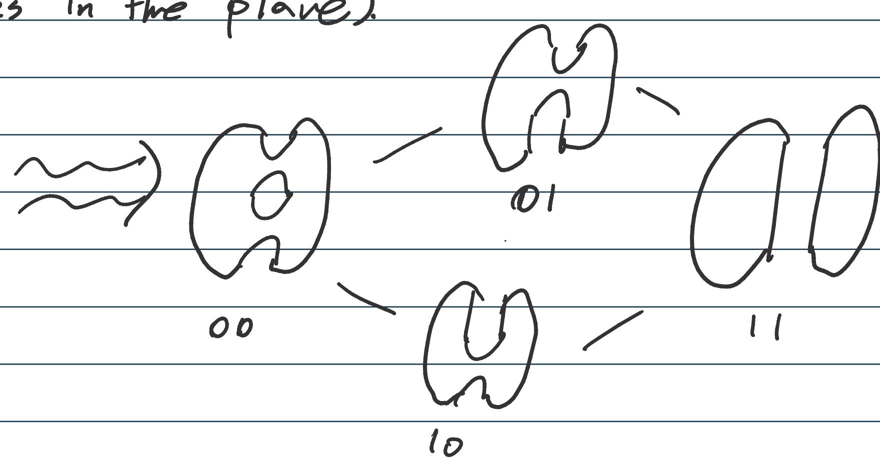
Given $D$, order its crossings. For each crossing we have a 0-res. and a 1-res.




If $n = n_+ + n_-$, every elt. of $\{0,1\}^n$ gets a complete resolution (collection of circles in the plane).

E.g. $H =$ 

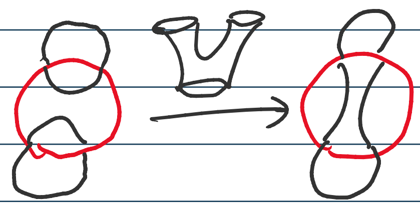

$n_+ = 0$, $n_- = 2$

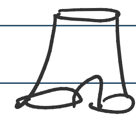


$$\langle H \rangle = (q+q^{-1})^2 - q(q+q^{-1}) - q(q+q^{-1}) + q^2(q+q^{-1})$$
$$= q^4+q^2+1+q^{-2}$$

$$J(H) = (-1)^2 q^{-4} \langle H \rangle = 1+q^{-2}+q^{-4}+q^{-6}$$

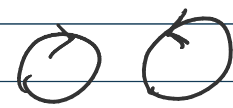
Changing a 0-res. to a 1-res. is given by a saddle.





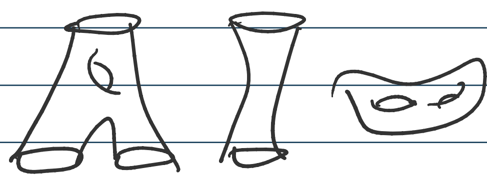
We will interpret the Khovanov cx as a cx in a category of pictures.

Let $\overline{Cob^3}$ be the category with



Objects: Collections of oriented circles in $\mathbb{R}^2$

Morphisms: Oriented cobordisms in $\mathbb{R}^2 \times [0,1]$



For $\Sigma \in Mor(O_1, O_2)$, define $\deg \Sigma = \chi(\Sigma)$.

Add objects $O\{m\}$ for each $O \in Obj(\overline{Cob^3})$ $\rightsquigarrow$ Graded cat. $\underline{Cob^3}$.

$m \in \mathbb{Z}$

Let $Mat(Cob^3)$ denote the additive closure of $Cob^3$

$Obj$ = formal direct sums of objects in $Cob^3$

$Mor$ = matrices of $\mathbb{Z}$-linear combinations of morphisms.

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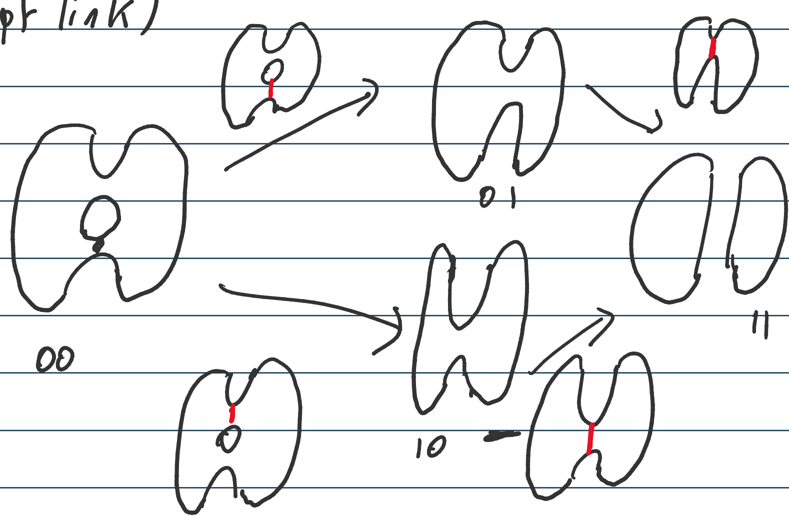
$Kom(Mat(Cob^3))$ = Cat. of chain complexes over $Mat(Cob^3)$.

For $\alpha \in \{0,1\}^n$, assign the object $\Theta\{m\}$, $\Theta$ is the res. of $L$ gotten from the $\alpha_i$-res. at the $i^{th}$ crossing, $m = \sum \alpha_i$.
For each edge, assign the saddle cobordism b/w objects at each edge.

(Hopf link)




$d^2 = 0$ mean that all square faces in the diagram anti-commute

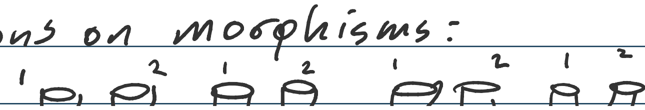
Signs: Add signs to the edges s.t. each square face gets an odd # of minus signs.

lives $Kom(Mat(Cob^3))$

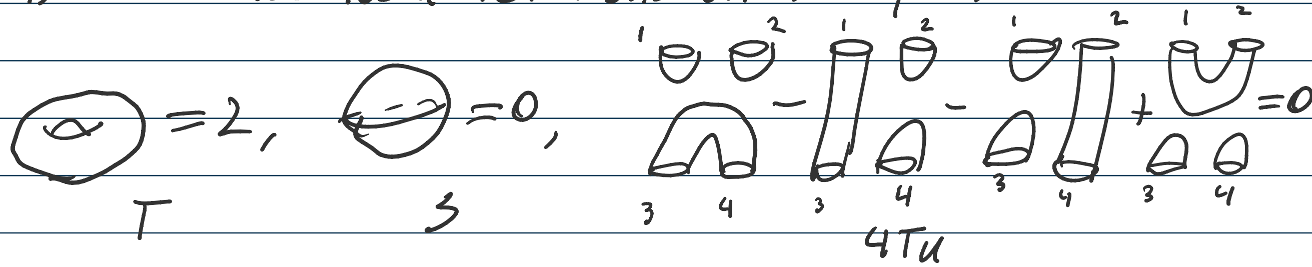
The $m^{th}$ chain gp. $[\![L]\!]^m$ of $[\![L]\!]$ is $\bigoplus_{\alpha \in \{0,1\}^n : \sum \alpha_i = m}$ (resolution over $\alpha$)

The chain htpy type of $[\![L]\!]$ is a link invariant after passing to a quotient of $Cob^3$.

$Cob^3_{/l}$ is $Cob^3$ mod local relations on morphisms:

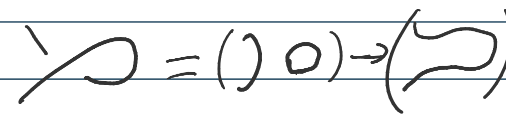


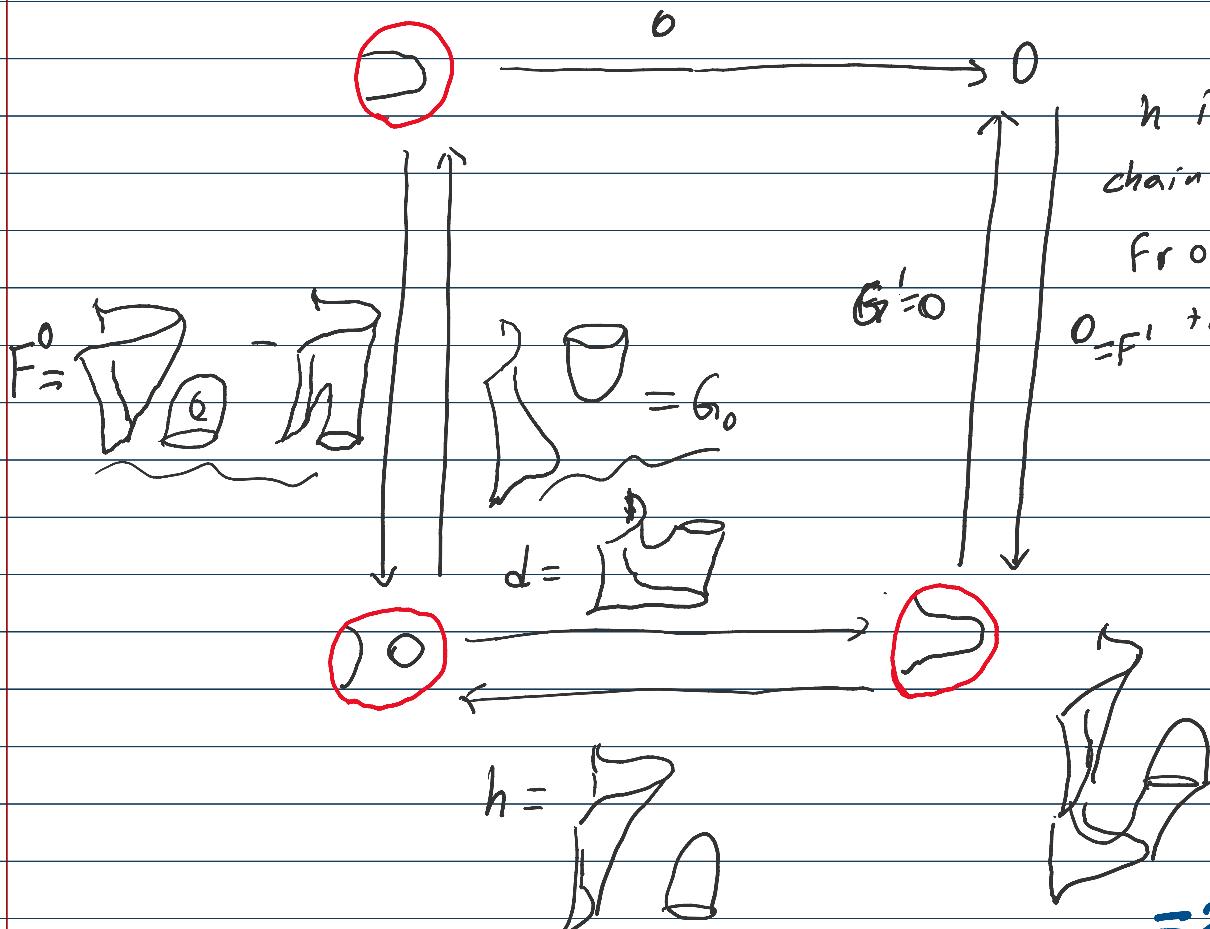
$Cob^3_{/\ell}$ is $Cob^3$ mod local relations on morphisms:



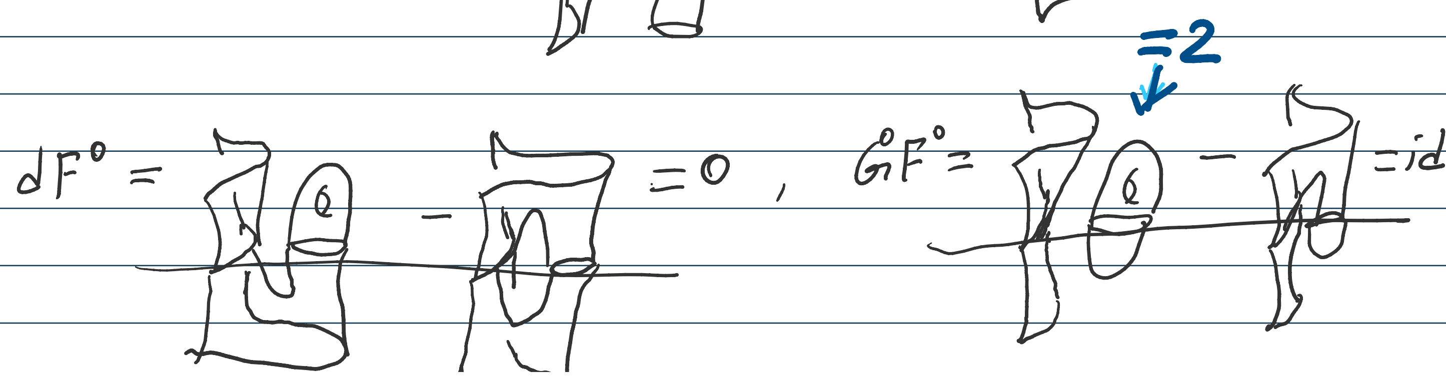

Bar-Natan category $BN := Kom(Mat(Cob^3_{/\ell}))$.

Proof of Reidemeister I invariance:



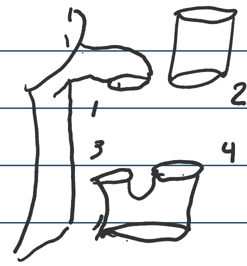

$h$ is the chain htpy from $F \circ G$ to id



Clearly $\underset{110}{\underbrace{F^1G^1}} - id + dh = 0$.

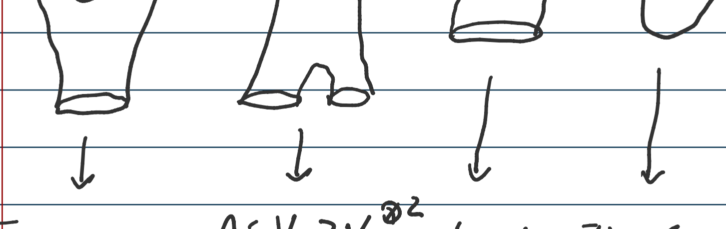
$F^0G^0 - id + hd =$ 

$= 0$ $\qquad$ $= F^0 G$

by 4Tu applied to 


Any functor from $Cob^3_{/\ell} \longrightarrow$ Abelian Category gives a knot invariant.

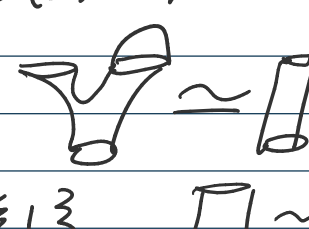
Any cobordism in $Mor(Cob^3_{/\ell})$ can be decomposed into:



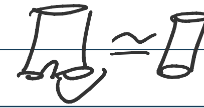
by Morse theory.

(for $\mathcal{F}$ a TQFT) $\mathcal{F}(S^1) = V$

$\mathcal{F}$: $m : V \otimes V \to V$ $\quad \Delta : V \to V^{\otimes 2}$ $\quad \iota : V \to \mathbb{Z}$ $\quad \epsilon : V \to \mathbb{Z}$



$\mathcal{F}(\sqcup_k \bigcirc) = V^{\otimes k}$ ; $V = \mathbb{Z}\{-1\} \oplus \mathbb{Z}\{1\}$ 

$\mathcal{F}(\amalg_k \bigcirc) = V^{\otimes k}$ ; $V = \underset{\langle v_- \rangle}{\mathbb{Z}\{-1\}} \oplus \underset{\langle v_+ \rangle}{\mathbb{Z}\{1\}}$ 

$$m = \begin{cases} v_- \otimes v_- \mapsto 0, & v_- \otimes v_+ \mapsto v_- \\ v_+ \otimes v_+ \mapsto v_+, & v_+ \otimes v_- \mapsto v_- \end{cases} \qquad \iota = \{1 \mapsto v_+$$

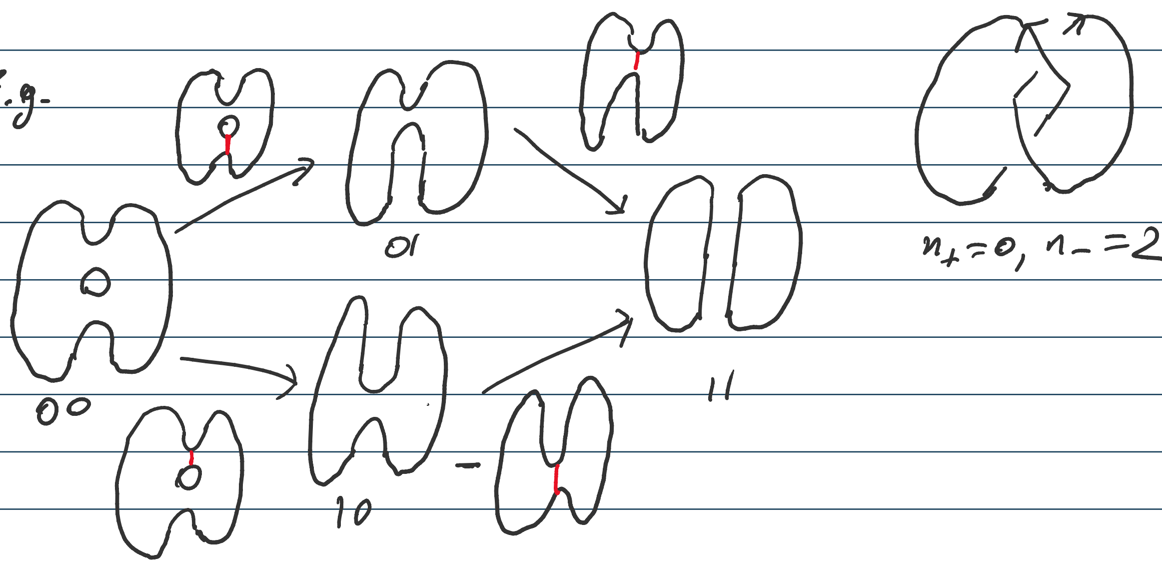
$$\Delta = \begin{cases} v_- \mapsto v_- \otimes v_- \\ v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \end{cases} \qquad \varepsilon : \begin{cases} v_- \mapsto 1 \\ v_+ \mapsto 0 \end{cases}$$

The grading on $V$ is called the Jones grading

Define $Kh(L) = H^*(\mathcal{F}([\![L]\!])[-n_-]\{n_+ - 2n_-\})$

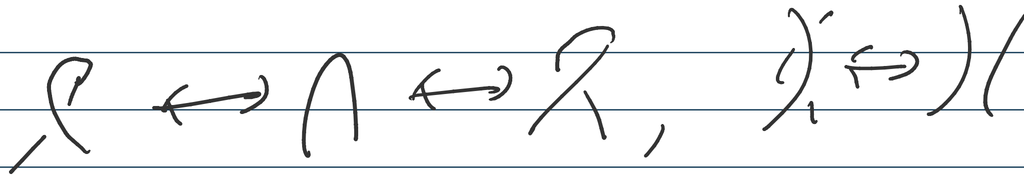
<u>Thm:</u> $J(L) = \sum_{i,j} (-1)^i q^j \, rk_{\mathbb{Z}} Kh^{i,j}(L)$.

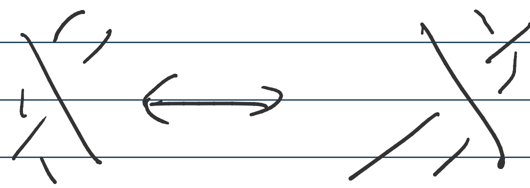
E.g.



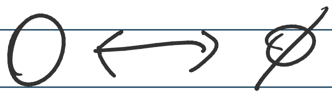

$$(V \otimes V)\{-4\} \xrightarrow{m \oplus m} V\{-3\} \oplus V\{-3\} \xrightarrow{\Delta^1 - \Delta^2} \underline{(V \otimes V)\{-2\}}$$

<u>Functoriality</u>: Cobordisms in $\mathbb{R}^3 \times [0,1]$ decompose into local moves

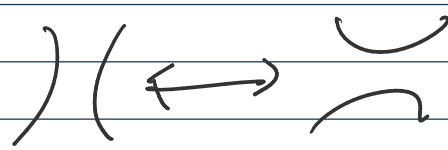




RI, II, III



cup/cap



saddles

<u>Thm</u> (Jacobsson, Bar-Natan) The map on $[\![L]\!]$ associated to a cobordism in $\mathbb{R}^3 \times [0,1]$ only is independent of its decomposition into elementary cobordisms up to a sign and chain homotopy in the Bar-Natan Category.