1. You could have discovered the HOMFLY-PT polynomial (if you know Hecke algebras)

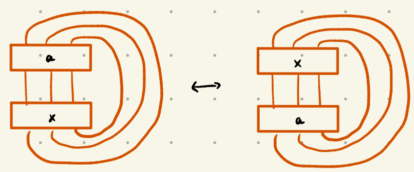
HOMFLY-PT polynomial: oriented link $L \mapsto P(L) \in \mathbb{Q}[t^{\pm 1}, z^{\pm 1}]$ defined by the local relation:

$$\boxed{t \diagup\!\!\!\!\diagdown + t^{-1} \diagup\!\!\!\!\diagdown = z \;)(} \;, \qquad \bigcirc = 1$$

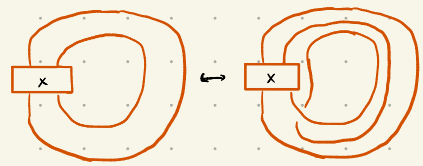
How? Braid group: $Br_n = \langle \underbrace{\times\,|\cdots|}_{s_1}, |\times|\cdots|, \underbrace{|\cdots\times}_{s_{n-1}} \rangle$

Fact 1: every link is the closure of a braid:  = trefoil.

Fact 2: two braids give rise to isotopic links iff they are related by Markov moves:

MI: 

i.e. conjugation by a

MII: 

i.e. x and $xs_n^{\pm 1}$

In rep theory, $\mathbb{Z}[q^{\pm 1}]\, Br_n \twoheadrightarrow H(S_n)$ (quotient by quadratic rel. e.g. $s_i^2 = (q-1)s_i + 1$)

(why $H(S_n)$? $\mathrm{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes n})$)

Idea: define a trace $\mathrm{Tr} : \bigcup_{n \geq 1} H(S_n) \to \mathbb{Z}[q^{\pm 1}, z^{\pm 1}]$ so that $\mathrm{Tr}(xs_n) = z\,\mathrm{Tr}(x)$ for $x, y \in H_n$.

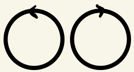
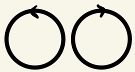
It turns out $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ (just a few cases to check). Normalizing $\mathrm{Tr}(1) = 1$, we get the Ocneanu trace.

Rmk: setting $z = 0$ one recovers the usual trace on $H(S_n)$.

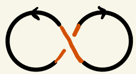
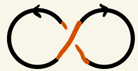
Now what about xs_n^{-1}? In order to satisfy MII, we can rescale the generators so that both moves affect the trace in the same way. This leads to the new basis $s_n' = \frac{\overbrace{1-q+z}^{\lambda}}{qz} s_n$. This yields: for $\alpha \in Br_n$,

$$\left(-\frac{(1-\lambda q)}{\sqrt{\lambda}(1-q)} \right)^{n-1} \mathrm{tr}(\sqrt{\lambda}\, \hat{\alpha})$$ only depends on the link $\hat{\alpha}$. Setting $t = \sqrt{\lambda}\sqrt{q}$, $x = \sqrt{q} - 1/\sqrt{q}$, we get the usual relation.

Essentially: rescale and normalize $\rightsquigarrow$ link invariant

Example: $\langle$  $\rangle = z^{-1} z \, \langle$  $\rangle$

(Skip)

$= z^{-1} t \, \langle$  $\rangle + z^{-1} t^{-1} \, \langle$  $\rangle$

$$= \frac{t + t^{-1}}{z}$$

2. Triply graded homology (via Soergel bimodules)

Khovanov - Rozansky (2004) : KhR from matrix factorizations.
Khovanov (2005): HHH from Soergel bimodules
Elias - Hogancamp (2016) : HHH (torus (n,n) - link)
Mellit (2017): HHH (torus (m,n) - knots)
Hogancamp - Mellit (2019) : HHH (torus (m,n) - link) $\}$ $m,n \geq 0$

Soergel bimodules in type A

$S_m = \langle s_1, \dots, s_{m-1} \rangle$

Example: $S_2 = \{1, s\}$

- $R = \mathbb{Q}[x_1, \dots, x_m]$, $\deg x_i = 2$
- $R = \mathbb{Q}[x_1, x_2]$
- Usual $S_m \curvearrowright R$, $R^{s_i} = \{ r \in R : s(r) = r \}$
- $R^s = \mathbb{Q}[\underbrace{x_1 + x_2}_{r}, \underbrace{(x_1 - x_2)^2}_{t}]$
- $B_{s_i} = R \otimes_{R^{s_i}} R(1)$ R-bimodule
- Bott - Samelson $BS(s_{i_1} \dots s_{i_n}) = B_{s_{i_1}} \otimes \dots \otimes B_{s_{i_n}}$
- $BS(s^2) = B_s^{\otimes 2} = R \otimes_{R^s} R \otimes_R R \otimes_{R^s} R\,(2) = R \otimes_{R^s} R \otimes_{R^s} R(2)$
- Soergel bimodules $:= \otimes, \oplus, (1), \subseteq_{\oplus}$ of Bott - Samelson bimodules.

Soergel's Categorification theorem

$H(S_n)$ has a special basis "KL" : $\{b_w : w \in W\}$

Theorem : $\{$ Indecomposables in SBim $\}$ $\longleftrightarrow$ KL - basis

Multiplication $\longleftrightarrow$ $\otimes$

Example: $b_{s_i}^2 = q b_{s_i} + q^{-1} b_{s_i}$

$$
\begin{aligned}
B_s^{\otimes 2} &= R \otimes_{R^s} R \otimes_{R^s} R\,(2) \\
&= R \otimes_{R^s} (\underbrace{\mathbb{Q}[r, t^2]}_{R^s} \oplus t \underbrace{\mathbb{Q}[r, t^2]}_{R^s}) \otimes_{R^s} R(2) \\
&= R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R \\
&= B_s(1) \oplus B_s(-1)
\end{aligned}
$$

"Categorification of the braid group"

We have $b_s = \delta_s + q$. How do we categorify δ_s? "Subtractions are cones": $B_{s_i} \longrightarrow R$ (only nonzero map up to scalars, shifts)
$f \otimes g \longmapsto fg$

Define the complexes $F_s = \quad 0 \longrightarrow B_{s_i} \xrightarrow{\P_i} R(1)$
$F_{s^{-1}} = \quad R(-1) \xrightarrow{\text{\rotatebox{180}{$\P$}}_i} B_{s_i} \longrightarrow 0$

Define a map $\quad s_{i_1}^{\pm 1} \cdots s_{i_n}^{\pm 1} \mapsto F_{s_{i_1}^{\pm 1}} \otimes \cdots \otimes F_{s_{i_n}^{\pm 1}}$ "Rouquier complexes"

$s_1 s_1 \mapsto \delta_1 \delta_1$

Theorem (Rouquier): this map is a well defined group homomorphism
(braid rels, $F_s F_{s^{-1}}$)

Write $F(\omega) = \cdots \to F^{-1}(\omega) \to F^0(\omega) \to F^1(\omega) \to \cdots$, and let HH_n be the nth Hochschild homology functor.
Apply HH_* to $F(\omega)$. The cohomology of this complex is the triply graded homology $HHH(\sigma)$.

Theorem (Khovanov): • $HHH(\omega)$ is an invariant of oriented links (up to an overall grading shift), as it is $\cong$ KhR.
• Its Euler characteristic, after some normalization + change of variables, is the HOMFLYPT.

The three gradings:
- Homological grading in the complex (cohom at $HH_*(F^i(\sigma))$)
- Hochschild grading (HH_n)
- Internal grading of each bimodule ($HH_n(F^i(\sigma))$ is a graded vector space)

Remarks: • Functorial but not functorial

• Story is analogous with HH^*, because there is a (graded) Poincaré duality $HH_n \leftrightarrow HH^{m-n}$

• There is a colored HOMFLY homology. Gorsky, Gukov, Stošić (2013) conjecture a fourth grading on it.
Also a y-ified version.

3. Hochschild homology is easier for polynomial rings.

Hochschild homology: R k-algebra, $R^{env} = R \otimes_k R^{op}$. Then $HH_n(-) = \mathrm{Tor}_n^{R^{env}}(R, -)$.

(Usually bar res., bad idea)

Resolution for R? If R is a polynomial ring then Koszul complex works: if $V = \mathrm{Span}_k(x_1, \dots, x_m)$, then

$K^\bullet = \Lambda^m V \otimes R^{env} \to \Lambda^{m-1} V \otimes R^{env} \to \dots \to \Lambda^2 V \otimes R^{env} \to V \otimes R^{env} \to R^{env} \longrightarrow R \to 0$ is a free resolution for R.

So $HH_n(M)$ = nth homology group of $K^\bullet \otimes M$.

<u>Example</u>: $R = \mathbb{Q}[x] = M$, graded: $\deg x = 2$.

$$K^\bullet = R^{env}(-2) \xrightarrow{x\otimes 1 - 1\otimes x} R^{env} \quad \text{(explain sign)} \qquad \Rightarrow \qquad K^\bullet \otimes R = R(-2) \xrightarrow{0} R$$

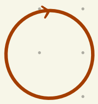
$$\text{So } HH_*(R) = \begin{cases} R & * = 0 \\ R(-2), & * = 1 \end{cases}$$

<u>Example</u>: $R = \mathbb{Q}[x,y]$ $\deg x = \deg y = 2$. M graded R-module

$$K^\bullet = R^{env}(-4) \xrightarrow{\begin{pmatrix} x\otimes 1 - 1\otimes x \\ 1\otimes y - y\otimes 1 \end{pmatrix}} \begin{matrix} R^{env}(-2) \\ \oplus \\ R^{env}(-2) \end{matrix} \xrightarrow{(y\otimes 1 - 1\otimes y,\; x\otimes 1 - 1\otimes x)} R^{env}$$

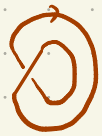
$$K^\bullet \otimes M = M(-4) \xrightarrow{\begin{pmatrix} x\cdot() - ()\cdot x \\ ()\cdot y - y\cdot () \end{pmatrix}} \begin{matrix} M(-2) \\ \oplus \\ M(-2) \end{matrix} \xrightarrow{(y\cdot() - ()\cdot y,\; x\cdot() - ()\cdot x)} M$$

4. Triple example of triply graded homology

 is the closure of $\mathrm{id} \in Br_1$, hence we want $HHH_{***}(F^\bullet(\mathrm{id}))$. Here $F^\bullet(\mathrm{id}) = 0 \to \underline{R} \to 0$, where $R = \mathbb{Q}[x]$

The only nonzero term in the complex occurs at 0, and $HHH_{0**}(F) = HH_*(R) = \begin{cases} \mathbb{Q}[x] \\ \mathbb{Q}[x](-2) \end{cases}$ Hochschild deg: 0, 1

The Euler char is $(1+q^2+q^4+\dots) + a(q^2+q^4+q^6+\dots) = \dfrac{1+aq^2}{1-q^2}$. $\rightsquigarrow$ HOMFLY
change of vars, normalization

 is the closure of $s_1 \in Br_2$. $F^\bullet(s_1) = 0 \to \underline{B_s} \xrightarrow{\varphi} R(1) \to 0$, where $R = \mathbb{Q}[x,y]$. Here $\varphi(f \otimes g) = fg$

We want to compute the cohomology of the complex $HH_*(B_s) \to HH_*(R)$

The Koszul complex for R is simply $R(-4) \xrightarrow{0} R(-2) \oplus R(-2) \xrightarrow{0} \underline{R}$ (scroll up)

Let $r = x+y$, $t = x-y$. Then $R = \mathbb{Q}[r,t]$, $B_s = R \otimes_{\mathbb{Q}[r,t^2]} R(1)$ and the Koszul complex for B_s is:

$$B_s(-4) \xrightarrow{\binom{f}{f}} \begin{matrix} B_s(-2) \\ \oplus \\ B_s(-2) \end{matrix} \xrightarrow{(-f,\, f)} \underline{B_s} \qquad \text{where } f(m) := \tfrac{1}{2}mt - \tfrac{1}{2}tm$$

Easy check: $\mathrm{Ker}(f) = \{ mt + tm : m \in B_s \}$
$\mathrm{Im}(f) = \{ mt - tm : m \in B_s \}$

The induced maps from $B_s \xrightarrow{\varphi} R(1)$ are:

cohom deg: 1 $\qquad$ 0

Thus $HH_0(B_s) = B_s / \mathrm{Im}(f) \cong R(1) \xrightarrow{\mathrm{id}} R(1) = HH_0(R)$

$HH_1(B_s) = \{ (m,n) \in B_s(-2)^{\oplus 2} : m-n \in \mathrm{Ker}(f) \} / \{(tm-mt,\, tm-mt)\} \cong R(-1) \oplus \mathrm{Ker}(f)(-2) \xrightarrow{(\mathrm{id},\varphi)} R(-1)^{\oplus 2} = HH_1(R)$

$HH_2(B_s) = \mathrm{Ker}(f)(-4) \xrightarrow{\varphi} R(-3) = HH_2(R)$

Therefore, $HHH_{0**}(s_1)$ $\qquad$ $HHH_{1**}(s_1)$

Hochschild deg:	$HHH_{0**}(s_1)$	$HHH_{1**}(s_1)$
0	0	0
1	0	$\mathbb{Q}[r](-1)$
2	0	$\mathbb{Q}[r](-3)$

$\cong$ $HHH(1)$ in previous example after shift

 is the closure of $s_1^2 \in Br_2$. Here $F^\bullet(s_1^2) = (\underline{B_s} \xrightarrow{\varphi} R(1))^{\otimes 2} =$

$$\begin{tikzcd} & B_s(1) \arrow[dr, "\varphi"] & \\ B_s^{\otimes 2} \arrow[ur, "1|\varphi"] \arrow[dr, "\varphi|1"'] & \oplus & R(2) \\ & B_s(1) \arrow[ur, "-\varphi"'] & \end{tikzcd}$$

Rmk: can replace $R = \mathbb{Q}[r,t]$ by $S = \mathbb{Q}[t]$, so B_s becomes $C_s = \mathbb{Q}[t] \otimes_{\mathbb{Q}[t^2]} \mathbb{Q}[t](1)$, etc.

So get
$$\begin{tikzcd} & C_s(1) \arrow[dr, "\varphi"] & \\ C_s^{\otimes 2} \arrow[ur, "1|\varphi"] \arrow[dr, "\varphi|1"'] & & \mathbb{Q}[t](2) \\ & C_s(1) \arrow[ur, "-\varphi"'] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} & C_s(1) \arrow[dr, "\varphi"] & \\ \begin{matrix} C_s(-1) \\ \oplus \\ C_s(1) \end{matrix} \arrow[ur, "\binom{1\,t}{1}"] \arrow[dr, "\binom{t}{1}"'] & & \mathbb{Q}[t](2) \\ & C_s(1) \arrow[ur, "-\varphi"'] & \end{tikzcd}$$

The point: Gaussian elimination can simplify the complex

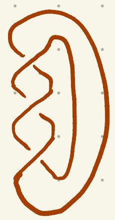
Gaussian Elimination: $\simeq \underline{C_s(-1)} \xrightarrow{(|t-t|)} C_s(1) \xrightarrow{1} S(2)$

Taking HH_*:

$$\begin{array}{llllll} HH_0 & \underline{S} & \xrightarrow{0} & S(2) & \xrightarrow{id} & S(2) \\ HH_1 & \underline{S(-4)} & \xrightarrow{0} & S(-2) & \xrightarrow{2t} & S \end{array}$$

	HHH_{0**}	HHH_{1**}	HHH_{2**}
Hoch deg 0	$\mathbb{Q}[r,t]$	0	0
1	$\mathbb{Q}[r,t](-4)$	0	$\mathbb{Q}[r]$

For good measure, here is the trefoil:



$$F^\bullet(s_1^3) = F^\bullet(s_1) \otimes F^\bullet(s_1^2)$$

$$= \underline{B_s^2(-1)} \longrightarrow \begin{array}{c} B_s^2(1) \\ \oplus \\ B_s \end{array} \longrightarrow \begin{array}{c} B_s(2) \\ \oplus \\ B_s(2) \end{array} \longrightarrow R(3)$$

Some Gaussian elimination later...

$$\simeq \underline{B_s(-2)} \xrightarrow{0} B_s(0) \xrightarrow{|t-t|} B_s(2) \xrightarrow{1} R(3)$$

After taking HH_0, HH_1:

	HHH_{0**}	HHH_{1**}	HHH_{2**}	HHH_{3**}
Hoch deg 0	0	0	0	0
1	0	$\mathbb{Q}[r](-3)$	0	$\mathbb{Q}[r](1)$

Remark: $m=2$ is easy (this pattern continues). There exist implementations; performance depends on choosing minimal complexes cleverly. Seems to work up to 7 crossings.

5. Knot homology meets geometry (future talks?)

Braid varieties

Let $B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & z & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$

Then an easy computation shows $B_i(z_1) B_{i+1}(z_2) B_i(z_3) = B_{i+1}(z_3) B_i(z_2 - z_1 z_3) B_{i+1}(z_1)$

Given a positive braid $\beta = s_{i_1} \cdots s_{i_r}$, $B_\beta(z_1, \ldots, z_r) = B_{i_1}(z_1) \cdots B_{i_r}(z_r)$ "braid matrix", indep of braid relation (up to change of variables)

The braid variety is $\{ (z_1, \ldots, z_r) : B_\beta(z_1, \ldots, z_r) \text{ is upper triangular} \}$, indep of braid word.

Example: $X(\text{trefoil}) = X(s^3) = \{ 1 + z_1 z_2 = 0 \} = \mathbb{C}^\times_{z_1} \times \mathbb{C}_{z_3}$

There is a torus action $(\mathbb{C}^\times)^{n-1} \curvearrowright X(\beta)$, $n = \#$ strands. Example: $(z_1, z_2, z_3) \mapsto (z_1, t z_2, t^{-1} z_3)$. For knots the action is free.

<u>Theorem</u> (Trinh, 2021) $H^T_{BM,*}(X(\beta))$ has a nontrivial weight filtration. Its associated graded is $\simeq HHH_{0,*,*}(\beta)$

Example: $H^T_{BM,*}(X(\text{trefoil})) \cong H_*(S^1)$ (has a grading on it)

Remark: Trinh has a version of this for all a-degrees using Springer theory.

Hilbert schemes

- Oblomkov – Shende Conjecture (2012). (Roughly) C integral (plane) curve, p singularity, $C_p^{[l]}$ Hilbert scheme of points, length l, supported at p.

  Then $\mathrm{HOMFLY}(\mathrm{link}(p)) = \left(\frac{a}{q}\right)^{\mu-1} \sum_{l,m} q^{2l}(-a^2)^m \chi(C_p^{[l,l+m]})$.

  Milnor number

- Oblomkov – Rasmussen – Shende Conjecture (2018): replace χ by cohomology of $C_p^{[l,l+m]}$, which has a filtration, giving a new parameter. This agrees with HHH. Concrete example: $HHH_n(\omega) = \bigoplus_{k=0}^{\infty} H^*(\mathrm{Hilb}^{[k]}(C_p))$

  Verified for torus knots by Mellit, using Elias – Hogancamp.

  skip | Crucial fact: homology is supported in even homological degrees $\Rightarrow$ A certain spectral sequence collapsing to E^2 allows the computation

  Open question: verify for algebraic links with positivity conditions.

- Oblomkov – Rozansky homology (2018): Define:

  $V = \mathbb{C}^n \Rightarrow \mathrm{Hilb}_{1,n}^{free} = \{(X,Y,v) \in \mathfrak{b}_n \times \mathfrak{n}_n \times V : \mathbb{C}\langle X,Y\rangle v = V\}/B$, $\quad \circlearrowleft GL_n$

  "non-comm analog of the nested Hilbert scheme"

  skip probably

  Action $\mathbb{C}^\times \times \mathbb{C}^\times \circlearrowright \mathrm{Hilb}_{1,n}^{free}$. let $\mathcal{B}$ trivial bundle over $\mathrm{Hilb}_{1,n}^{free}$

  Construct $\mathcal{S}_\beta \in K^{2\text{-per}}(\mathrm{QCoh}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}_{1,n}^{free}))$, $\mathbb{H}^k(\mathcal{S}_\beta) = \mathbb{H}(\mathcal{S}_\beta \otimes \Lambda^k \mathcal{B})$

  Theorem: this categorifies HOMFLY

  <u>Conjecture</u>: this is the same as HHH.

- Gorsky – Negut – Rasmussen (2016)

  dg version of the Flag Hilbert scheme: $X = \mathrm{FHilb}_n^{dg}(\mathbb{A}^2_{\mathbb{C}})$

  <u>Conjecture</u>: Monoidal functor

$$\begin{tikzcd} K^b(\mathrm{SBim}_n) \arrow[r, "L_*"] \arrow[d, "HHH"'] \arrow[dr, phantom, "\circlearrowright"] & D^b(\mathrm{Coh}_{\mathbb{C}^\times \times \mathbb{C}^\times}(X)) \arrow[dl] \\ \mathbb{Z}^{\oplus 3}\text{-graded } \mathbb{Q}\text{-vect} & \end{tikzcd}$$

6. Questions?

$H^*(\mathbb{C}^*)$

$\uparrow$

$\mathbb{C}[x,y^{\pm 1}]$

$\mathbb{C}[x, \frac{1}{x}]$

$$\operatorname{Spec}\left(\underline{\mathbb{C}[x,y] / (xy-1)} \right)$$

$$0 \to (xy-1)\,\mathcal{O}_{\mathbb{A}^2} \longrightarrow \mathcal{O}_{\mathbb{A}^2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$\begin{array}{ccc}
(xy-1) & \mathbb{C}[x,y] & \\
\shortparallel & \shortparallel & \\
H^0(\mathcal{I}) & H^0(\mathbb{A}^2) & H^0(\mathcal{O}_X) \\
 & & \\
H^1(\mathcal{I}) & H^1(\mathbb{A}^2) & H^1(\mathcal{O}_X) \\
 & & \\
H^2(\mathcal{I}) & H^2(\mathbb{A}^2) & H^2(\mathcal{O}_X)
\end{array}$$