The Toda lattice and its ultradiscretization

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0.1 Intro

This talk will introduce briefly classical integrable systems and then focus on the example of the Toda lattice, together with its solitonic and periodic solutions. Then, following [IKT12, ch.4 and ch. 6] the ultradiscretization of this classical system and its relation with the box-ball system (BBS) cellular automaton will be discussed, with solitonic and periodic solutions.

0.2 Integrable systems

A classical mechanical system is a state space M (possibly infinite dimensional, but let's assume $2n$) that is a symplectic manifold (equivalently, Poisson manifold with bracket $\{-,-\}$) together with a Hamiltonian function $H : M \to \mathbb{R}$. The Hamiltonian generates time flow via equation (on coordinates p_i, q_i) given by

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} && = \{q_i, H\} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} && = \{p_i, H\} \end{aligned}$$

and H is preserved on orbits of this flow.

The system is called integrable if there are $n-1$ *other* functions $H_2, \dots, H_n$ ($H_1 = H$) such that $\{H_i, H_j\} = 0$, from which it follows that

1. The flows generated by the H_i commute
2. The orbits generated by the flows through any point are Lagrangian submanifolds.

0.3 Toda lattice

The Toda lattice is a particular example of an integrable system, which is a model of a waves in a discrete chain of particles with nearest neighbor interaction. There are innumerable variations of the model with various boundary conditions or limits but we will primarily focus on the case of N particles with displacements from their frozen positions $u_i(t)$ and momenta $p_i(t) = \dot{u_i}$ for $t \in \mathbb{R}$ such that the Hamiltonian has the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{j=1}^{n-1} e^{2(u_j(t)-u_{j+1}(t))}$$

up to an additive constant (which is needed for infinite lattice size).

The flow generated by this is the equation

$$\frac{d^2u_j}{dt^2} = e^{u_{j+1}-uj} - e^{u_j-u_{j-1}}, \qquad i = 1,\dots n \tag{0.1}$$

2d Toda The 1d Toda lattice hierarchy (like the KP hierarchy, KdV, ...) can be seen as a special case of the 2d Toda lattice defined in [UT84], where each u_i depends on 2 independent variables. There are many approaches to the 1d Toda equation but historically the solitons relating to the box-ball system were actually written in [TNS99] down using the reduction from the 2d Toda lattice τ-functions to 1d Toda solutions in [HIK88, Appendix B] so this is the approach we will take.

The 2d Toda lattice of length n with coordinates $u_i(x, y), i = 1, \dots, n$ is given by the equation

$$\frac{d^2u_j}{dxdy} = e^{u_{j+1}-uj} - e^{u_j-u_{j-1}}, \qquad i = 1,\dots n \tag{0.2}$$

0.3.1 τ-function

This section basically follows [Oko00, Appendix A].

Infinite wedge, free fermions, boson-fermion correspondence Let V be a vector space with basis $\{..., \underline{-\frac{1}{2}}, \underline{\frac{1}{2}}, \underline{\frac{3}{2}}, \dots\}$. Let $\Lambda^{\frac{\infty}{2}}V$ be the half-infinite wedge space spanned by

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \cdots$$

with $s_i > s_{i+1}$ for all i where the set $\{s_i\}$ differs from $\mathbb{Z}_{\leq 0} - \frac{1}{2}$ by only a finite set. The natural inner product on $\Lambda^{\frac{\infty}{2}}V$ has $\{v_S\}$ as an orthonormal basis.

Define operators

$$\psi_k(-) = \underline{k} \wedge -, \qquad k \in \mathbb{Z} + \frac{1}{2}$$

with adjoint $\psi_k^*(-) = \underline{k} \neg -$. These satisfy Clifford relations $\{\psi_k, \psi_j^*\} = \delta_{jk}$.

Combine these into series

$$\psi(z) = \sum_{i \in \mathbb{Z}} z^{i+\frac{1}{2}} \psi_i, \qquad \psi^*(w) = \sum_{j \in \mathbb{Z}} w^{-j-\frac{1}{2}} \psi_j^*.$$

Let $H = \sum_k k : \psi_k \psi_k^* :$ and $C = \sum_k : \psi_k \psi_k^* :$ where $: \psi_k \psi_k^* :$ first does the operation which annihilates the vacuum

$$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \ldots$$

We have a direct sum decomposition into eigenspaces of C $\Lambda^{\frac{\infty}{2}} V = \bigoplus_k R^k \Lambda_0$ where R shifts $\underline{k}$ to $\underline{k+1}$ and Λ_0 consists of vectors disagreeing from $v_\emptyset$ in only finitely many places. A basis of Λ_0 is indexed by partitions $\lambda = [\lambda_1, \ldots, \lambda_k]$ so that $v_\lambda = \underline{\lambda_1 - 1 + \frac{1}{2}} \wedge \underline{\lambda_2 - 2 + \frac{1}{2}} \wedge \cdots$.

The bosonic operators

$$\alpha_n = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n} \psi_k^*, \qquad n \in \mathbb{Z}$$

generate a Heisenberg algebra, with $\alpha_n^* = \alpha_{-n}$. Now given any sequence $s = (s_1, s_2, \ldots)$ (usually an infinite number of variables) define

$$\Gamma_\pm = \exp\left(\sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right).$$

When $s = \{z\} := (z, z^2/2, z^3/3, \ldots)$ these are related to vertex operators, and in particular we can invert the boson-fermion correpsondence using the formulas

$$\psi(z) = z^C R \Gamma_-(\{z\}) \Gamma_+(-\{1/z\}) \tag{0.3}$$

$$\psi^*(z) = R^{-1} z^{-C} \Gamma_-(-\{z\}) \Gamma_+(\{1/z\}). \tag{0.4}$$

Plücker relations See [Miw+00] for an excellent exposition of this material. $\mathrm{GL}(V)$ acts on $\Lambda^{\infty/2} V$, where we omit the definition of $\mathrm{GL}(V)$. It's image is cut out in $\mathrm{Aut}(\Lambda^{\infty/2} V)$ by the infinite-Grassmanian Plücker relations

$$[A \otimes A, \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_k \otimes \psi_k^*] = 0. \tag{0.5}$$

We can now simultaneously construct the hierarchy of differential equations extending the 2d Toda Lattice equation and also its solution. Let $x_1, x_2, \ldots$ be a sequence of variables extending $x = x_1$ and let $y_1, y_2, \ldots$ be a second with $y = y_1$. These x_i, y_i for $i > 1$ will be the 2d analogue of the"times" for the flows associated with the higher conserved energy functions, i.e. the coordinates along the Lagrangians mentioned above.

Then define the τ-function to be the matrix coefficient

$$\tau_n(x, y; A) := \langle v_n | \Gamma_+(x) A \Gamma_-(y) | v_n \rangle. \tag{0.6}$$

Then any other matrix coefficient of $\widetilde{A} = \Gamma_+(x) A \Gamma_-(y) \in \mathrm{GL}(V)$ can be obtained from the partial derivatives of the $\tau_n(x, y; A)$. The Plücker relations (0.5) then give an infinite number of PDEs satisfied by the $\tau_n(x, y; A)$.

Specifically these include

$$\frac{\partial^2}{\partial x \partial y}(\log \tau_n(x, y)) = \frac{\tau_{n+1}(x, y)\tau_{n-1}(x, y)}{\tau_n(x, y)^2}.$$

Writing

$$u_n(x, y) = \log \frac{\tau_{n+1}}{\tau_n}$$

recovers

$$\begin{aligned} \frac{d^2 u_j}{dxdy} &= \frac{\tau_{n+2}\tau_n}{\tau_{n+1}^2} - \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \\ &= e^{u_{j+1} - uj} - e^{u_j - u_{j-1}}. \end{aligned}$$

Picking solutions only depending on $t = x + y$ recovers to 1d Toda equation.

Periodic solutions Associated to the spectrum of the Lax operator (in the Lax formalism for the integrability and solutions of the 1d Toda equation) there is a hyperelliptic curve

$$w^2 = (z - E_0) \cdots (z - E_{2g+1}).$$

Periodic solutions of the 1d Toda equation can be expressed in terms of this curve as follows: Let $\mathbf{z}_0 \in \mathbb{R}^g$, $\theta : \mathbb{R}^g \to \mathbb{R}$ the Riemann theta function associated to this curve and let $a, b \in \mathbb{R}, \mathbf{A}, \mathbf{c} \in \mathbb{R}^g$ are constants depends on the curve only. Then [Bul+98] there are periodic solutions of the form

$$u_j(t) = u_0 - 2(tb + j\log(2a)) - \log \frac{\theta(\mathbf{z}_0 - 2j\mathbf{A} - 2t\mathbf{c})}{\theta(\mathbf{z}0 - 2(j-1)\mathbf{A} - 2t\mathbf{c})}.$$

0.4 Tropicalization and ultradiscretization

Discrete 1d Toda We convert the 1d Toda lattice equation from a differential equation to a difference equation.

The resulting equation is

$$q_j^{t+1} = q_j^t + w_j^t - w_{j-1}^{t+1} \tag{0.7}$$

$$w_j^{t+1} = \frac{q_{j+1}^t w_j^t}{q_j^{t+1}}. \tag{0.8}$$

and it's easiest to see this gives the Toda lattice by going the other way: let

$$w_j^t = \delta^2 e^{u_{j+1}(\delta t) - u_j(\delta t)}, \qquad q_j^t = 1 + \delta \dot{u}_j(\delta t).$$

For example if $T = n_0 \delta$, this becomes

$$\delta \dot{u}_j(T + \delta) = \delta \dot{u}_j(T) + \delta^2 e^{u_{j+1}(T) - u_j(T)}$$

which tends to the Toda lattice equation in the limit $\delta \to 0$.

Molecule boundary condition Consider the boundary conditions on a length $N+1$ Toda lattice given by $u_0(t) = +\infty$ and $u_N(t) = -\infty$, which we interpret as boundary conditions for a length $N-1$ Toda lattice for the middle coordinates.

This reduces in the discretization to the conditions $w_0^t = w_{N+1}^t = 0$.

Bilinear expression The τ-function also descends to a discrete bilinear equation which encodes solutions to these equations, namely if τ_n^t are sequences satisfying the bilinear equation

$$\tau_j^{t+1} \tau_j^{t-1} = (\tau_j^t)^2 + \tau_{j+1}^{t-1} \tau_{j-1}^{t+1}$$

and the boundary conditions for the molecule equation

$$\tau_{-1}^* = \tau_{N+1}^* = 0$$

then

$$q_j^t = \frac{\tau_{j-1}^t \tau_j^{t+1}}{\tau_j^t \tau_{j-1}^{t+1}}, \qquad w_j^t = \frac{\tau_{j+1}^t \tau_{j-1}^{t+1}}{\tau_j^t \tau_j^{t+1}}$$

solve the discrete Toda molecule equation. Solutions to this can be written as Casoratian (= difference Wronskian) determinants based on initial data.

0.4.1 Tropicalization

The tropicalization process is a certain limit of algebraic expressions sending things like polynomials to piecewise linear equations involving addition and the $\min(-,-)$ function.

Specifically we can reduce subtraction-free rational functions based on the algebra $(\mathbb{R}_{>0}, +, \times)$ in some variables $a, b, c, \dots$ to expressions in the min-plus algebra $(\mathbb{R}\cup\{\infty\}, \oplus, \odot)$ in the same number of variables $A, B, C, \dots$. This is accomplished by the substitution

$$\mathrm{Log}_\epsilon : a \mapsto A := -\epsilon \log a$$

followed by the limit $\epsilon \to 0$. Thus

$$\mathrm{Log}_\epsilon(a + b) = \min(A, B), \qquad \mathrm{Log}_\epsilon(a \times b) = A + B.$$

In practice we apply the above rules to reduce subtraction-free rational functions to min-plus expressions. E.g.

$$\prod_{i=1}^{n} \frac{1 + q_i}{1 + a_i} \mapsto \sum_{i=1}^{n} \min(0, Q_i) - \min(0, A_i).$$

Tropicalization of Toda We now consider the tropicalization of discrete Toda (0.7) - (0.8). Because (0.7) contains a subtraction we need to rewrite.

Write (0.7) as

$$\begin{aligned} q_j^{t+1} - w_j^t &= q_j^t - w_{j-1}^{t+1} \\ &= q_j^t - \frac{q_j^t w_{j-1}^t}{q_{j-1}^{t+1}} \\ &= \frac{q_j^t}{q_{j-1}^{t+1}}(q_{j-1}^{t+1} - w_{j-1}^t) \\ &= \prod_{k=2}^{j} \frac{q_k^t}{q_{k-1}^{t+1}}(q_1^t + 0). \end{aligned}$$

Then our equations become

$$\begin{aligned} q_j^{t+1} &= \frac{\prod_{k=1}^{j} q_k^t}{\prod_{k=1}^{j-1} q_k^{t+1}} + w_j^t \\ w_j^{t+1} &= \frac{q_{j+1}^t w_{j-1}^{t+1}}{q_j^{t+1}} \end{aligned}$$

which are subtraction-free. We can thus write down the tropicalization of these equations, with Q_j^t associated to q_j^t and likewise for W_j^t. They are the equations

$$Q_j^{t+1} = \min(W_j^t, \sum_{k=1}^{j} Q_k^t - \sum_{k=1}^{j-1} Q_k^{t+1}), \qquad j = 1, \dots, N \tag{0.9}$$

$$W_j^{t+1} = Q_{j+1}^t + W_{j-1}^{t+1} - Q_j^{t+1}, \qquad j = 1, \dots, N-1. \tag{0.10}$$

Because these equations are defined over $\mathbb{Z} \cup \{\infty\}$ while a priori they are expected to be only defined over $\mathbb{R} \cup \{\infty\}$, these equations are referred to as the *ultradiscretization* of the Toda molecule equation rather than just their tropicalization.

0.5 Relation to BBS

Now consider the $\widehat{\mathfrak{sl}_2}$ BBS with N solitons. Define quantities

- Q_j^t = the number of balls in the jth soliton at time t
- W_j^t = the number of empty boxes between solitons j and $j+1$.

Then it is proven in [TNS99] that the equations describing the evolution of these quantities is exactly equations (0.9) and (0.10) of the ultradiscrete Toda molecule equation. As a brief remark, the ultradiscretization of the discreke Lotka–Volterra equations also has a description in terms of the BBS.

0.6 Tropical geometry, trop-pToda, periodic BBS

The reference for this section is [IKT12, Ch. 6] and the papers cited therein but we will very breifly describe periodic solutions to a tropical version of the Toda lattice using the tropical geometry of the spectral curve.

Just as we can reduce some rational functions to min-plus expressions we can sometimes reduce an *algebraic variety* defined as the zero loci of algebraic expressions $X = V(I)$ to a *tropical variety* denoted $\mathrm{Trop}(X)$ which is the locus where $\mathrm{Trop}(f)$ is indifferentiable for an expression $\mathrm{Trop}(f)(A, B, C, \ldots)$ in the tropical semiring.

Very briefly, the N-periodic discrete Toda lattice has and ultradiscretization in variables Q_j^t, W_j^t for $j \in \mathbb{Z}/N\mathbb{Z}$, such that subject to the condition on the geometric-mean density of solitons $\prod_{j=1}^N w_j^t/q_j^t < 1$ (which is preserved under time evolution) the time evolution has the form

$$\begin{aligned} Q_j^{t+1} &= \min(W_j^t, Q_j^t - \min_{0 \le k < N}(\sum_{\ell=1}^{k} W_{j-\ell}^t - Q_{j-\ell}^t)) \\ W_j^{t+1} &= Q_{j+1}^t + W_j^t - Q_j^{t+1}. \end{aligned}$$

Assoicated to this system is a *tropic spectral curve* Γ_C depending on spectral data C associated to conserved quantities and therefore the initial conditions. Subject to the simplicity of the spectrum, the tropical spectral curve is smooth and the genus of Γ_C is $g = N - 1$. Let Ξ be the $g \times g$ period matrix. Then define the tropical Riemann theta function defined for $\mathbf{Z} \in \mathbb{R}^g$ as

$$\Theta(\mathbf{Z}; \Xi) = \min_{\mathbf{n} \in \mathbb{Z}^g}(\mathbf{n} \cdot (\frac{1}{2}\Xi\mathbf{n} + \mathbf{Z})).$$

Then we have two facts concerning the solution to trop-pToda and its relation to the periodic BBS. Given starting data such that the spectral data is simple, there are (explicit) constants $\lambda, \mathbf{Z}_0, \mathbf{c} \in \mathbb{R}^g$ and C_1 and C_2 such that if $T_n^t = \Theta(\mathbf{Z}_0 + t\lambda + \mathbf{c}n)$ then the solution to trop-pToda is

$$
\begin{aligned}
Q_n^t &= T_{n-1}^t + T_n^{t+1} - T_{n-1}^{t+1} - T_n^t + C_1 \\
W_n^t &= T_{n-1}^{t+1} + T_{n+1}^t - T_n^t - T_n^{t+1} + C_2.
\end{aligned}
$$

Further, up to a rotational shift in the location of the N solitions, this also gives the solution to the periodic BBS.

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