ODEs 5-3-21

Organizational info:
On Syllabus

What is a differential equation?

Like an equation in algebra

$$y^2 = x \quad \text{has solution} \quad y = \sqrt{x} \quad \text{or} \ -\sqrt{x}$$

But they may involve derivatives of functions

$$\frac{dy(x)}{dx} = x \qquad \text{or} \qquad \frac{dy(x)}{dx} = f(x,y)$$

We solve the equation by finding a expression for $y(x)$ that doesn't involve derivatives

Eg: $\frac{dy(x)}{dx} = x$ has solution given by an integral

Solution $y(x) = \int x \, dx + C$ for any C

Goal of the course: · Understand different types of differential equations
· Whether they have solutions
· How many / which solutions
If we can't come up with elementary functions as solutions
· Describe the behavior of solutions

Examples of differential equations

1) Newton's second law $(F = ma)$ ms

$$\frac{d^2y}{dt^2} = \frac{1}{m} F(y, t, \dots)$$

Motion of rigids objects / particles under forces

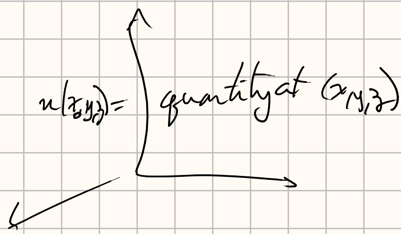
2) Natural law exponential growth of a quantity $Q(t)$

$$\frac{\partial Q}{\partial t} = kQ$$

$k > 0$ Exponential growth

$k < 0$ Exponential decay

3) Laplace equation for $u(x,y,z)$



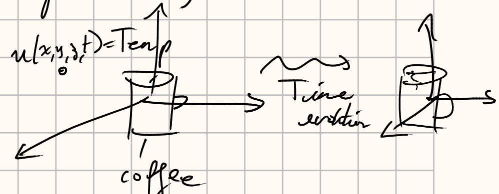
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

This is a (relatively simple) <u>Partial differential equation</u>

or (PDE)

4) Heat equation (models heat dissipation) $u = $ Temp.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$



complexity

5) Other PDEs

- Einstein's equations for General relativity
- Navier-Stokes equation for fluid dynamics

<u>Takeaway</u>: All sorts of phenomena are modelled by differential equations

This course focuses on the simplest differential equations:
Differential equations of <u>one</u> independent variable
(usually, think of this as some quantity that changes over time, not time & space)

These are <u>Ordinary</u> <u>differential equations</u> or ODEs.

Def: An Ordinary differential equation for a function $y(t)$ is an equation of the form

$$F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt}, \ldots, \frac{d^Ny}{dt^N}\right) = 0$$

Here

- N is the order of the equation (if it can't be written as an equation of lower order)

Eg: $\frac{dQ}{dt} = kQ$ is ODE of 1st order

Eg: $\frac{d^2y}{dt^2} = \frac{1}{m}F$ is a 2nd order ODE

Solving differential equations

- General methods
- Tricks that work in special situations

Graphical / Numerical methods

- work in the most general situations
- Especially if there is no closed form solution to the ODE.

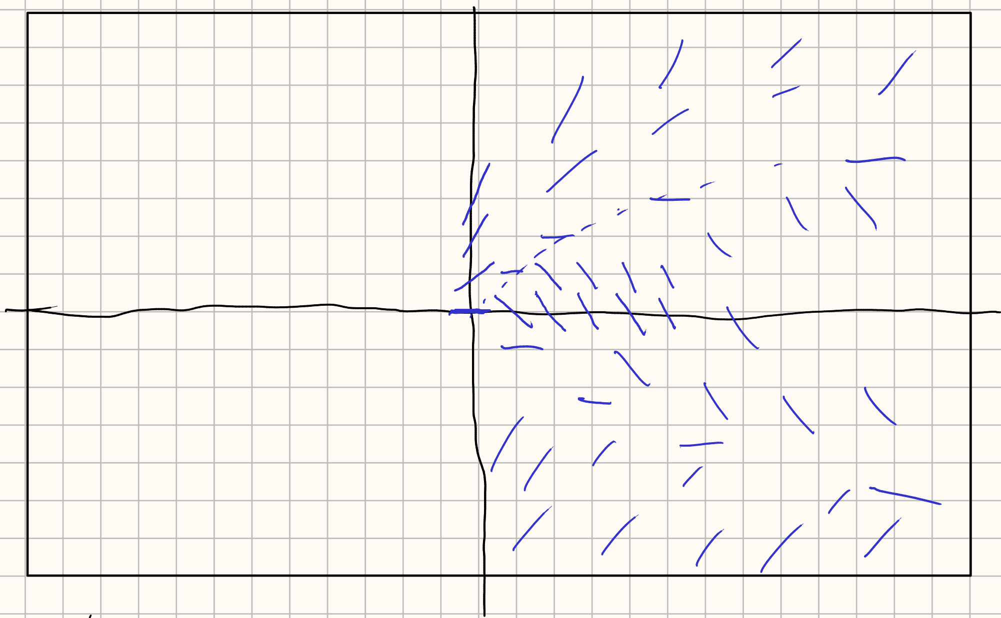
Eg: $\frac{dy}{dx} = y^2 - x$ doesn't have solution written in terms of $\{y^3, x^2, \ln(x), xy, e^x, \sin(x) \ldots\}$

Eg: $\frac{dQ}{dt} = kQ$ has solution $Q_0 e^{kt} = Q$ b/c $\frac{dQ}{dt} = kQ_0 e^{kt} = kQ$

We can approach the problem graphically & numerically

Def: The slope field of a ODE $\frac{dy}{dx} = f(x,y)$ plots a line segment with slope $f(x,y)$ at point (x,y)

Eg: Slope field for $\frac{dy}{dx} = y^2 - x$

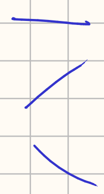


(x,y)	$f(x,y) = \frac{dy}{dx}$
0,0	0
0,1	1
1,0	-1
2,0	-2
0,2	4
0,n	n^2
n,0	$-n$
1,1	0

along $y^2 - x = 0$ - slope $= 0$

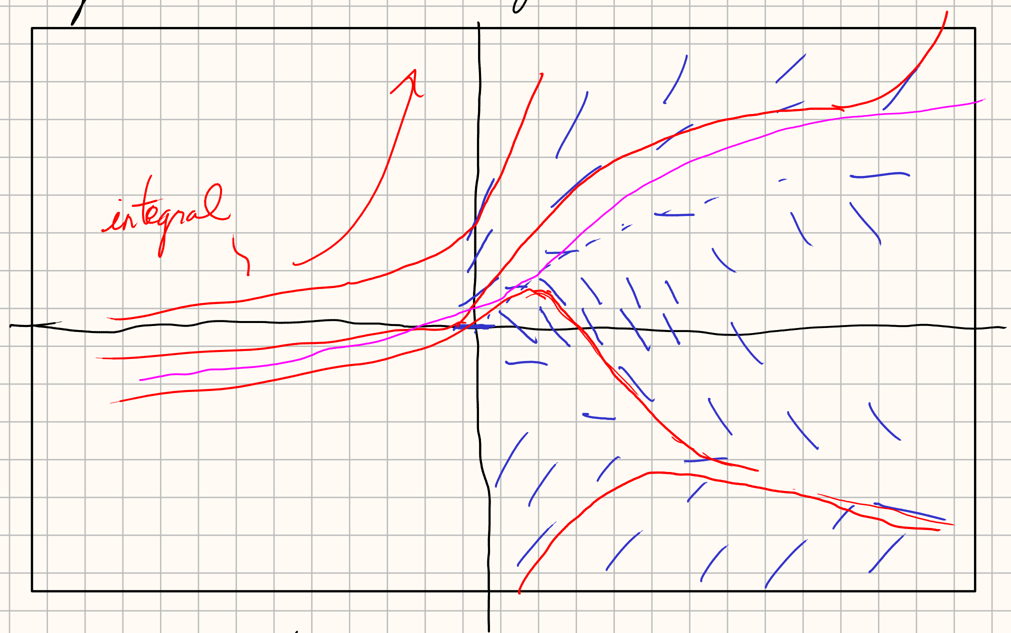
$y^2 - x > 0$ slope > 0

$y^2 - x < 0$ slope < 0



Def: An integral curve of the ODE is the plot $(x, y(x))$ of a solution

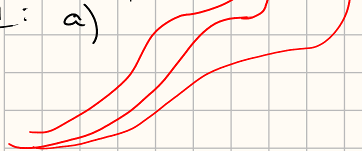
Fact: Integral curves are tangent the the slope field



You can also compute these using a computer.

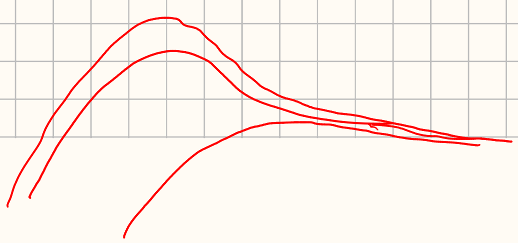
The graphical method lets us understand the qualitative behavior of solutions:

Either: a)



$y(x) \to \infty$ as $x \to \infty$

b) $y(x) \to$ fixed asymptotic as $x \to \infty$



c) There is a single other solution between those two