ODEs 5-5

Goal: - Understand the Existence and Uniqueness theorem (for 1^{st} order ODEs)

An ODE $y' = f(y,x)$ can have many solutions, usually parametrized by a single constant: E.g.

$\frac{dy}{dt} = ky$ has a family of solutions $y(t) = Ae^{kt}$ where A is arbitrary. I.e. one can solve the

IVP $y' = ky$ $y(0) = y_0$ $\rightsquigarrow$ $y(t) = y_0 e^{kt}$

$$y_0 = y(0) = Ae^{k0} = A$$

Ideal hope: (it doesn't work out this way)

Nth order ODE

$$F\left(\frac{d^N y}{dt^N}, \dots\right) = \frac{d^N y}{dt^N} + \dots f_1 \frac{dy}{dt} + \dots = 0$$

If we specify $\underbrace{y(0)}_{c_0}, \underbrace{y'(0)}_{c_1}, \dots \underbrace{y^{(n-1)}(0)}_{c_{n-1}}$, we would like to have a unique to the IVP $F\left(\frac{d^N y}{dt^N}, \dots\right) = 0$, $y(0) = c_0$, $\dots$ $y^{(n-1)}(0) = c_{n-1}$

In the first order case, this is correct with some caveats

Theorem (Existence and Uniqueness of 1^{st} order ODEs)

Let $\frac{dy}{dx} = f(x,y)$ be an ODE (most general 1^{st} order ODE)

If
- $f(x,y)$ to be continuous $x \in [x_0, x_1]$, $y \in [y_0, y_1]$
- $\frac{\partial f}{\partial y}$ to be continuous $x \in [x_0, x_1]$ $y \in [y_0, y_1]$
- $y(x_i) = c_i$ where $x_i \in [x_0, x_1]$, $c_i \in [y_0, y_1]$

Then There is a unique solution $y(t)$ to the ODE w/ IVP $y(x_i) = c_i$ ONLY for $x \in (x_i - \varepsilon, x_i + \varepsilon)$ for some $\varepsilon > 0$

Why are the caveats necessary?

Some of them are related to the non-linearity of a general ODE

For linear ODEs we have a better theorem

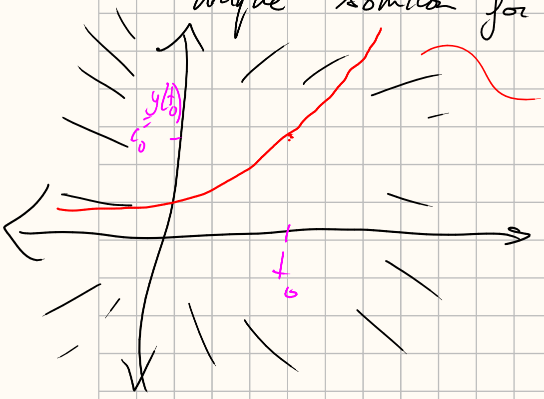
Our most general 1^{st} order linear ODE

$$\frac{dy}{dt} + p(t)y = q(t) \qquad \left(\text{i.e. } f(x,y) = q(t) - p(t)y \quad t=x\right)$$

Theorem (Linear 1^{st} order E/U)

If
- $p(t)$ and $q(t)$ are continuous on $t \in [a,b]$
- $t_0 \in [a,b]$ $\quad c_0 = \text{anything}$

Then the IVP $y' + p(t)y = q(t)$, $y(t_0) = c_0$ has a unique solution for $t \in [a,b]$



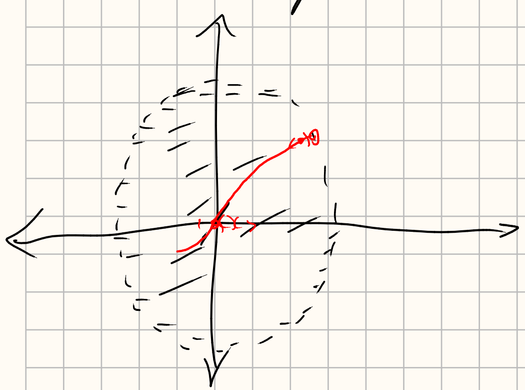
Uniqueness also implies that $y(t)$ is a function (i.e. single valued)

Eg: (Why we don't solutions for all times in the non-linear case)

$$\frac{dy}{dt} = \sqrt{1 - t^2 - y^2}$$

this is non-linear

We don't get solutions for all t



Eg: HW#1 P.6 Conditions not satisfied $\Rightarrow$ conclusion not necessarily true.

Eg: (Non-linearities mean that our solution isn't single valued for all time)

separable

$$\frac{dy}{dx} = \frac{x}{y}$$

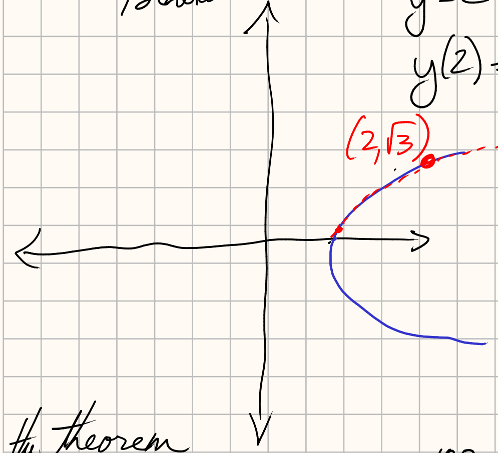
– if $y \neq 0$, $f(x,y) = \frac{-x}{y}$ has $f(x,y)$ continuous, $\frac{\partial f}{\partial y}$ continuous

General solution

$y^2 = C + x^2$

$y(2) = \sqrt{3}$

$y^2 = -1 + x^2$



Only for $x \geq 1$ does there exist a unique value for $y(x)$

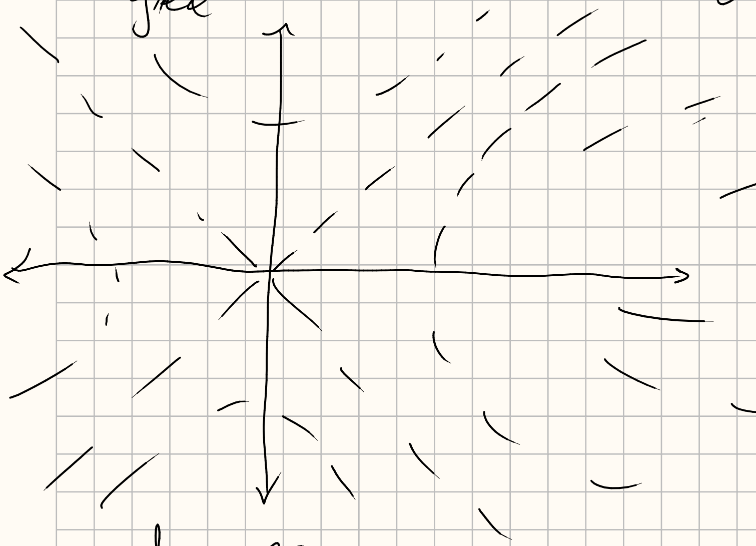
From the theorem ε could be 1×10^{-100}, but since we know the solution, ε as large as 1

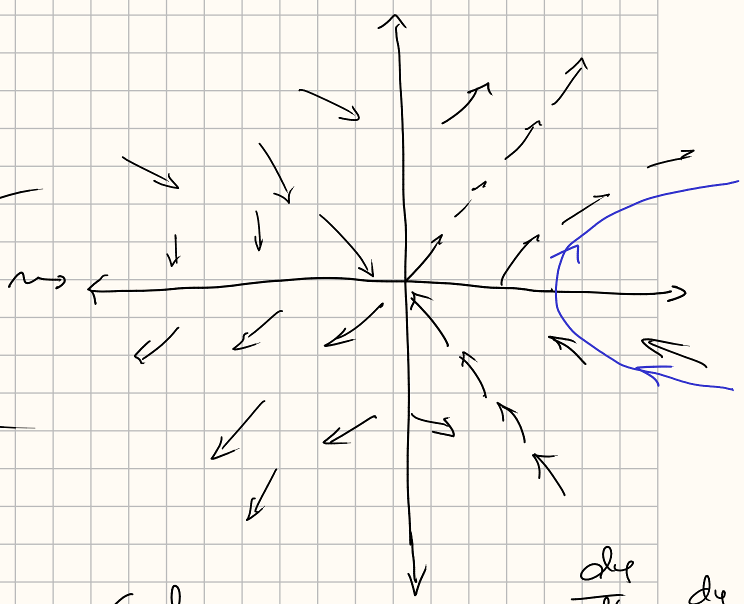
$x \in (2-1, 2+1)$

There are other ways to handle situations where our solution is only an implicit equation

A 1st order ODE (general) – can be turned into a system of 1st order autonomous ODEs

Graphically corresponds to replacing a slope field by a vector field





$$\frac{dy}{dx} = \frac{x}{y} \quad \rightsquigarrow \quad \begin{cases} \frac{dy}{dt} = x \\ \frac{dx}{dt} = y \end{cases}$$

one example

there are many more examples

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

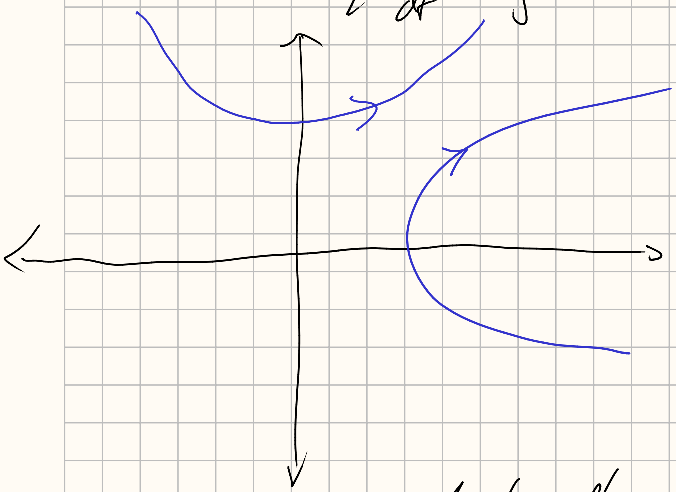
E.g.: "flip the arrows" $\frac{dy}{dt} \longmapsto -\frac{dy}{dt}$, $\frac{dx}{dt} \longmapsto -\frac{dx}{dt}$

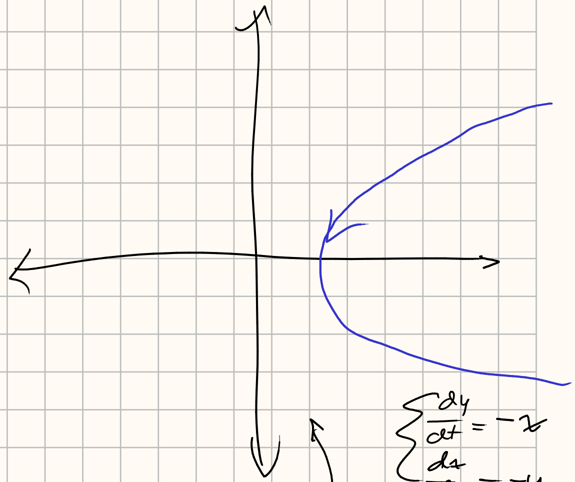
is another v. field w/ the same slope field

Even though the ODE $\frac{dy}{dx} = \frac{x}{y}$ only has solutions as functions $y(x)$ for some values of x for any initial condition (x_0, y_0) the solution to the autonomous system of ODEs

$$\begin{cases} \frac{dy}{dt} = x \\ \frac{dx}{dt} = y \end{cases}$$

exist for all time





$$\begin{cases} \frac{dy}{dt} = -x \\ \frac{dx}{dt} = -y \end{cases}$$

but the parametrization isn't unique

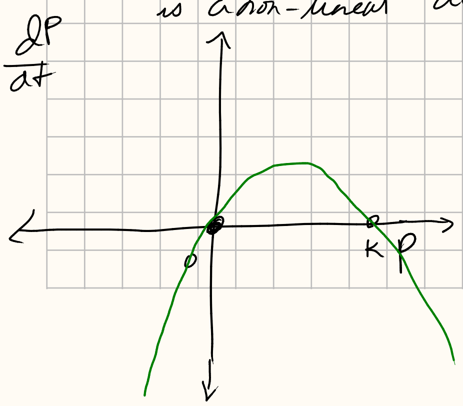
Vector field analysis of autonomous equations

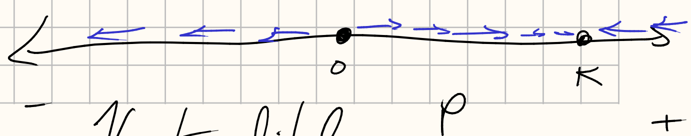
Goal: - Understand qualitative behavior of autonomous ODEs based on the direction of the corresponding vector field

This is particularly nice for 1D autonomous ODE
1-dim

Eg: $$\frac{dP}{dt} = k\,P\left(1 - \frac{P}{K}\right)$$ the logistic model of population dynamics

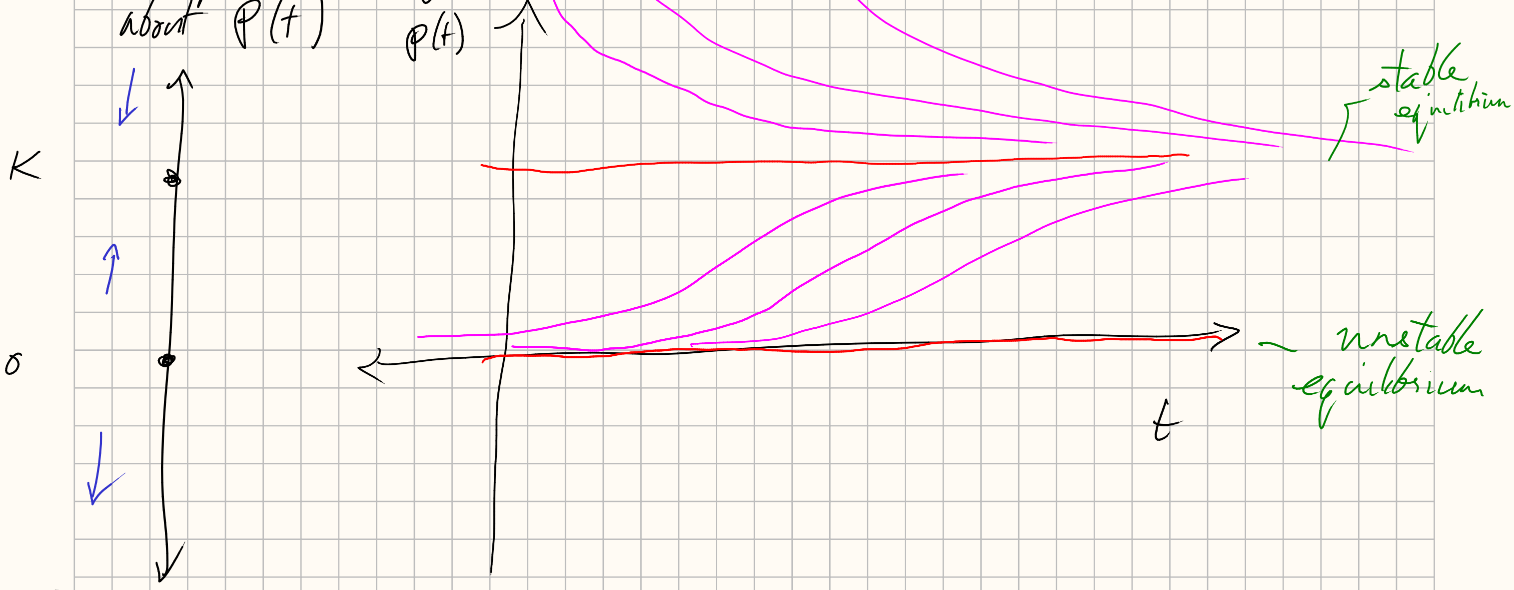
is a non-linear autonomous ODE. Draw its vector field





Vector field

The graphical info about the v. field is enough to answer qualitative questions about $P(t)$



Def A solution to an autonomous system is an <u>equilibrium solution</u> if it is constant for all t (= independent variable)

it is <u>stable</u> if nearby integral curves asymptotically approach the solution

<u>unstable</u> — — — are repelled —

<u>semistable</u> otherwise (i.e. some attracted, some repelled).