ODE 5-6

Goal:
- Solve the general 1st order linear ODE
- Understand why this works
- Be able to identify vector fields as conservative and find their scalar potential
- Identify and solve exact differential equations.

The solution to the following ODE seems like magic.

Eg: $\frac{dy}{dt} + y = e^{t/3}$ This is a 1st order linear ODE

Magic: Multiply both sides by e^t

$$e^t\left(\frac{dy}{dt} + y\right) = e^{t/3}e^t = e^{\frac{4}{3}t}$$

$$e^t\frac{dy}{dt} + e^t y = e^{4/3 t}$$

$\frac{d}{dt}e^t y + e^t \frac{dy}{dt}$

$$\Leftarrow \frac{d}{dt}\left(e^t y(t)\right) = e^{4/3 t}$$

$$e^t y(t) = \int e^{4/3 t}\,dt + c$$

$$y(t) = \frac{1}{e^t}\left(\frac{3}{4}e^{4/3 t} + c\right)$$

This works in general:

Method to solve linear 1st order ODE

$$y' + p(t)y = q(t)$$

multiply both sides by $\mu(t) = e^{\int p(t)\,dt}$

This is called an *integrating factor*

$$e^{\int p(t)dt}\left(y' + p(t)y\right) = e^{\int p(t)dt} q(t)$$

$\frac{d}{dt}\left(e^{\int p(t)dt} y\right) = p(t)e^{\int p(t)dt} y + e^{\int p(t)dt}\frac{dy}{dt} = \mu(t)\left(y' + p(t)y\right)$

$$\frac{d}{dt}\left(e^{\int p(t)dt} y\right) = e^{\int p(t)dt} q(t)$$

LHS is a total derivative

$$e^{\int p(t)dt} y = \int e^{\int p(t)dt} q(t)\,dt + C$$

$$\boxed{y = \frac{1}{\mu(t)}\left[\int \mu(t) q(t)\,dt + C\right]}$$

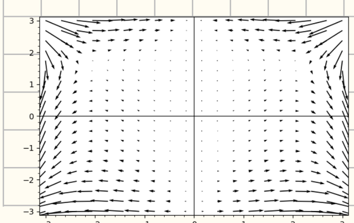
Solution to linear 1st order ODE.

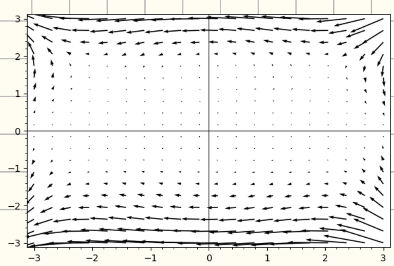
Why does this work?

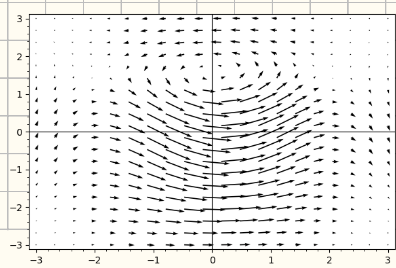
Recall: A given slope field may correspond to many different vector fields

If we have
- any vector field $\mathbf{v}(x,y) = (v_1(x,y), v_2(x,y))$
- any function $\mu(x,y)$

The slope of $\mu(x,y)\mathbf{v}(x,y)$ is the same as the slope of $\mathbf{v}(x,y)$







all have the same slope field

A given 1st order ODE corresponds to many autonomous systems of ODEs

Eg: $\dfrac{dy}{dx} = \dfrac{(4x-x^3)}{(4+y^2)} \rightarrow \begin{cases} \dfrac{dy}{dt} = 4x-x^3 \\ \dfrac{dx}{dt} = 4+y^2 \end{cases}$

$\neq$

$\| \qquad \searrow \begin{cases} \dfrac{dy}{dt} = \mu(x,y)(4x-x^3) \\ \dfrac{dx}{dt} = \mu(x,y)(4+y^2) \end{cases}$

$\dfrac{\mu(x,y)}{\mu(x,y)} \dfrac{(4x-x^3)}{(4+y^2)}$

Key idea There are ∞ of autonomous systems corresponding to an ODE. Find the "best one" which allows you to solve.

There is a class of vector fields called conservative vector fields

Def: A vector field $\mathbf{w}(x,y)$ is conservative if

$$\mathbf{w}(x,y) = \underbrace{\nabla\phi}_{\text{"grad } \phi\text{"}} := \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

ϕ is called the scalar potential

It happens that $\mathbf{w}(x,y)$ is not conservative, but $\mu(x,y)\mathbf{w}(x,y)$ is.
In general, this is hard: in our case there is a solution.

Fact: $\mathbf{w}(x,y)$ is conservative iff

$$\frac{\partial}{\partial y} w_1 = \frac{\partial}{\partial x} w_2 \qquad \text{where} \qquad \mathbf{w} = (w_1, w_2)$$

Proof: If $\mathbf{w}(x,y)$ is conservative with scalar potential ϕ

$\mathbf{w} = (\underbrace{\tfrac{\partial}{\partial x}\phi}_{w_1}, \underbrace{\tfrac{\partial}{\partial y}\phi}_{w_2})$ then $\dfrac{\partial}{\partial y} w_1 = \dfrac{\partial}{\partial y}\dfrac{\partial}{\partial x}\phi$
$\dfrac{\partial}{\partial x} w_2 = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y}\phi$ $\Rightarrow$ ✓

The other direction lets us find the scalar potential
Suppose $\dfrac{\partial}{\partial y} w_1 = \dfrac{\partial}{\partial x} w_2$

Find ϕ. We know that

$$\frac{\partial}{\partial x}\phi = w_1 \quad \text{so} \quad \text{if } \phi \text{ exists, it is of the form}$$

$$\phi = \int w_1(x,y)\,dx + c(y) \quad . \text{ But then}$$

$$w_2 = \frac{\partial}{\partial y}\phi = \int \frac{\partial}{\partial y} w_1\,dx + c(y)$$

$$w_2 = \int \frac{\partial}{\partial x} w_2\,dx + c(y) \quad . \text{ But } \left(w_2 - \int \frac{\partial}{\partial x} w_2\,dx\right) \text{ doesn't}$$

depend on x so we can solve $c(y) = w_2 - \int \frac{\partial}{\partial x} w_2\,dx$.

Eg: $\mathbf{W}(x,y) = (1+2xy^2,\ 2x^2y+2y)$

This is exact b/c $\frac{\partial}{\partial y} w_1 = 4xy$

$$\frac{\partial}{\partial x} w_2 = 4xy \quad \checkmark$$

Find ϕ: $\phi = \int 1+2xy^2\,dx + c(y)$

$$= x + x^2y^2 + c(y)$$

$$w_2 = \frac{\partial}{\partial y}\phi = 2x^2y + \frac{dc}{dy} = 2x^2y + 2y$$

$$\frac{dc}{dy} = 2y$$

$$c = y^2$$

$\phi = x + x^2y^2 + y^2$ is a scalar potential

Exact differential equations $\longleftrightarrow$ conservative vector fields

Def Sometimes written as

$M(x,y)\cancel{dx} + N(x,y)\frac{dy}{\cancel{dx}} = 0 \longleftrightarrow$ The vector field

is an exact ODE $\quad \mathbf{W}(x,y) = (M(x,y),\ N(x,y))$

is conservative

In terms of the corresponding autonomous system:

$$M(x,y)+N(x,y)y'=0 \longleftrightarrow \begin{cases} \frac{dy}{dt} = -M(x,y) \\ \frac{dx}{dt} = N(x,y) \end{cases}$$ is governed by a divergence-free vector field

$$\mathbf{v}(x,y) = (N(x,y), -M(x,y))$$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

So if we rescale the autonomous system $\mathbf{v}(x,y) \mapsto \mu(x,y)\mathbf{v}(x,y)$ so $\mu(x,y)\mathbf{v}(x,y)$ is <u>divergence-free</u> (= solenoidal) then

$$\mu(x,y)\mathbf{w}(x,y) = (\mu M(x,y), \mu N(x,y))$$ will be <u>conservative</u>

<u>Generally</u>: an exact ODE $M + Ny' = 0$ has solution

$\phi(x,y) = c$ where ϕ is the scalar potential of $\mathbf{w}(x,y) = (M(x,y), N(x,y))$.

<u>Solving exact ODE</u>

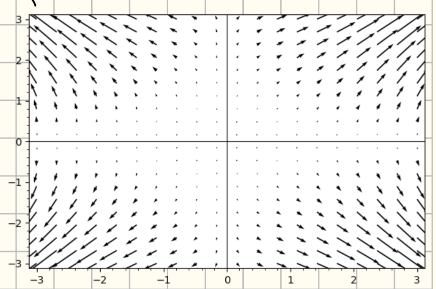
Our conservative vector field from before

$$\mathbf{w}(x,y) = (1+2xy^2, 2x^2y+2y)$$

$$\mathbf{v}(x,y) = (\underbrace{-2x^2y-2y}_{\frac{dx}{dt}}, \underbrace{1+2xy^2}_{\frac{dy}{dt}})$$

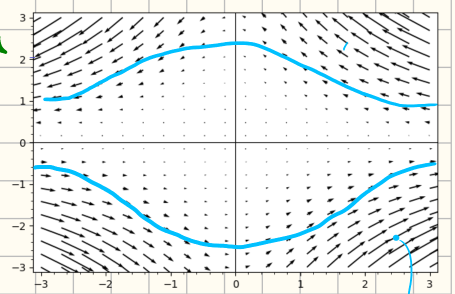
$$\frac{dy}{dx} = \frac{1+2xy^2}{-2x^2y-2y} \quad \text{OK}$$

$\mathbf{W}$ conservative - no curl



$\mathbf{V}$

$\operatorname{div}\mathbf{V}=0$ (solenoidal)



integral curves

$$1+2xy^2 + (2x^2y+2y)y' = 0.$$

$\mathbf{W}$ has scalar potential

$$\phi = x + x^2y^2 + y^2.$$

Then we can calculate

$$\frac{d}{dx}\left(\phi(x,y(x))\right) = \underbrace{\frac{\partial}{\partial x}\phi + \frac{\partial}{\partial y}\phi \frac{dy}{dx}}_{\text{Our equation}} = 0$$

So $\phi(x,y) = \int 0\, dx + c$ or $x + x^2y^2 + y^2 = c$ is our general solution.

Why does this work for 1st order linear ODEs?

We can find an integrating factor so that the corresponding vector field is conservative $\Rightarrow$ general solution

For $y' + p(t) y = q(t)$ or

$p(t) y - q(t) + y' = 0$ the integrating factor $e^{\int p(t)\,dt} = \mu(t)$

results in the exact equation

$\mu(t) p(t) y - \mu(t) q(t) + \mu(t) y' = 0$, with solution

$\phi(y,t) = c$ for the scalar potential ϕ