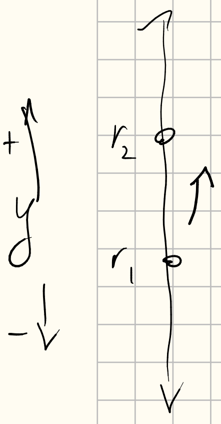
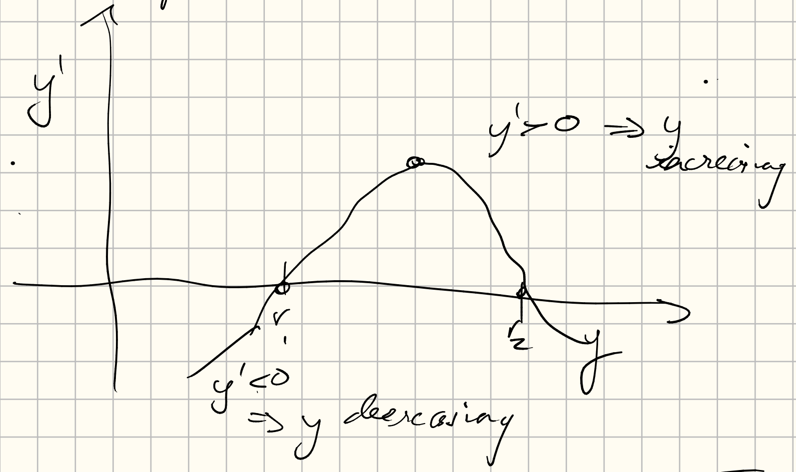
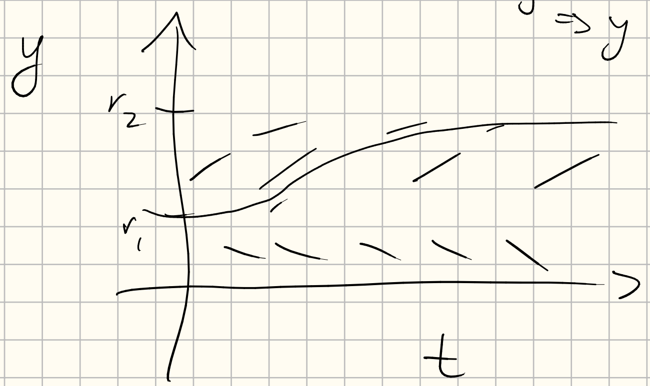
ODE 5-10

Q about 1d Autonomous systems

Stability of 1d autonomous systems







$h(t) = y(-t)$

$h' = \underbrace{-y'(-t)}$

Review exact ODEs & integrating factors

An exact equation is

$$M(x,y)\,dx + N(x,y)\,\frac{dy}{dx} = 0 \quad \text{if v. field}$$

$W(x,y) = (M, N)$ is conservative

w/ scalar potential $\phi(x,y)$. i.e. $W = D\phi$, or

$M = \frac{\partial \phi}{\partial x}$, $N = \frac{\partial \phi}{\partial y}$

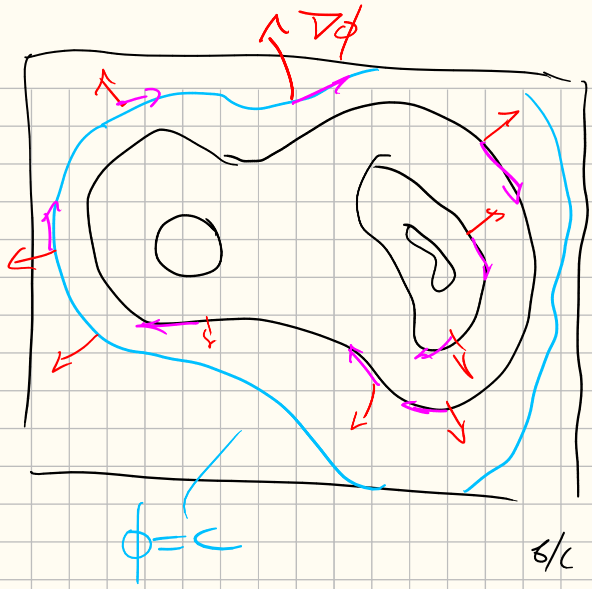
If this is the case:

- $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ ← check if eq. is exact
- Solutions are $\boxed{\phi(x,y) = C}$ b/c

$$\frac{d}{dx}\left(\phi(x, y(x))\right) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$$

- We find $\phi(x,y)$ by $\int M(x,y)\,dx + c(y)$ solving $c(y)$ b/c

$$\frac{\partial}{\partial y}\left(\int M(x,y)\,dx - c(y)\right) = N(x,y).$$



$W = (M, N) = \nabla\phi$

orthogonal to level sets $\phi = c$

Rotate each vector 90°

$$(M,N) \longrightarrow (N,-M)$$

$$V(x,y) = (N(x,y), -M(x,y))$$

is solenoidal ($\Leftrightarrow$ div $V = 0$)

b/c $\operatorname{div} V = \dfrac{\partial N}{\partial x} + \dfrac{\partial(-M)}{\partial y} = 0$

W is conservative

The v. field $V(x,y)$ is field associated to

$$\begin{cases} \dfrac{dy}{dt} = N \\ \dfrac{dx}{dt} = -M \end{cases}$$

Rescale the system $V \longrightarrow \mu(x,y)V$

$$(N,-M) \longmapsto (\mu(x,y)N, \mu(x,y)(-M))$$

s.t. V is solenoidal $\Leftrightarrow$ W is conservative

eq. $\mu(x,y)M + \mu(x,y)N\dfrac{dy}{dx} = 0$ becomes exact

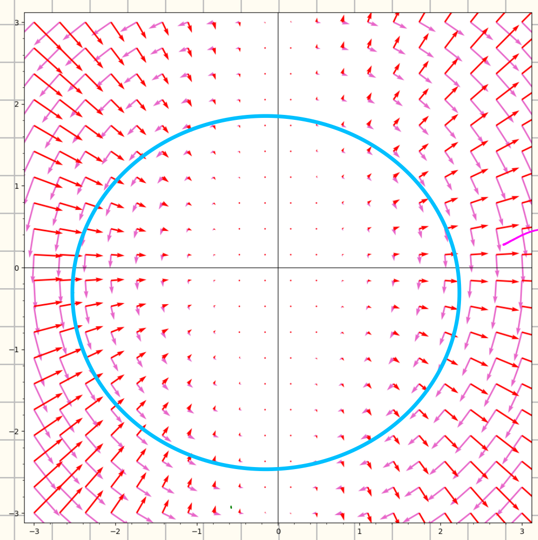
Eg.

$$x^2 + xy\frac{dy}{dx} = 0$$

Not exact b/c

$$\frac{\partial M}{\partial y} = 0$$

$$\frac{\partial N}{\partial x} = y$$



Multiply by integrating factor

$$\frac{1}{x} = \mu(x,y)$$

equation becomes

$$\frac{1}{x}(x^2) + \frac{1}{x}(xy)\frac{dy}{dx} = 0$$

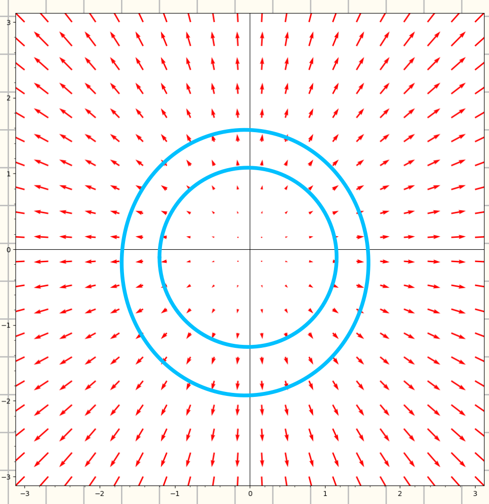
OK

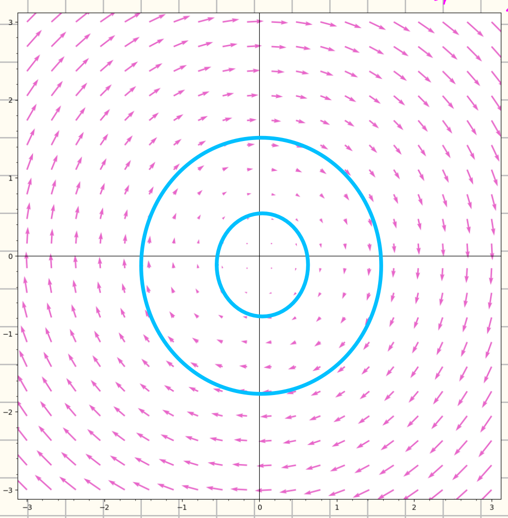
$$x + y\frac{dy}{dx} = 0$$

Now it's exact

no div (no sources/sinks)

"no 2D curl"





$(\mu M, \mu N) = (x, y)$ is conservative

$(\mu N, -\mu M) = (y, -x)$ solenoidal

$M = x \qquad N = y$

$$\phi = \int x\,dx + c(y)$$
$$= \frac{1}{2}x^2 dx + c(y)$$
$$\frac{\partial}{\partial y}\phi = \frac{dc}{dy} = y = N(x,y) \Rightarrow c = \frac{1}{2}y^2$$
$$\phi = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

solutions are $\phi(x,y) = \boxed{\frac{1}{2}x^2 + \frac{1}{2}y^2 = C}$

Today: 2nd Order Linear Homogeneous equations

New adjective in taxonomy: homogeneous

Linear — Non-linear

Homogeneous

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

Non-homogeneous

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y + r(t) = 0$$

Why do we care?

The coefficient $r(t)$ usually says something about "external inputs" or "external forcing"

Eg: 1st order linear

$$y' = p(t)y + q(t)$$

- $p(t)$ is variable exponential growth rate
- $q(t)$ is inputs

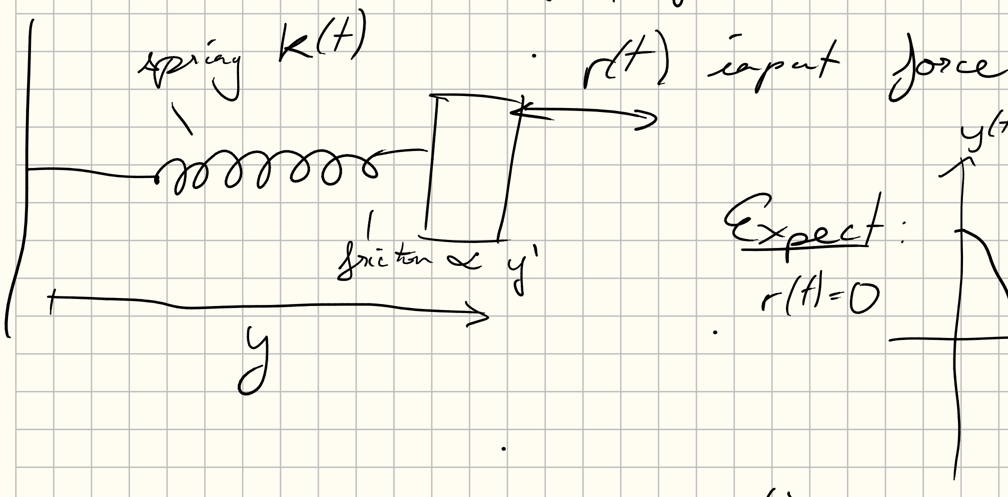
Eg: $y(t)$ = Bank account balance
$p(t)$ = interest rate
$q(t)$ = Deposits $q(t)=0 \iff$ homogeneous

Eg:

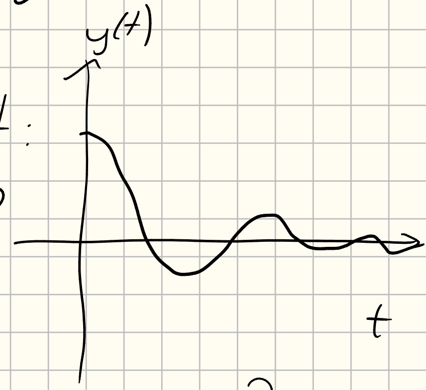
$$y'' + m(t)y' + k(t)y + r(t) = 0$$

General 2nd order linear homogeneous

$m \geq 0, k > 0$ models a spring w/ friction, external forcing



Expect: $r(t)=0$



$r(t) \neq 0 \rightsquigarrow$?

so we do homogeneous case first $\rightsquigarrow$ use to solve $r(t) \neq 0$.

For now: restrict to $k(t) = \text{const.}$ $m(t) = \text{const.}$ $r(t) = 0$

2nd order linear constant coeff

homogeneous equation

$$\frac{d^2y}{dt^2} + m\frac{dy}{dt} + ky = 0$$

Eg: $\dfrac{d^2y}{dt^2} = y$ has solution. $y(t) = c_1 e^t$

has another solution. $y(t) = c_2 e^{-t}$

Eg: $\dfrac{d^2y}{dt^2} = -y$

- $c_1 \cos(t)$
- $c_2 \sin(t)$

How do we get the general solution? Because our equation

$$L(y) = \frac{d^2}{dt^2}y + m\frac{d}{dt}y + ky = 0$$

is linear

Principle of superposition If $L(y)$ is a linear differential operator and y_1 & y_2 are solutions $L(y_1)=0$, $L(y_2)=0$ then

$L(c_1y_1(t) + c_2y_2(t))$ is also a solution i.e. $L(c_1y_1+c_2y_2)=0$.

Rmk.
- $c_1y_1(t)+c_2y_2(t)$ is a linear combination of y_1 & y_2
- $L(y)$ is a linear operator if
$$L(c_1y_1+c_2y_2) = c_1L(y_1)+c_2L(y_2)$$
- ↑ lets us conclude principle of superposition since
$$L(c_1y_1+c_2y_2) = c_1L(y_1)+c_2L(y_2) = 0+0=0$$
- Eg: Linear operators $\Leftrightarrow$ linear equations

$\dfrac{d^2}{dt^2}y + p(t)\dfrac{dy}{dt} + q(t)y$ is linear since

$$\frac{d^2}{dt^2}(c_1y_1+c_2y_2) = c_1\frac{d^2}{dt^2}y_1 + c_2\frac{d^2}{dt^2}y_2, \quad p(t)\frac{d}{dt}(c_1y_1+c_2y_2) = c_1p(t)\frac{d}{dt}y_1 + c_2p(t)\frac{d}{dt}y_2.$$

Two solutions y_1 & y_2 of a 2nd order linear homogeneous equation $\Rightarrow$

$c_1 y_1 + c_2 y_2$ is also a solution

$L(y) = y' + 2$ not linear

since $L(y_1 + y_2) = y_1' + y_2' + 2$
$\neq y_1' + 2 + y_2' + 2$

Inhomogeneous eq.

$$\underbrace{y'' + p(t) y' + q(t) y}_{\underset{\text{linear}}{L(y)}} = r(t)$$

$L'(y) =$

$$\underbrace{y'' + p(t) y' + q(t) y - r(t)}_{L'(y) \text{ - not linear}} = 0$$

Eg: $\frac{d^2}{dt^2} y - y = 0$ General solution:

$$c_1 e^{t} + c_2 e^{-t}$$

Eg: $\frac{d^2}{dt^2} y + y = 0$ General solution:

$$c_1 \cos(t) + c_2 \sin(t).$$

How do one solve

$$y'' + my' + ky = 0 \quad ?$$

Answer: make ansatz ..guess

$$y(t) = e^{rt}$$

$y'' = r^2 e^{rt} = r^2 y$

$y' = r e^{rt} = ry$

$y = y$

$y'' + my' + ky = 0$

$r^2 y + mry + ky = 0$

$y(r^2 + mr + k) = 0 \Rightarrow r^2 + mr + k = 0$ then e^{rt} is a solution.

Def

$r^2 + mr + k$ is called <u>characteristic polynomial</u> of the ODE $y'' + my' + ky = 0$

If r_1 is a root of the char. poly then $e^{r_1 t}$ is a solution to the eq.

Eg: $y'' + 4y' + 3y = 0$ Find general solution

char. poly. $r^2 + 4r + 3 = 0$ has roots $r_1 = -3$, $r_2 = -1$

$(r+3)(r+1) = 0$

General solution: $c_1 e^{-3t} + c_2 e^{-t}$.

Eg: Solve IVP $y(0) = 1$, $y'(0) = 0$

$y(0) = c_1 e^0 + c_2 e^0 = 1$

$y'(0) = c_1(-3)e^0 + c_2(-1)e^0 = 0$

$$c_1 + c_2 = 1$$
$$-3c_1 + -1c_2 = 0$$

$$\begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right.$$

$$= \frac{1}{1(-1) - 1\cdot(-3)}\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

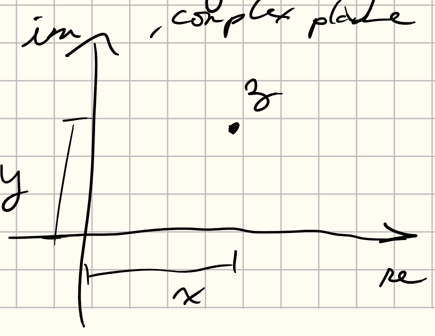
$$= \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1\cdot 1 + -1\cdot 0 \\ 3\cdot 1 + 1\cdot 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$ sol. $\boxed{y(t) = -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}}$.

What about complex roots?

Recall the following about complex #s $z \in \mathbb{C}$



$z = x + iy$ $i = \sqrt{-1}$

- Any degree n polynomial

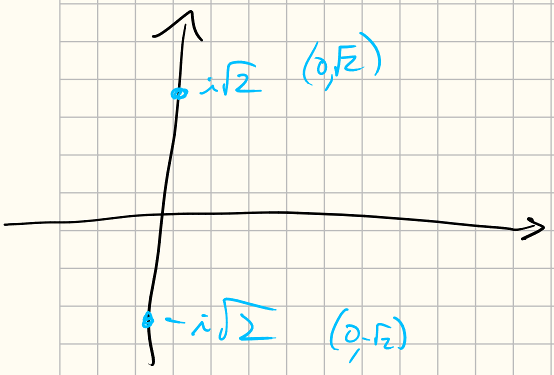
$$z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$$

has n roots (count w/ multiplicity)

Eg: $z^2 + 2 = 0$ roots $i\sqrt{2}$, $-i\sqrt{2}$

b/c $(i\sqrt{2})^2 + \sqrt{2} = -1\sqrt{2} + \sqrt{2} = 0$

$+ (-i)$



\ real polynomial roots pair off as conjugates

$a_i \in \mathbb{R}$

$$z \mapsto \bar{z}$$

$$x+iy \mapsto x-iy$$

Eg: (Roots of unity)

$z^n - 1 = 0$ $\quad z = \sqrt[n]{1}$ has n solutions

$e^{\frac{2ki\pi}{n}}$ $\quad k = 0, \ldots, n-1$ solutions

$$\left(e^{\frac{2ki\pi}{n}}\right)^n = e^{2\pi i k} = \left(e^{2\pi i}\right)^k$$

$$= 1$$

use $\underline{e^{2\pi i} = 1}$

- We can define $\exp(z)$ (and many other functions) for $z \in \mathbb{C}$, usually using power series (e.g. use power rule for differentiation)

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

- Euler's formula

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

Exercise

Prove this formula using

$\exp(z) = \frac{1}{0!} + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots$

$\sin(z) = z + \frac{-1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$

$\cos(z) = \cdots$

2nd order homogeneous c. coeff. linear ODE w/ complex roots?

Eg: $y''+y=0$

$r^2+1=0$ has solutions $r_1 = i$, $r_2=-i$

General solution (over $\mathbb{C}$) is

$c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{it} + c_2 e^{-it}$ where $c_1 \in \mathbb{C}$, $c_2 \in \mathbb{C}$.

Find the general real solution (use Euler's formula)

$c_1\left(\cos(t)+i\sin(t)\right) + c_2\left(\cos(-t)+i\sin(-t)\right)$

Take $c_1=\frac{1}{2}$ $c_2=\frac{1}{2}$

$\frac{1}{2}\cos(t) + \frac{1}{2}\cos(-t) \quad + i\left(\left(\frac{1}{2}\sin(t)\right) + \frac{1}{2}\sin(-t)\right)$

$\frac{1}{2}\left(\cos(t)+\cos(t)\right) \quad + i\left(\frac{1}{2}\sin(t) - \frac{1}{2}\sin(t)\right)$

$=\cos(t)$

Take $c_1 = \frac{i}{2}$ $\quad c_2 = \frac{-i}{2}$

$\frac{i}{2}\left(\cos(t)+i\sin(t)\right) - \frac{i}{2}\left(\cos(-t)+i\sin(-t)\right)$

$\frac{i}{2}\left(\cancel{\cos(t) - \cos(-t)}\right) + \frac{i}{2} i\sin(t) - \frac{i}{2} i \sin(-t)$

$+ \frac{-1}{2}\sin(t) - \frac{(-1)}{2}\left(-\sin(t)\right)$

$= -1\sin(t)$.

get $-\sin(t)$, $\cos(t)$. Principle of superposition $\Rightarrow$

General real solution: $d_1\cos(t) + d_2\sin(t)$ $\quad d_1, d_2 \in \mathbb{R}$.

Finish other roots tomorrow: