ODE 5-11

Goal:
- Finish 2nd Order constant coeff homogeneous eq. solutions
- Deal with inhomogeneous equations & solutions

Recall: $y''+my'+k=0$

has solutions $e^{r_1 t}$ & $e^{r_2 t}$ if r_1, r_2 are roots of $\overbrace{r^2+mr+k}^{\text{characteristic polynomial}}$

What happens if $m=0$ $k=1$? r^2+1 has roots $\pm\sqrt{-1} = \pm i$

Saw: Complex solution: $y''+y=0$

$$c_1 e^{it} + c_2 e^{-it} \qquad c_1, c_2 \in \mathbb{C}$$

uses $e^{it} = \cos(t) + i\sin(t)$

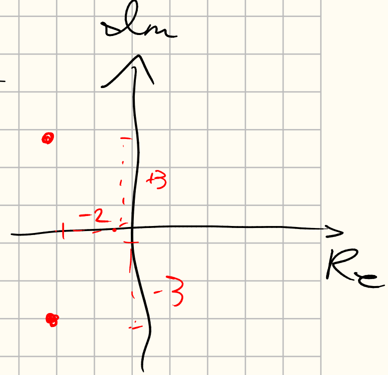
$$d_1 \cos(t) + d_2 \sin(t) \qquad d_1, d_2 \in \mathbb{R}$$

Eg: $y''+4y'+13y=0$ char. poly. $r^2+4r+13$

$$r_1, r_2 = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i$$

$\mathbb{C}$ - solution

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad c_1, c_2 \in \mathbb{C}$$



$$= c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t}$$

$$= e^{-2t}\left(c_1 e^{i3t} + c_2 e^{-i3t}\right)$$

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$

$$e^{-2t}\left(\frac{1}{2} e^{i3t} + \frac{1}{2} e^{-i3t}\right) = e^{-2t}\left(\frac{1}{2}\left(\cos(3t) + \cancel{i\sin(3t)}\right) + \frac{1}{2}\left(\cos(-3t) + \cancel{i\sin(-3t)}\right)\right)$$

$$= e^{-2t}\left(\frac{1}{2}\cos(3t) + \frac{1}{2}\cos(3t)\right)$$

$$= e^{-2t}\cos(3t)$$

We can also get $\sin(3t)$:

Abstractly

$$Re(z) = \frac{1}{2}z + \bar{z} = \left(\frac{1}{2}x + iy + \frac{1}{2}x - iy\right) = x$$

$$Im(z) = \frac{-i}{2}(z - \bar{z}) = \left(\frac{-i}{2}(x + iy - x + iy)\right) = \frac{-i}{2}(2iy) = y$$

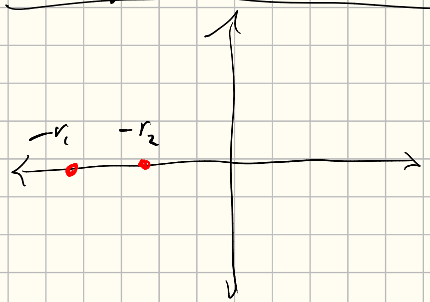
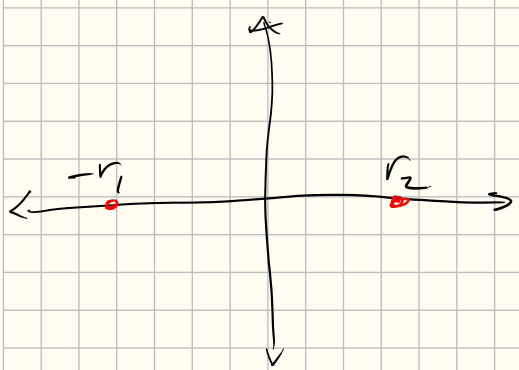
Since $\overline{e^{it}} = e^{-it}$ $\left(\overline{\cos(t) + i\sin(t)} = \cos(t) - i\sin(t) = \cos(-t) + i\sin(-t)\right)$

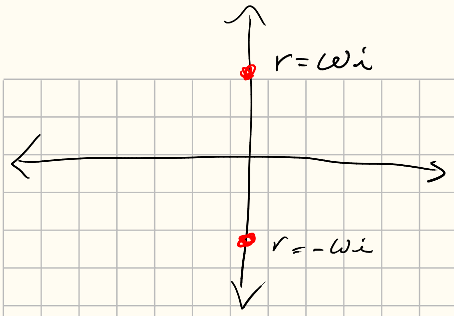
$$e^{-2t}\sin(3t) = Im\left(e^{-2t}e^{3it}\right) = \underset{c_1}{\underset{\shortparallel}{\frac{-i}{2}}}\left(e^{-2t}e^{3it}\right) + \underset{c_2}{\underset{\shortparallel}{\frac{i}{2}}}\left(e^{-2t}e^{3it}\right)$$

Superposition $\Rightarrow$

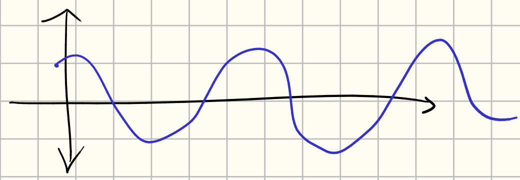
$$d_1 e^{-2t}\sin(3t) + d_2 e^{-2t}\cos(3t) \qquad d_1, d_2 \in \mathbb{R}$$

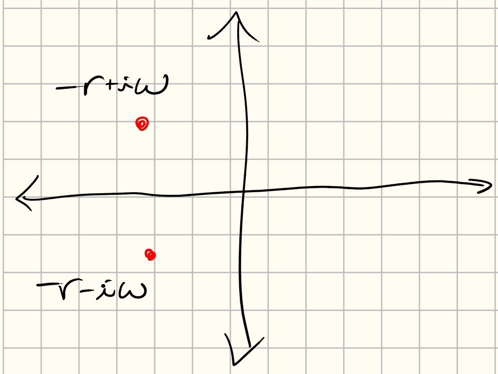
is a solution

Solution roots of $r^2 + mr + k$	Solution $(d_1, d_2 \in \mathbb{R})$
	$d_1 e^{-r_1 t} + d_2 e^{-r_2 t}$ 
	$d_1 e^{-r_1 t} + d_2 e^{r_2 t}$ 

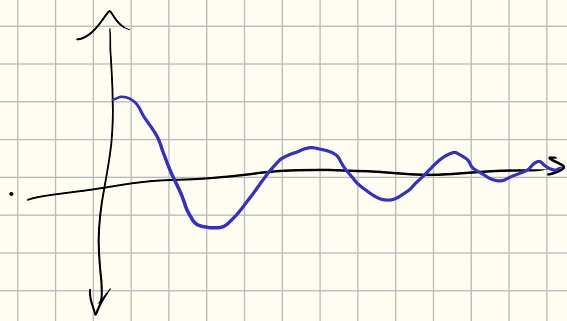


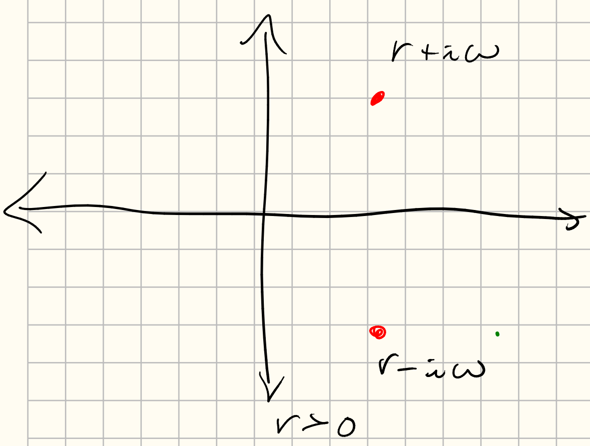
$d_1\cos(\omega t) + d_2 \sin(\omega t)$





$d_1 e^{-r}\cos(\omega t) + d_2 e^{-r}\sin(\omega t)$





$d_1 e^{r}\cos(\omega t) + d_2 e^{r}\sin(\omega t)$

One other case: multiple roots

Eg: $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 = (r-3)(r-3)$$

Solution: $c_1 e^{3t} + c_2 e^{3t} = (c_1 + c_2) e^{3t}$ Not to solve general IVP $y(0) = y_0$ $y'(0) = y_1$

There is one more solution

Make ansatz $y(t) = u(t)e^{3t}$ ← Method of reduction of order

$$y'(t) = u'(t)e^{3t} + u(t)\cdot 3e^{3t}$$

$$y''(t) = u''(t)e^{3t} + 3u'(t)e^{3t} + 3u'(t)e^{3t} + 9u(t)e^{3t}$$

$$y'' - 6y' + 9y = 0$$

$$e^{3t}\left[(u'' + \cancel{6u'} + \cancel{9u}) - 6(\cancel{u'} + \cancel{3u}) + \cancel{9u}\right] = 0$$

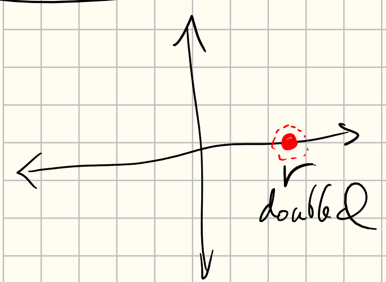
$u'' = 0$ w/ solutions $u(t) = c_2 t + c_3$

Original eq. has solutions

$$c_1 e^{3t} + (c_2 t + c_3) e^{3t} \quad \text{OR}$$

$$\boxed{c_1 e^{3t} + c_2 t e^{3t}}$$

Complete chart



$$d_1 e^{rt} + d_2 t e^{rt}$$

→ How can we be sure we have all of the solutions?

Answer: The existence / uniqueness thm 2nd order linear eqs

+ Calculation of Wronskian

Thm: Given linear 2nd order ODE $y'' + p(t)y' + q(t)y = 0$

- if $p(t)$ and $q(t)$ are continuous on $[a,b]$
- $t_0 \in [a,b]$

Then there exist a unique solution $y(t)$ to the IVP $y(t_0) = y_0$, $y'(t_0) = y_0'$ defined on $[a,b]$

Rmk: Relax linearity to $y'' = f(t, y, y')$

- Now $\frac{\partial f}{\partial y}$ continuous & $\frac{\partial f}{\partial y'}$ continuous on a rectangle $\Rightarrow$ solution on $(t_0 - \varepsilon, t_0 + \varepsilon)$

The Wronskian lets us say that a given set of solutions can solve any IVP:

Def. The Wronskian of 2 solutions y_1 & y_2 of the 2nd order ODE

$$y'' + p(t)y' + q(t) = 0$$

is

$$W[y_1, y_2](t_0) = \det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_1 y_2' - y_2 y_1'.$$

Def: y_1 and y_2 are a <u>fundamental set of solutions</u> to the ODE if

$$W[y_1, y_2](t_0) = 0$$

Why does this definition make sense?

If $\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0$ then for any vector $\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$, then we can always find a_1 and a_2 so that

$$\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = a_1 \begin{pmatrix} y_1(t_0) \\ y_1'(t_0) \end{pmatrix} + a_2 \begin{pmatrix} y_2(t_0) \\ y_2'(t_0) \end{pmatrix}.$$

In this case, the function

$$Y(t) = a_1 y_1(t) + a_2 y_2(t)$$

solves the IVP $\begin{matrix} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{matrix}$.

since
$$Y(t_0) = a_1 y_1(t_0) + a_2 y_2(t_0) = y_0$$
$$Y'(t_0) = a_1 y_1'(t_0) + a_2 y_2'(t_0) = y_0'$$

<u>Conclusion:</u> If y_1 & y_2 satisfy $W[y_1, y_2](t_0) \neq 0$ for some t_0 then $c_1 y_1 + c_2 y_2$ is the general solution to the ODE.

Eg: $y'' + 4y' + 13$

We said $c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$ is the general solution

Justify w/ the Wronskian:

$$y_1 = e^{-2t}\cos(3t) \qquad y_1' = -3e^{-2t}\sin(3t) - 2e^{-2t}\cos(3t)$$
$$y_2 = e^{-2t}\sin(3t) \qquad y_2' = 3e^{-2t}\cos(3t) - 2e^{-2t}\sin(3t)$$

$$W[y_1,y_2](t_0) = \det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0)\end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$= e^{-2t}\cos(3t)\left(3e^{-2t}\cos(3t) - 2e^{-2t}\sin(3t)\right) - e^{-2t}\sin(3t)\left(-3e^{-2t}\sin(3t) - 2e^{-2t}\cos(3t)\right)$$

$$= e^{-4t}\left[3\cos^2(3t) - \cancel{2\cos(3t)\sin(3t)} + 3\sin^2(3t) + \cancel{2\sin(3t)\cos(3t)}\right]$$

$$= 3e^{-4t}$$ This $\neq 0 \Rightarrow$

y_1 & y_2 are a fundamental set of solutions

Eg: $y'' - 6y' + 9y$

e^{3t} and $2e^{3t}$ is a fundamental set

because "y_1" "y_2"

$$y_1 = e^{3t} \qquad y_1' = 3e^{3t}$$
$$y_2 = 2e^{3t} \qquad y_2' = 6e^{3t}$$

$$W[y_1,y_2](t_0) = \det\begin{pmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t}\end{pmatrix}$$

$$= e^{3t}\cdot 6e^{3t} - 2e^{3t}\cdot 3e^{3t} = 0$$

y_1 & y_2 are NOT a fundamental set. We need $y = te^{3t}$ as well.

Higher order?

nth order homogeneous constant coeff. ODE:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

- characteristic polynomial (from ansatz e^{rt})

  $r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$ w/ roots $r_1, \ldots, r_n$

- Complex solution is

  $c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}$ $\quad c_i \in \mathbb{C}$ $\quad (r_i \neq r_j)$

  OR

  $c_1 e^{rt} + c_2 t e^{rt} + c_3 t^3 e^{rt} + \cdots + c_k t^k e^{rt} + \cdots$

  if r multiplicity $k+1$.

- Real solution

  $c_1 e^{rt + i\omega t} + c_2 e^{rt - i\omega t} \rightsquigarrow d_1 e^{rt} \cos(\omega t) + d_2 e^{rt} \sin(\omega t)$

  $c_i \in \mathbb{C}$ $\qquad d_i \in \mathbb{R}$.

  The Wronskian is

- $W[y_1, \ldots, y_n](t) = \det \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1^{(1)} & & & & \\ y_1^{(2)} & & \ddots & & \\ \vdots & & & \ddots & \\ y_1^{(n-1)} & & & & y_n^{(n-1)} \end{pmatrix}$.

- A fundamental set of solutions is $y_1, \ldots, y_n$ s.t.

  $W[y_1, \ldots, y_n](t_0) \neq 0$.

Inhomogeneous equations

$$y'' + m(t)y' + k(t)y = r(t)$$

How do we solve them? Answer comes from superposition

Corresponding homogeneous equation

$y'' + m(t)y' + k(t)y = 0$ has a fundamental set of solutions $\{y_1(t), y_2(t)\}$ so general solution $c_1y_1(t) + c_2y_2(t)$.

If we can find 1 particular solution $P(t)$

to $y'' + m(t)y' + k(t)y = r(t)$ then

Fact

$$\boxed{\underbrace{c_1y_1(t) + c_2y_2(t)}_{\text{Complementary soln}} + \underbrace{P(t)}_{\text{particular solution}} \text{ is the general soution}}$$

Why? If two solutions $P_1(t)$ & $P_2(t)$ to

$L[y] = r(t)$, (i.e. $L[P_1] = r(t)$ $L[P_2] = r(t)$)

linearity of L implies that

$$L[P_1 - P_2] = L[P_1] - L[P_2] = r(t) - r(t) = 0$$

i.e. $(P_1 - P_2)$ is a solution to $L[y] = 0$ or

$\underline{y'' + m(t)y' + k(t)y = 0}$ so $P_1 - P_2 = c_1y_1 + c_2y_2$ for some c_1 & c_2.

homogeneous

Suppose we have P_2, then

$$P_1 = P_2 + c_1y_1 + c_2y_2 \text{ for some } c_1 \,\&\, c_2.$$

Problem becomes: how to find a $P(t)$?

Methods:

1) There is a formula (method of variation of parameters)

y_1 & y_2 are a fundamental set for the homogeneous eq.

then

$$\boxed{P(t) = -y_2(t)\int_{t_0}^{t}\frac{y_1(s)r(s)}{W[y_1,y_2](s)}ds + y_1(t)\int_{t_0}^{t}\frac{y_2(s)r(s)}{W[y_1,y_2](s)}ds}$$

is a particular solution.

Method 2: Method of undetermined coefficients

"Guess" a formula for $P(t)$

Eg: $y'' + \omega_F^2 y = \cos(\omega_I t)$

Associated homogeneous:

$y'' + \omega_F^2 y = 0$ with solutions

$c_1\cos(\omega_F t) + c_2\sin(\omega_F t)$

Ansatz for particular solution

$$P(t) = A\cos(\omega_I t) + B\sin(\omega_I t)$$

$$P'(t) = -\omega_I A\sin(\omega_I t) + B\omega_I\cos(\omega_I t)$$

$$P''(t) = -\omega_I^2 A\cos(\omega_I t) - \omega_I^2 B\sin(\omega_I t)$$

$$P'' - \omega_F^2 P = \cos(\omega_I t)$$

$$-\omega_I^2\left(A\cos(\omega_I t) + B\sin(\omega_I t)\right) + \omega_F^2\left(A\cos(\omega_I t) + B\sin(\omega_I t)\right) = \cos(\omega_I t)$$

sol'n: $A = \dfrac{1}{\omega_F^2 - \omega_I^2}$, $B = 0$

$$P(t) = \frac{1}{\omega_F^2 - \omega_I^2}\cos(\omega_I t) \quad \text{if } \omega_F \neq \omega_I.$$